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Generalized Semi-Symmetric Non-Metric Connections of Non-Integrable Distributions

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Abstract: In this work, the cases of non-integrable distributions in a Riemannian manifold with the first generalized semi-symmetric non-metric connection and the second generalized semi-symmetric non-metric connection are discussed. We obtain the Gauss, Codazzi, and Ricci equations in both cases. Moreover, Chen's inequalities are also obtained in both cases. Some new examples based on non-integrable distributions in a Riemannian manifold with generalized semi-symmetric non-metric connections are proposed.

Keywords: non-integrable distributions; semi-symmetric non-metric connections; Chen's inequalities; Einstein distributions; distributions with constant scalar curvature

MSC: 53C40; 53C42

1. Introduction

In [1], the notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by H. A. Hayden. Some properties of a Riemannian manifold endowed with a semi-symmetric metric connection were studied by K. Yano [2]. Later, the properties of the curvature tensor of a semi-symmetric metric connection in a Sasakian manifold were also investigated by T. Imai [3,4]. Z. Nakao [5] studied the Gauss curvature equation and the Codazzi–Mainardi equation with respect to a semi-symmetric metric connection on a Riemannian manifold and a submanifold. The idea of studying the tangent bundle of a hypersurface with semi-symmetric metric connections was presented by Gozutok and Esin [6]. In [7], Demirbag investigated the properties of a weakly Ricci-symmetric manifold admitting a semi-symmetric metric connection. N. S. Agashe and M. R. Chafle showed some properties of submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection in [8,9]. In [10,11], the study of non-integrable distributions, as a generalized version of distributions, was initiated by Synge. In [12], a regular distribution was shown in a Riemannian manifold.

Besides this, in [13–15], an important inequality was established by B. Y. Chen, called Chen inequality. In geometry, by studying different submanifolds in various ambient spaces, we can obtain similar results. In [16,17], Mihai and Özgü presented the relationships between the mean curvature associated with the semi-symmetric metric connection, scalar, and sectional curvatures and the k-Ricci curvature. In this paper, we obtain the Chen inequalities of non-integrable distributions of real-space forms endowed with the first generalized semi-symmetric non-metric connection and the second generalized semi-symmetric non-metric connection.

In the literature, we find several works that were conducted with Einstein manifolds and manifolds involving a constant scalar curvature. In [18], Dobarro and Unal studied Ricci-flat and Einstein Lorentzian multiply-warped products and constant scalar curvatures for this class of warped products. In [19–21], the authors obtained some results with Einstein warped-product manifolds with a semi-symmetric non-metric connection.

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In Section 2, we obtain the Gauss, Codazzi, and Ricci equations for non-integrable distributions with the first generalized semi-symmetric non-metric connection by establishing the Gauss formula and the Weingarten formula. Meanwhile, the result of the Chen inequality is presented. In Section 3, we obtain the Gauss, Codazzi, and Ricci equations for non-integrable distributions by establishing the Gauss formula and the Weingarten formula and the second generalized semi-symmetric non-metric connection. Meanwhile, we obtain the result of the Chen inequality. Finally, in Section 4, some examples based on non-integrable distributions in a Riemannian manifold with generalized semi-symmetric non-metric connections are presented.

2. Non-Integrable Distributions with the First Generalized Semi-Symmetric Non-Metric Connection

Let (M, g) be a m-dimensional smooth Riemannian manifold, where g is the Riemannian metric and ∇ is the Levi–Civita connection on (M, g). For $X, Y \in \Gamma(M)$, denote $\nabla_X Y$ the covariant derivative of Y with respect to X and represent by $\Gamma(M)$ the $C^\infty(M)$-module of vector fields on M.

Definition 1. *If there are $X, Y \in \Gamma(D)$ such that $[X, Y]$ is not in $\Gamma(D)$, we say that D is a non-integrable distribution, where D is a sub-bundle of the tangent bundle TM with a constant rank n and $\Gamma(D)$ is the space of sections of D.*

Let g^D be a metric tensor field in the distribution D and let $g^{D^\perp}$ be a metric tensor field in the orthogonal distribution to D, such that $g = g^D \oplus g^{D^\perp}$.

Definition 2. *Let $\pi^D : TM \to D$, $\pi^{D^\perp} : TM \to D^\perp$ be the projections associated to the tangent bundle TM; then, $\nabla^D_X Y = \pi^D(\nabla_X Y)$ and $[X, Y]^D = \pi^D([X, Y])$ and $[X, Y]^{D^\perp} = \pi^{D^\perp}([X, Y])$ for any $X, Y \in \Gamma(D)$.*

By [12], we obtain

$$\nabla^D_{fX} Y = f \nabla^D_X Y, \ \ \nabla^D_X (fY) = X(f)Y + f \nabla^D_X Y, \tag{1}$$

where $X, Y \in \Gamma(D)$ and $f \in C^\infty(M)$.

$$\nabla^D_X g^D = 0, \quad T(X, Y) := \nabla^D_X Y - \nabla^D_Y X - [X, Y] = -[X, Y]^{D^\perp}, \tag{2}$$

and

$$\nabla_X Y = \nabla^D_X Y + B(X, Y). \tag{3}$$

where $B(X, Y) = \pi^{D^\perp} \nabla_X Y$ and $B(X, Y) \neq B(Y, X)$.

Definition 3. *For any $V \in \Gamma(TM)$, let ω be a 1-form satisfying $\omega(V) = g(U, V)$, here $U \in \Gamma(TM)$ is a vector field. Let $\lambda_1, \lambda_2 \in C^\infty(M)$, we give the definition of the first generalized semi-symmetric non-metric connection on M*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \lambda_1 \omega(Y) X - \lambda_2 g(X, Y) U. \tag{4}$$

Let $U^D = \pi^D U$ and $U^{D^\perp} = \pi^{D^\perp} U$; then, $U = U^D + U^{D^\perp}$.

Definition 4. *Let*

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}^D_X Y + \widetilde{B}(X, Y), \ \ \widetilde{\nabla}^D_X Y = \pi^D \widetilde{\nabla}_X Y, \ \ \widetilde{B}(X, Y) = \pi^{D^\perp} \widetilde{\nabla}_X Y. \tag{5}$$

Then,

$$\widetilde{\nabla}^D_X Y = \nabla^D_X Y + \lambda_1 \omega(Y) X - \lambda_2 g(X,Y) U^D, \ \ \widetilde{B}(X,Y) = B(X,Y) - \lambda_2 g(X,Y) U^{D^\perp}, \quad (6)$$

where $\widetilde{B}(X,Y)$ is called the second fundamental form with the first generalized semi-symmetric non-metric connection.

Then, by (2) and (6), we obtain

$$\widetilde{\nabla}^D_X (g^D)(Y,Z) = (\lambda_2 - \lambda_1)[g(X,Z)\omega(Y) + g(X,Y)\omega(Z)], \quad (7)$$
$$\widetilde{T}^D(X,Y) = -[X,Y]^{D^\perp} + \lambda_1[\omega(Y)X - \omega(X)Y].$$

If $D = TM$, we obtain the following results:

Theorem 1. *If a linear connection $\widetilde{\nabla}^D : \Gamma(D) \times \Gamma(D) \to \Gamma(D)$ on D satisfies Equation (7), then this connection is unique.*

We choose $\{E_1, \ldots, E_n\}$ as an orthonormal basis of D and let $\widetilde{H} = \frac{1}{n}\sum_{i=1}^n \widetilde{B}(E_i, E_i) \in \Gamma(D^\perp)$ be the mean curvature vector associated to $\widetilde{\nabla}$ on D. Similarly, let $H = \frac{1}{n}\sum_{i=1}^n B(E_i, E_i)$; then, $\widetilde{H} = H - \lambda_2 U^{D^\perp}$. If $\widetilde{H} = 0$, we say that D is minimal with the first generalized semi-symmetric non-metric connection $\widetilde{\nabla}$. If $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, we say that curve γ is $\widetilde{\nabla}$-geodesic. If every $\widetilde{\nabla}$-geodesic with an initial condition in D is contained in D, we say that D is totally geodesic with the first generalized semi-symmetric non-metric connection $\widetilde{\nabla}$.

Let $h(X,Y) = \frac{1}{2}[B(X,Y) + B(Y,X)]$ and $\widetilde{h}(X,Y) = \frac{1}{2}[\widetilde{B}(X,Y) + \widetilde{B}(Y,X)]$; then, according to [12], we obtain the following:

Proposition 1. *(1) If D is totally geodesic with respect to the first generalized semi-symmetric non-metric connection $\widetilde{\nabla}$, then $\widetilde{B}(X,Y)$ is dissymmetrical.*
(2) When $U \in \Gamma(D)$, $H = \widetilde{H}$,$U \in \Gamma(D)$, and vice versa.
(3) If $h = Hg^D$ (or $\widetilde{h} = \widetilde{H}g^D$), then D is umbilical with respect to ∇ (resp. $\widetilde{\nabla}$).

Proposition 2. *If D is umbilical with respect to ∇, then D is umbilical with respect to $\widetilde{\nabla}$, and vice versa.*

Proof. For $X, Y \in D$, by $\widetilde{h}(X,Y) = h(X,Y) - \lambda_2 g(X,Y) U^{D^\perp}$ and $\widetilde{H} = H - \lambda_2 U^{D^\perp}$, then $\widetilde{H} g^D(X,Y) = H g^D(X,Y) - \lambda_2 U^{D^\perp} g^D(X,Y)$. Therefore, we obtain Proposition 1. □

Thus, by Definition 4, we obtain

$$\widetilde{\nabla}_X \eta = \nabla_X \eta + \lambda_1 \omega(\eta) X, \quad (8)$$

where $\eta \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$. We define

$$g^D(A_\eta X, Y) := g^{D^\perp}(B(X,Y), \eta), \quad (9)$$

where $A_\eta : \Gamma(D) \to \Gamma(D)$ is the shape operator with respect to ∇. Let $L^\perp_X \eta = \pi^{D^\perp} \nabla_X \eta$; then, $\nabla_X \eta = \pi^D \nabla_X \eta + L^\perp_X \eta$, and so we can get the Weingarten formula with respect to ∇

$$\pi^D \nabla_X \eta = -A_\eta X, \ \ \nabla_X \eta = -A_\eta X + L^\perp_X \eta, \quad (10)$$

where $L^\perp_X \eta : \Gamma(D) \times \Gamma(D^\perp) \to \Gamma(D^\perp)$ is a metric connection on $D^\perp$ along $\Gamma(D)$. Let $\widetilde{A}_\eta = (A_\eta - \lambda_1 \omega(\eta))I$; then, by (8) and (10), we have the Weingarten formula with respect to $\widetilde{\nabla}$

$$\widetilde{\nabla}_X \eta = -\widetilde{A}_\eta X + L^\perp_X \eta, \quad (11)$$

Given $X_1, X_2, X_3 \in \Gamma(TM)$, we define the curvature tensor $\widetilde{R}$ with respect to $\widetilde{\nabla}$

$$\widetilde{R}(X_1, X_2)X_3 := \widetilde{\nabla}_{X_1}\widetilde{\nabla}_{X_2}X_3 - \widetilde{\nabla}_{X_2}\widetilde{\nabla}_{X_1}X_3 - \widetilde{\nabla}_{[X_1,X_2]}X_3. \tag{12}$$

Given $X_1, X_2, X_3 \in \Gamma(D)$, we define the curvature tensor $\widetilde{R}^D$ on D with respect to $\widetilde{\nabla}^D$

$$\widetilde{R}^D(X_1, X_2)X_3 := \widetilde{\nabla}^D_{X_1}\widetilde{\nabla}^D_{X_2}X_3 - \widetilde{\nabla}^D_{X_2}\widetilde{\nabla}^D_{X_1}X_3 - \widetilde{\nabla}^D_{[X_1,X_2]^D}X_3 - \pi^D[[X_1, X_2]^{D^\perp}, X_3]. \tag{13}$$

In (13), $\widetilde{R}^D$ is a tensor field created by adding the extra term $-\pi^D[[X_1, X_2]^{D^\perp}, X_3]$.

Given $X_1, X_2, X_3, X_4 \in \Gamma(D)$, similarly, we define the Riemannian curvature tensor $\widetilde{R}$ and $\widetilde{R}^D$

$$\widetilde{R}(X_1, X_2, X_3, X_4) = g(\widetilde{R}(X_1, X_2)X_3, X_4), \;\; \widetilde{R}^D(X_1, X_2, X_3, X_4) = g(\widetilde{R}^D(X_1, X_2)X_3, X_4). \tag{14}$$

Theorem 2. *If $X_1, X_2, X_3, X_4 \in \Gamma(D)$, we obtain the Gauss equation for D with respect to $\widetilde{\nabla}$*

$$\begin{aligned}\widetilde{R}(X_1, X_2, X_3, X_4) = {} & \widetilde{R}^D(X_1, X_2, X_3, X_4) + g(B(X_2, X_4), B(X_1, X_3)) - g(B(X_1, X_4), B(X_2, X_3)) \\ & + g(B(X_3, X_4), [X_1, X_2]) - \lambda_1\omega(\widetilde{B}(X_1, X_3))g(X_2, X_4) + \lambda_1\omega(\widetilde{B}(X_2, X_3))g(X_1, X_4) \\ & - \lambda_2 g(X_1, X_3)\omega(B(X_2, X_4)) + \lambda_2 g(X_2, X_3)\omega(B(X_1, X_4)).\end{aligned} \tag{15}$$

Proof. From (5) and (11), for $X_1, X_2, X_3 \in \Gamma(D)$, we have

$$\begin{aligned}\widetilde{\nabla}_{X_1}\widetilde{\nabla}_{X_2}X_3 = {} & \widetilde{\nabla}^D_{X_1}\widetilde{\nabla}^D_{X_2}X_3 + \widetilde{B}(X_1, \widetilde{\nabla}^D_{X_2}X_3) - A_{\widetilde{B}(X_2,X_3)}X_1 \\ & + \lambda_1\omega(\widetilde{B}(X_2, X_3))X + L^\perp_{X_1}(\widetilde{B}(X_2, X_3)),\end{aligned} \tag{16}$$

$$\begin{aligned}\widetilde{\nabla}_{X_2}\widetilde{\nabla}_{X_1}X_3 = {} & \widetilde{\nabla}^D_{X_2}\widetilde{\nabla}^D_{X_1}X_3 + \widetilde{B}(X_2, \widetilde{\nabla}^D_{X_1}X_3) - A_{\widetilde{B}(X_1,X_3)}X_2 \\ & + \lambda_1\omega(\widetilde{B}(X_1, X_3))Y + L^\perp_{X_2}(\widetilde{B}(X_1, X_3)).\end{aligned} \tag{17}$$

For $X_1, X_2 \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_{X_1}X_2 = \widetilde{\nabla}_{X_2}X_1 + [X_1, X_2] + \lambda_1\omega(X_2)X_1 - \lambda_1\omega(X_1)X_2. \tag{18}$$

Then, by (11) and (18), we have

$$\widetilde{\nabla}_{[X_1,X_2]^{D^\perp}}X_3 = -A_{[X_1,X_2]^{D^\perp}}X_3 + L^\perp_{X_3}([X_1, X_2]^{D^\perp}) + \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp} + [[X_1, X_2]^{D^\perp}, X_3]. \tag{19}$$

By (19) and (5), we get

$$\begin{aligned}\widetilde{\nabla}_{[X_1,X_2]}X_3 &= \widetilde{\nabla}_{[X_1,X_2]^D}X_3 + \widetilde{\nabla}_{[X_1,X_2]^{D^\perp}}X_3 \\ &= \widetilde{\nabla}^D_{[X_1,X_2]^D}X_3 + \widetilde{B}([X_1, X_2]^D, X_3) - A_{[X_1,X_2]^{D^\perp}}X_3 \\ &\quad + L^\perp_Z([X_1, X_2]^{D^\perp}) + \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp} + [[X_1, X_2]^{D^\perp}, X_3].\end{aligned} \tag{20}$$

By (12)–(20), we have

$$\begin{aligned}\widetilde{R}(X_1, X_2)X_3 = {} & \widetilde{R}^D(X_1, X_2)X_3 - \pi^{D^\perp}[[X_1, X_2]^{D^\perp}, X_3] + \widetilde{B}(X_1, \widetilde{\nabla}^D_{X_2}X_3) \\ & - \widetilde{B}(X_2, \widetilde{\nabla}^D_{X_1}X_3) - \widetilde{B}([X_1, X_2]^D, X_3) - A_{\widetilde{B}(X_2,X_3)}X_1 + A_{\widetilde{B}(X_1,X_3)}X_2 \\ & + L^\perp_{X_1}(\widetilde{B}(X_2, X_3)) - L^\perp_{X_2}(\widetilde{B}(X_1, X_3)) + \lambda_1\omega(\widetilde{B}(X_2, X_3))X_1 - \lambda_1\omega(\widetilde{B}(X_1, X_3))X_2 \\ & + A_{[X_1,X_2]^{D^\perp}}X_3 - L^\perp_Z([X_1, X_2]^{D^\perp}) - \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp}.\end{aligned} \tag{21}$$

By the second equality in (6) and (9), (14), (21), we get Theorem 2. □

Corollary 1. *If $U = 0$, then $\omega = 0$ and $\widetilde{\nabla} = \nabla$, and we have*

$$\begin{aligned} R(X_1, X_2, X_3, X_4) =& R^D(X_1, X_2, X_3, X_4) - g(B(X_1, X_4), B(X_2, X_3)) \\ &+ g(B(X_2, X_4), B(X_1, X_3)) + g(B(X_3, X_4), [X_1, X_2]). \end{aligned} \tag{22}$$

Theorem 3. *If $X_1, X_2, X_3 \in \Gamma(D)$, we get the Codazzi equation with respect to $\widetilde{\nabla}$*

$$\begin{aligned} (\widetilde{R}(X_1, X_2)X_3)^{D^\perp} =& (L^\perp_{X_1}\widetilde{B})(X_2, X_3) - (L^\perp_{X_2}\widetilde{B})(X_1, X_3) - \lambda_1\omega(X_1)\widetilde{B}(X_2, X_3) \\ &+ \lambda_1\omega(X_2)\widetilde{B}(X_1, X_3) - \pi^{D^\perp}[[X_1, X_2]^{D^\perp}, X_3] - L^\perp_Z([X_1, X_2]^{D^\perp}) \\ &- \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp}, \end{aligned} \tag{23}$$

where $(L^\perp_{X_1}\widetilde{B})(X_2, X_3) = L^\perp_{X_1}(\widetilde{B}(X_2, X_3)) - \widetilde{B}(\widetilde{\nabla}^D_{X_1}X_2, X_3) - \widetilde{B}(X_2, \widetilde{\nabla}^D_{X_1}X_3)$.

Proof. By (21), we have

$$\begin{aligned} (\widetilde{R}(X_1, X_2)X_3)^{D^\perp} =& -\pi^{D^\perp}[[X_1, X_2]^{D^\perp}, X_3] + \widetilde{B}(X_1, \widetilde{\nabla}^D_{X_2}X_3) \\ &- \widetilde{B}(X_2, \widetilde{\nabla}^D_{X_1}X_3) - \widetilde{B}([X_1, X_2]^D, X_3) + L^\perp_{X_1}(\widetilde{B}(X_2, X_3)) \\ &- L^\perp_{X_2}(\widetilde{B}(X_1, X_3)) - L^\perp_Z([X_1, X_2]^{D^\perp}) - \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp}. \end{aligned} \tag{24}$$

By (18), (24) and (25), we get

$$\begin{aligned} (\widetilde{R}(X_1, X_2)X_3)^{D^\perp} =& -\pi^{D^\perp}[[X_1, X_2]^{D^\perp}, X_3] + \widetilde{B}(X_1, \widetilde{\nabla}^D_{X_2}X_3) - \widetilde{B}(X_2, \widetilde{\nabla}^D_{X_1}X_3) \\ &- \widetilde{B}(\widetilde{\nabla}^D_{X_1}X_2 - \widetilde{\nabla}^D_{X_2}X_1 - \lambda_1\omega(X_2)X_1 + \lambda_1\omega(X_1)X_2, X_3) + L^\perp_X(\widetilde{B}(X_2, X_3)) \\ &- L^\perp_{X_2}(\widetilde{B}(X_1, X_3)) - L^\perp_Z([X_1, X_2]^{D^\perp}) - \lambda_1\omega(X_3)[X_1, X_2]^{D^\perp}. \end{aligned} \tag{25}$$

Thus, (23) holds. □

Corollary 2. *If $U = 0$, then we have*

$$\begin{aligned} (R(X_1, X_2)X_3)^{D^\perp} =& (L^\perp_{X_1}B)(X_2, X_3) - (L^\perp_{X_2}B)(X_1, X_3) \\ &- \pi^{D^\perp}[[X_1, X_2]^{D^\perp}, X_3] - L^\perp_Z([X_1, X_2]^{D^\perp}). \end{aligned} \tag{26}$$

Theorem 4. *If $X_1, X_2 \in \Gamma(D)$, $\eta \in \Gamma(D^\perp)$, we get the Ricci equation for D with respect to $\widetilde{\nabla}$*

$$(\widetilde{R}(X_1, X_2)\eta)^{D^\perp} = -\widetilde{B}(X_1, \widetilde{A}_\eta X_2) + \widetilde{B}(X_2, \widetilde{A}_\eta X_1) + \widetilde{R}^{L^\perp}(X_1, X_2)\eta \tag{27}$$

where

$$\widetilde{R}^{L^\perp}(X_1, X_2)\eta := L^\perp_{X_1}L^\perp_{X_2}\eta - L^\perp_{X_2}L^\perp_{X_1}\eta - L^\perp_{[X_1,X_2]^D}\eta - \pi^{D^\perp}\widetilde{\nabla}_{[X_1,X_2]^\perp}\eta.$$

Proof. From (5) and (11), we have

$$\widetilde{\nabla}_{X_1}\widetilde{\nabla}_{X_2}\eta = -\widetilde{\nabla}^D_{X_1}(\widetilde{A}_\eta X_2) - \widetilde{B}(X_1, \widetilde{A}_\eta X_2) - \widetilde{A}_{L^\perp_{X_2}\eta}X_1 + L^\perp_{X_1}L^\perp_{X_2}\eta, \tag{28}$$

$$\widetilde{\nabla}_{X_2}\widetilde{\nabla}_{X_1}\eta = -\widetilde{\nabla}^D_{X_2}(\widetilde{A}_\eta X_1) - \widetilde{B}(X_2, \widetilde{A}_\eta X_1) - \widetilde{A}_{L^\perp_{X_1}\eta}X_2 + L^\perp_{X_2}L^\perp_{X_1}\eta, \tag{29}$$

By $\widetilde{\nabla}_{[X_1,X_2]}\eta = \widetilde{\nabla}_{[X_1,X_2]^D}\eta + \widetilde{\nabla}_{[X_1,X_2]^{D^\perp}}\eta$, we have

$$\widetilde{\nabla}_{[X_1,X_2]}\eta = -\widetilde{A}_\eta([X_1,X_2]^D) + L^\perp_{[X_1,X_2]^D}\eta + \pi^D\widetilde{\nabla}_{[X_1,X_2]^{D^\perp}}\eta + \pi^{D^\perp}\widetilde{\nabla}_{[X_1,X_2]^{D^\perp}}\eta. \tag{30}$$

So (27) holds. □

Corollary 3. *If $U = 0$, then we have*

$$(R(X_1,X_2)\eta)^{D^\perp} = -B(X_1, A_\eta X_2) + B(X_2, A_\eta X_1) + R^{L^\perp}(X_1,X_2)\eta, \tag{31}$$

where

$$R^{L^\perp}(X_1,X_2)\eta := L^\perp_{X_1}L^\perp_{X_2}\eta - L^\perp_{X_2}L^\perp_{X_1}\eta - L^\perp_{[X_1,X_2]^D}\eta - \pi^{D^\perp}\nabla_{[X_1,X_2]^\perp}\eta.$$

Now, we present the proof of the Chen inequality with respect to D and $\widetilde{\nabla}$. By $(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$, we let

$$\alpha(X_1,X_2) = (\nabla_{X_1}\omega)(X_2) - \lambda_1\omega(X_1)\omega(X_2) + \frac{\lambda_2}{2}g(X_1,X_2)\omega(U),$$

$$\beta(X_1,X_2) = \frac{\omega(U)}{2}g(X_1,X_2) + \omega(X_1)\omega(X_2).$$

where $X_1, X_2 \in \Gamma(TM)$. In [16], we get

$$\begin{aligned}\widetilde{R}(X_1,X_2,X_3,X_4) = {} & R(X_1,X_2,X_3,X_4) + \lambda_1\alpha(X_1,X_3)g(X_2,X_4) - \lambda_1\alpha(X_2,X_3)g(X_1,X_4) \\ & - \lambda_2(\lambda_1-\lambda_2)g(X_2,X_3)\beta(X_1,X_4) + \lambda_2(\lambda_1-\lambda_2)g(X_1,X_3)\beta(X_2,X_4) \\ & + \lambda_2 g(X_1,X_3)\alpha(X_2,X_4) - \lambda_2 g(X_2,X_3)\alpha(X_1,X_4).\end{aligned} \tag{32}$$

Let $\{E_1,\ldots,E_n,E_{n+1},\ldots,E_m\}$ be a local orthonormal frame in M and $D = span\{E_1,\ldots,E_n\}$. And let $\lambda = \sum_{i=1}^n \alpha(E_i,E_i)$,$\mu = \sum_{i=1}^n \beta(E_i,E_i)$. Let M be an m-dimensional real space form of constant sectional curvature c endowed with the first generalized semi-symmetric non-metric connection $\widetilde{\nabla}$. The curvature tensor R with respect to the Levi–Civita connection on M is expressed by

$$R(X_1,X_2,X_3,X_4) = c\{g(X_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(X_2,X_4)\}. \tag{33}$$

By (33) and (35), we get

$$\begin{aligned}\widetilde{R}(X_1,X_2,X_3,X_4) = {} & c\{g(X_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(X_2,X_4) - \lambda_2 g(X_2,X_4)\alpha(X_1,X_4) \\ & + \lambda_2(\lambda_1-\lambda_2)g(X_1,X_3)\beta(X_2,X_4) - \lambda_2(\lambda_1-\lambda_2)g(X_2,X_3)\beta(X_1,X_4) \\ & - \lambda_1\alpha(X_2,X_3)g(X_1,X_4) + \lambda_2 g(X_1,X_3)\alpha(X_2,X_4).\end{aligned} \tag{34}$$

Let $\Pi \subset D$, be a two-plane section. Denote by $\widetilde{K}^D(\Pi)$ the sectional curvature of D with the induced connection $\widetilde{\nabla}^D$ defined by

$$\widetilde{K}^D(\Pi) = \frac{1}{2}[\widetilde{R}^D(E_1,E_2,E_2,E_1) - \widetilde{R}^D(E_1,E_2,E_1,E_2)], \tag{35}$$

where E_1, E_2 are orthonormal bases of Π and $\widetilde{K}^D(\Pi)$ is independent of the choice of E_1, E_2. For any orthonormal basis $\{E_1,\ldots,E_n\}$ of D, the scalar curvature $\widetilde{\tau}^D$ with respect to D and $\widetilde{\nabla}^D$ is defined by

$$\widetilde{\tau}^D = \frac{1}{2}\sum_{1\le i,j\le n}\widetilde{R}^D(E_i,E_j,E_j,E_i). \tag{36}$$

Let E_1, E_2 be the orthonormal bases of $\Pi \subset D$ such that the following definitions are independent of the choice of the orthonormal bases:

$$A^D = \frac{1}{2}\sum_{1\le i,j\le n} g(B(E_j,E_i),[E_j,E_i]), \tag{37}$$

$$\begin{aligned}\Omega^{\Pi} = {} & \frac{\lambda_1+\lambda_2}{2}[\alpha(E_1,E_1)+\alpha(E_2,E_2)] - \frac{1}{2}g(B(E_1,E_2)-B(E_2,E_1),[E_1,E_2]) \\ & + \frac{\lambda_2}{2}(\lambda_1-\lambda_2)[\beta(E_1,E_1)+\beta(E_2,E_2)] + \frac{\lambda_2}{2}[\omega(B(E_1,E_1))+\omega(B(E_2,E_2))] \\ & + \frac{\lambda_1}{2}[\omega(\widetilde{B}(E_1,E_1))+\omega(\widetilde{B}(E_2,E_2))].\end{aligned} \tag{38}$$

Theorem 5. *Let $TM = D \oplus D^{\perp}$, $\mathrm{dim}D = n \ge 3$, and let M be a manifold with constant sectional curvature c endowed with a connection $\widetilde{\nabla}$; then, we get the Chen inequality:*

$$\begin{aligned}\widetilde{\tau}^D - \widetilde{K}^D(\Pi) \le {} & \frac{(n+1)(n-2)}{2}c - \frac{\lambda_1+\lambda_2}{2}(n-1)\lambda - \frac{\lambda_2}{2}(\lambda_1-\lambda_2)(n-1)\mu \\ & - \frac{\lambda_2}{2}(n-1)n\omega(B) - \frac{\lambda_2}{2}(n-1)n\omega(\widetilde{B}) + A^D + \Omega^{\Pi} + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}||B||^2.\end{aligned} \tag{39}$$

where $||B||^2 = \sum_{i,j=1}^n g(B(E_i,E_j),B(E_i,E_j))$ is the squared length of B and $||\widetilde{B}||^2 = \sum_{i,j=1}^n g(\widetilde{B}(E_i,E_j),\widetilde{B}(E_i,E_j))$ is the squared length of $\widetilde{B}$.

Proof. We choose the orthonormal bases $\{E_1,\ldots,E_n\}$ and $\{E_{n+1},\ldots,E_m\}$ of D and $D^{\perp}$, respectively, such that $\Pi \subset D = span\{E_1,E_2\}$. By Theorem 2, (34) and (35), we obtain

$$\widetilde{K}^D(\Pi) = c - \Omega^{\Pi} + \sum_{r=n+1}^{m}[h_{11}^r h_{22}^r - h_{12}^r h_{21}^r]. \tag{40}$$

Then, we get

$$\begin{aligned}\widetilde{\tau}^D = {} & \frac{1}{2}\sum_{1\le i\ne j\le n}\widetilde{R}^D(E_i,E_j,E_j,E_i) \\ = {} & \frac{n(n-1)}{2}c - \frac{\lambda_1+\lambda_2}{2}(n-1)\lambda - \frac{\lambda_2}{2}(\lambda_1-\lambda_2)(n-1)\mu - \frac{\lambda_2}{2}(n-1)n\omega(B) \\ & - \frac{\lambda_1}{2}(n-1)n\omega(\widetilde{B}) + A^D + \sum_{r=n+1}^{m}\sum_{1\le i<j\le n}[h_{ii}^r h_{jj}^r - h_{ij}^r h_{ji}^r].\end{aligned} \tag{41}$$

Thus,

$$\begin{aligned}\widetilde{\tau}^D - \widetilde{K}^D(\Pi) = {} & \frac{(n+1)(n-2)}{2}c - \frac{\lambda_1+\lambda_2}{2}(n-1)\lambda - \frac{\lambda_2}{2}(\lambda_1-\lambda_2)(n-1)\mu \\ & - \frac{\lambda_2}{2}(n-1)n\omega(B) - \frac{\lambda_1}{2}(n-1)\omega(\widetilde{B}) + A^D + \Omega^{\Pi} \\ & + \sum_{r=n+1}^{m}[(h_{11}^r+h_{22}^r)\sum_{3\le j\le n}h_{jj}^r + \sum_{3\le i<j\le n}h_{ii}^r h_{jj}^r - \sum_{1\le i<j\le n}h_{ij}^r h_{ji}^r + h_{12}^r h_{21}^r].\end{aligned}$$

By Lemma 2.4 in [22], we get

$$\sum_{r=n+1}^{m}[(h_{11}^r+h_{22}^r)\sum_{3\le j\le n}h_{jj}^r + \sum_{3\le i<j\le n}h_{ii}^r h_{jj}^r] \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2. \tag{42}$$

We note that

$$\sum_{r=n+1}^{m}[-\sum_{1\le i<j\le n} h^r_{ij}h^r_{ji} + h^r_{12}h^r_{21}] \tag{43}$$
$$= \sum_{r=n+1}^{m}[-\sum_{3\le j\le n} h^r_{1j}h^r_{j1} - \sum_{2\le i<j\le n} h^r_{ij}h^r_{ji}]$$
$$\le \sum_{r=n+1}^{m}[\sum_{3\le j\le n} \frac{(h^r_{1j})^2+(h^r_{j1})^2}{2} + \sum_{2\le i<j\le n} \frac{(h^r_{ij})^2+(h^r_{ji})^2}{2}]$$
$$\le \sum_{r=n+1}^{m}[\sum_{3\le j\le n} \frac{(h^r_{1j})^2+(h^r_{j1})^2}{2} + \sum_{2\le i<j\le n} \frac{(h^r_{ij})^2+(h^r_{ji})^2}{2} + \sum_{i=1}^{n}\frac{(h^r_{ii})^2}{2} + \frac{(h^r_{12})^2+(h^r_{21})^2}{2}]$$
$$= \frac{\|B\|^2}{2}.$$

Thus, (39) holds. □

Remark 1. *When $U \in \Gamma(D)$, that is $B = \widetilde{B}$, we get the following inequality*

$$\widetilde{\tau}^D - \widetilde{K}^D(\Pi) \le \frac{(n+1)(n-2)}{2}c - \frac{\lambda_1+\lambda_2}{2}(n-1)\lambda - \frac{\lambda_2}{2}(\lambda_1-\lambda_2)(n-1)\mu$$
$$- \frac{\lambda_1+\lambda_2}{2}(n-1)n\omega(B) + A^D + \Omega^\Pi + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}||B||^2. \tag{44}$$

Corollary 4. *If D is totally geodesic with respect to $\widetilde{\widetilde{\nabla}}$ and $h^r_{12} = h^r_{21} = 0$, then the equality case of (39) holds, and vice versa.*

Proof. From the equality case of (42) and the equality case of (43), Corollary 3 holds. □

Corollary 5. *If D is an integrable distribution—that is if $X, Y \in \Gamma(D)$—then $[X, Y]$ is in $\Gamma(D)$. Then,*

$$\widetilde{\tau}^D - \widetilde{K}^D(\Pi) \le \frac{(n+1)(n-2)}{2}c - \frac{\lambda_1+\lambda_2}{2}(n-1)\lambda - \frac{\lambda_2}{2}(\lambda_1-\lambda_2)(n-1)\mu$$
$$- \frac{\lambda_2}{2}(n-1)n\omega(B) - \frac{\lambda_2}{2}(n-1)n\omega(\widetilde{B}) + \Omega^\Pi + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}||B||^2. \tag{45}$$

where

$$\Omega^\Pi = \frac{\lambda_1+\lambda_2}{2}[\alpha(E_1,E_1)+\alpha(E_2,E_2)] + \frac{\lambda_2}{2}(\lambda_1-\lambda_2)[\beta(E_1,E_1)+\beta(E_2,E_2)] \tag{46}$$
$$+ \frac{\lambda_2}{2}[\omega(B(E_1,E_1))+\omega(B(E_2,E_2))] + \frac{\lambda_1}{2}[\omega(\widetilde{B}(E_1,E_1))+\omega(\widetilde{B}(E_2,E_2))].$$

We choose the orthonormal basis $\{E_1, \ldots, E_n\}$ of D and let $X = E_1$. We define

$$\widetilde{\mathrm{Ric}}^D(X) = \sum_{i=2}^{n}\widetilde{R}^D(X,E_i,E_i,X); \quad A^D(X) = \sum_{i=2}^{n} g(B(E_i,X),[E_i,X]); \tag{47}$$
$$\|B^X\|^2 = \sum_{i=2}^{n}[g(B(X,E_i),B(X,E_i)) + g(B(E_i,X),B(E_i,X))].$$

Theorem 6. *Let $TM = D \oplus D^{\perp}$, $\dim D = n \geq 2$, and let M be a manifold with constant sectional curvature c endowed with a connection $\widetilde{\nabla}$, then*

$$\begin{aligned}\widetilde{\mathrm{Ric}}^{D}(X) \leq\ & (n-1)c - \lambda_1\lambda + \lambda_1\alpha(X,X) + \lambda_2(1-n)\alpha(X,X) \\ & + \lambda_2(\lambda_1-\lambda_2)(n-1)\beta(X,X) - \lambda_2(n-1)\omega(B(X,X)) \\ & - \lambda_1 n\omega(\widetilde{B}) + \lambda_1\omega(B(X,X)) + \frac{n^2}{4}\|H\|^2 + \frac{\|B^X\|^2}{2} + A^D(X).\end{aligned} \tag{48}$$

Proof. By (34)–(36), we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}^{D}(X) =\ & (n-1)c - \lambda_1\lambda + \lambda_1\alpha(X,X) + \lambda_2(1-n)\alpha(X,X) \\ & - \lambda_2(\lambda_1-\lambda_2)(n-1)\beta(X,X) - \lambda_2(n-1)\omega(B(X,X)) - \lambda_1 n\omega(\widetilde{B}) \\ & + \lambda_1\omega(B(X,X)) + \sum_{r=n+1}^{m}\sum_{j=2}^{n}[h^r_{11}h^r_{jj} - h^r_{1j}h^r_{j1}] + A^D(X).\end{aligned} \tag{49}$$

From [22], we get

$$\sum_{r=n+1}^{n+p}\sum_{j=2}^{n} h^r_{11}h^r_{jj} \leq \frac{n^2}{4}\|H\|^2. \tag{50}$$

We note that

$$-\sum_{r=n+1}^{m}\sum_{j=2}^{n} h^r_{1j}h^r_{j1} \leq \sum_{r=n+1}^{m}\sum_{j=2}^{n}\frac{(h^r_{1j})^2 + (h^r_{j1})^2}{2} = \frac{\|B^X\|^2}{2}. \tag{51}$$

Thus, (48) holds. □

Corollary 6. *If $h^r_{1j} = -h^r_{j1}$ for $2 \leq j \leq n$ and $h^r_{11} - h^r_{22} - \cdots - h^r_{nn} = 0$, then the equality case of (48) holds, and vice versa.*

Corollary 7. *If D is an integrable distribution—that is if $X, Y \in \Gamma(D)$—then $[X,Y]$ is in $\Gamma(D)$. Then,*

$$\begin{aligned}\widetilde{\mathrm{Ric}}^{D}(X) \leq\ & (n-1)c - \lambda_1\lambda + \lambda_1\alpha(X,X) + \lambda_2(1-n)\alpha(X,X) + \lambda_2(\lambda_1-\lambda_2)(n-1)\beta(X,X) \\ & - \lambda_2(n-1)\omega(B(X,X)) - \lambda_1 n\omega(\widetilde{B}) + \lambda_1\omega(B(X,X)) + \frac{n^2}{4}\|H\|^2 + \frac{\|B^X\|^2}{2}.\end{aligned} \tag{52}$$

3. Non-Integrable Distributions with the Second Generalized Semi-Symmetric Non-Metric Connection

Definition 5. *For any $V \in \Gamma(TM)$, let ω be a one-form satisfying $\omega(V) = g(U,V)$; here, $U \in \Gamma(TM)$ is a vector field. Let $f_1, f_2 \in C^{\infty}(M)$; we give the definition of the second generalized semi-symmetric non-metric connection on M as follows:*

$$\overline{\nabla}_X Y = \nabla_X Y + f_1\omega(X) + f_2\omega(Y)X. \tag{53}$$

Similarly to (2.5), for $X, Y \in \Gamma(D)$

$$\overline{\nabla}_X Y = \overline{\nabla}^D_X Y + \overline{B}(X,Y), \;\; \overline{\nabla}^D_X Y = \pi^D\overline{\nabla}_X Y, \tag{54}$$

where $\overline{B}(X,Y) = \pi^{D^{\perp}}\overline{\nabla}_X Y$, and we call it the second fundamental form with respect to the second generalized semi-symmetric non-metric connection. Therefore, we have

$$\overline{\nabla}^D_X Y = \nabla^D_X Y + f_1\omega(X)Y + f_2\omega(Y)X, \;\; \overline{B}(X,Y) = B(X,Y), \tag{55}$$

where $f_1, f_2 \in C^\infty(M)$.

By (3.3), we have

$$\nabla^D_X(g^D)(Y,Z) = -2f_1\omega(X)g^D(Y,Z) - f_2\omega(Y)g^D(X,Z) - f_2\omega(Z)g^D(X,Y), \tag{56}$$
$$\overline{T}^D(X,Y) = -[X,Y]^{D^\perp} + (f_2 - f_1)[\omega(Y)X - \omega(X)Y].$$

If $D = TM$, we have the following results:

Theorem 7. *If a linear connection $\overline{\nabla}^D : \Gamma(D) \times \Gamma(D) \to \Gamma(D)$ on D satisfies the Equation (56), then this connection is the uniqueness.*

Proposition 3. *D is minimal (or umbilical) with respect to ∇ if and only if D is minimal (or umbilical) with respect to $\overline{\nabla}$.*

Let

$$\overline{\nabla}_X\eta = -\overline{A}_\eta X + L^\perp_X\eta, \tag{57}$$

where $\overline{A}_\eta = (A_\eta - f_2\omega(\eta))I$. Then, by the definition of $\overline{R}$ and $\overline{R}^D$, we get

Theorem 8. *If $X_1, X_2, X_3, X_4 \in \Gamma(D)$ and $\eta \in \Gamma(D)$, we have*

$$\begin{aligned}\overline{R}(X_1,X_2,X_3,X_4) = {} & \overline{R}^D(X_1,X_2,X_3,X_4) - \pi^{D^\perp}[[X_1,X_2]^{D^\perp},X_3] - g(B(X_1,X_3),B(X_2,X_4)) \\ & + g(B(X_3,X_4),[X_1,X_2]) - f_1\omega([X_1,X_2]^{D^\perp})g(X_3,X_4) - f_2 g(X_2,X_4)\omega(B(X_1,X_3)) \\ & + g(B(X_2,X_4),B(X_1,X_3)) + f_2 g(X_1,X_4)\omega(B(X_2,X_3)).\end{aligned} \tag{58}$$

$$\begin{aligned}(\overline{R}(X_1,X_2)X_3)^{D^\perp} = {} & (\overline{L}^\perp_{X_1}B)(X_2,X_3) - (\overline{L}^\perp_{X_2}B)(X_1,X_3) \\ & + (f_1 - f_2)\omega(X_1)B(V) + (f_2 - f_1)\omega(X_2)B(X_1,X_3) - \pi^{D^\perp}[[X_1,X_2]^{D^\perp},X_3] \\ & - L^\perp_{X_3}([X_1,X_2]^{D^\perp}) + (f_1 - f_2)\omega(Z)[X_1,X_2]^{D^\perp},\end{aligned} \tag{59}$$

where $(\overline{L}^\perp_{X_1}B)(X_2,X_3) = \overline{L}^\perp_{X_1}(B(X_2,X_3)) - B(\overline{\nabla}^D_{X_1}X_2,X_3) - B(X_2,\overline{\nabla}^D_{X_1}X_3)$.

$$(\overline{R}(X_1,X_2)\eta)^{D^\perp} = -B(X_1,\overline{A}_\eta X_2) + B(X_2,\overline{A}_\eta X_1) + \overline{R}^{L^\perp}(X_1,X_2)\eta, \tag{60}$$

where

$$\overline{R}^{L^\perp}(X_1,X_2)\eta := \overline{L}^\perp_{X_1}\overline{L}^\perp_{X_2}\eta - \overline{L}^\perp_{X_2}\overline{L}^\perp_{X_1}\eta - \overline{L}^\perp_{[X_1,X_2]^D}\eta - \pi^{D^\perp}\overline{\nabla}_{[X_1,X_2]^\perp}\eta.$$

Remark 2. *We use the equality $\overline{\nabla}_XY = \overline{\nabla}^D_XY + B(X,Y)$ to prove Theorem 7. We use the equality $\widetilde{\nabla}_XY = \widetilde{\nabla}^D_XY + \widetilde{B}(X,Y)$ to prove Theorems 1–3. This is the difference between the two cases.*

We may define $\overline{K}^D(\Pi)$, $\overline{\tau}^D$, and for $X, Y \in \Gamma(TM)$, we obtain

$$\alpha_1(X,Y) = (\nabla_X\omega)(Y).$$

Similarly, let $\lambda_1 = \sum_{i=1}^n \alpha_1(E_i,E_i)$. In [17], for $X_1, X_2, X_3, X_4 \in \Gamma(D)$, we have

$$\begin{aligned}\overline{R}(X_1,X_2,X_3,X_4) = {} & R(X_1,X_2,X_3,X_4) - f_1\alpha_1(,X_2,X_1)g(Z,W) + f_1\alpha_1(X_1,X_2)g(X_3,X_4) \\ & - f_2\alpha_1(X_2,X_3)g(X_1,X_4) + f_2\alpha_1(X_1,X_3)g(X_2,X_4) + f_2^2\omega(X_2)\omega(X_4)g(X_1,X_4) \\ & - f_2^2\omega(X_1)\omega(X_4)g(X_2,X_3).\end{aligned} \tag{61}$$

Let M be an m-dimensional real space form of the constant sectional curvature c endowed with the second generalized semi-symmetric non-metric connection $\overline{\nabla}$. By (33) and (61), we get

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) = & c\{g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4)\} - f_1\alpha_1(X_2, X_1)g(X_3, X_4) \\ & + f_1\alpha_1(X_1, X_2)g(X_3, X_4) - f_2\alpha_1(X_2, X_3)g(X_1, X_4) + f_2\alpha_1(X_1, X_3)g(X_2, X_4) \\ & + f_2^2\omega(X_2)\omega(Z)g(X_1, X_4) - f_2^2\omega(X_1)\omega(X_3)g(X_2, X_4).\end{aligned} \tag{62}$$

Let

$$\begin{aligned}&\mathrm{tr}(\alpha_1|_\Pi) = \alpha_1(E_1, E_1) + \alpha_1(E_2, E_2), \ \ \mathrm{tr}(B|_\Pi) = B(E_1, E_1) + B(E_2, E_2), \\ &\Omega^{\Pi*} = -\frac{1}{2}g(B(E_1, E_2) - B(E_2, E_1), [E_1, E_2]), \\ &\mathrm{tr}(\omega^2|_\Pi) = \omega(E_1^2) + \omega(E_2^2).\end{aligned} \tag{63}$$

Theorem 9. *Let $TM = D \oplus D^\perp$, $\mathrm{dim}D = n \geq 3$, and let M be a manifold with constant sectional curvature c endowed with a connection $\overline{\nabla}$, then*

$$\begin{aligned}\overline{\tau}^D - \overline{K}^D(\Pi) \leq & \frac{(n+1)(n-2)}{2}c - \frac{f_2}{2}(n-1)\lambda_1 - \frac{f_2}{2}n(n-1)\omega(H) + \frac{f_2^2}{2}(n-1)n \\ & + \frac{f_2}{2}\mathrm{tr}(\alpha_1|_\Pi) + \frac{f_2}{2}\omega(\mathrm{tr}(B|_\Pi)) + A^D + \Omega^{\Pi*} + \frac{f_2^2}{2}(n-1)\gamma \\ & - \frac{f_2}{2}tr(\omega\ |^2_\Pi) + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}||B||^2.\end{aligned} \tag{64}$$

Proof. We choose orthonormal bases $\{E_1, \cdots, E_n\}$ and $\{E_{n+1}, \cdots, E_m\}$ of D and $D^\perp$, respectively. Let E_1, E_2 be the orthonormal bases of $\Pi \subset D$. By (62), we obtain

$$\overline{R}(E_1, E_2, E_1, E_2) = -c + f_2\alpha_1(E_1, E_1) - f_2^2\omega(E_1)^2. \tag{65}$$

By (58), we have

$$\begin{aligned}\overline{R}^D(E_1, E_2, E_1, E_2) = & -c + f_2\alpha_1(E_1, E_1) - f_2^2\omega(E_1)^2 + g(B(E_1, E_2), B(E_2, E_1)) \\ & - g(B(E_1, E_1), B(E_2, E_2)) + f_2\omega(B(E_1, E_1)) - g(B(E_1, E_2), [E_1, E_2]).\end{aligned} \tag{66}$$

Similarly, we have

$$\begin{aligned}\overline{R}^D(E_1, E_2, E_2, E_1) = & c - f_2\alpha_1(E_2, E_2) + f_2^2\omega(E_2)^2 - g(B(E_1, E_2), B(E_2, E_1)) \\ & + g(B(E_1, E_1), B(E_2, E_2)) - f_2\omega(B(E_2, E_2)) - g(B(E_2, E_1), [E_1, E_2]).\end{aligned} \tag{67}$$

Thus, we obtain

$$\begin{aligned}\overline{K}^D(\Pi) = & c - \frac{f_2}{2}\mathrm{tr}(\alpha_1|_\Pi) - \frac{f_2}{2}g(\mathrm{tr}(B|_\Pi), U) \\ & - \Omega^{\Pi^*} + \frac{f_2}{2}tr(\omega\ |^2_\Pi) + \sum_{r=n+1}^{m}[h_{11}^r h_{22}^r - h_{12}^r h_{21}^r].\end{aligned} \tag{68}$$

Similarly to (67), we have

$$\begin{aligned}\overline{R}^D(E_i, E_j, E_j, E_i) = & c - f_2\alpha_1(E_j, E_j) + f_2^2\omega(E_j)^2 - g(B(E_i, E_j), B(E_j, E_i)) \\ & + g(B(E_i, E_i), B(E_j, E_j)) - f_2\omega(B(E_j, E_j)) - g(B(E_j, E_i), [E_i, E_j]).\end{aligned} \tag{69}$$

Then,

$$\begin{aligned}\overline{\tau}^D &= \frac{1}{2}\sum_{1\leq i\neq j\leq n} \overline{R}^D(E_i, E_j, E_j, E_i) \\ &= \frac{n(n-1)}{2}c - \frac{f_2}{2}(n-1)\lambda_1 - \frac{f_2}{2}n(n-1)\omega(H) + A^D \\ &+ \frac{f_2^2}{2}(n-1)\gamma + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}[h^r_{ii}h^r_{jj} - h^r_{ij}h^r_{ji}].\end{aligned} \tag{70}$$

where γ=$\sum_{j=1}^{n}\omega(E_j)^2$. Thus,

$$\begin{aligned}\overline{\tau}^D - \overline{K}^D(\Pi) &= \frac{(n+1)(n-2)}{2}c - \frac{f_2}{2}(n-1)\lambda - \frac{f_2}{2}n(n-1)\omega(H) \\ &+ \frac{f_2}{2}\mathrm{tr}(\alpha_1|_\Pi) + \frac{f_2}{2}g(\mathrm{tr}(B|_\Pi), U) + A^D + \Omega^{\Pi^*} + \frac{f_2^2}{2}(n-1)\gamma \\ &+ \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n} h^r_{ii}h^r_{jj} - h^r_{11}h^r_{22} - \sum_{1\leq i<j\leq n} h^r_{ij}h^r_{ji} + h^r_{12}h^r_{21}].\end{aligned} \tag{71}$$

Thus, (64) holds. □

Corollary 8. *If D is totally geodesic with respect to ∇ and $h^r_{12} = h^r_{21} = 0$, then the equality case of (3.12) holds, and vice versa.*

Corollary 9. *If D is an integrable distribution—that is, if $X, Y \in \Gamma(D)$—then $[X, Y]$ is in $\Gamma(D)$. Then,*

$$\begin{aligned}\overline{\tau}^D - \overline{K}^D(\Pi) &\leq \frac{(n+1)(n-2)}{2}c - \frac{f_2}{2}(n-1)\lambda_1 - \frac{f_2}{2}n(n-1)\omega(H) + \frac{f_2^2}{2}(n-1)n \\ &+ \frac{f_2}{2}\mathrm{tr}(\alpha_1|_\Pi) + \frac{f_2}{2}\omega(\mathrm{tr}(B|_\Pi)) + \frac{f_2^2}{2}(n-1)\gamma \\ &- \frac{f_2}{2}tr(\omega\ |^2_\Pi) + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}||B||^2.\end{aligned} \tag{72}$$

Theorem 10. *Let $TM = D \oplus D^\perp$, $\dim D = n \geq 2$, and let M be a manifold with constant sectional curvature c endowed with a connection $\overline{\nabla}$; then,*

$$\begin{aligned}\overline{\mathrm{Ric}}^D(X) &\leq (n-1)c - f_2\lambda_1 + f_2\alpha_1(X, X) - f_2 n\omega(H) \\ &+ f_2\omega(B(X, X)) + f_2^2\gamma + f_2^2\omega(X)^2 + \frac{n^2}{4}\|H\|^2 + \frac{\|B^X\|^2}{2} + A^D(X).\end{aligned} \tag{73}$$

Proof. By (69), we have

$$\begin{aligned}\overline{\mathrm{Ric}}^D(X) &= (n-1)c - f_2\lambda_1 + f_2\alpha_1(X, X) - f_2 n\omega(H) \\ &+ f_2\omega(B(X, X)) + f_2^2\gamma + f_2^2\omega(X)^2 + A^D(X) + \frac{\|B^X\|^2}{2} + \sum_{r=n+1}^{m}\sum_{j=2}^{n}[h^r_{11}h^r_{jj} - h^r_{1j}h^r_{j1}].\end{aligned} \tag{74}$$

Thus, (73) holds. □

Corollary 10. *If $h^r_{1j} = -h^r_{j1}$ for $2 \leq j \leq n$ and $h^r_{11} - h^r_{22} - \cdots - h^r_{nn} = 0$, then the equality case of (73) holds, and vice versa.*

Corollary 11. *If D is an integrable distribution—that is, if $X, Y \in \Gamma(D)$—then $[X, Y]$ is in $\Gamma(D)$. Then,*

$$\overline{\mathrm{Ric}}^{D}(X) \leq (n-1)c - f_2\lambda_1 + f_2\alpha_1(X,X) - f_2 n\omega(H) + f_2\omega(B(X,X)) + f_2^2\gamma + f_2^2\omega(X)^2 + \frac{n^2}{4}\|H\|^2 + \frac{\|B^X\|^2}{2}. \tag{75}$$

4. Examples

Example 1. *Let $\mathbb{S}^3$ be a unit sphere and $dim\mathbb{S}^3 = 3$, which we consider as a Riemannian manifold endowed with the metric induced from $\mathbb{R}^4$. Denote by $T\mathbb{S}^3 = 3$ the tangent space of $\mathbb{S}^3$; we choose an orthonormal basis X_1, X_2, X_3 of $T\mathbb{S}^3$ at each point, which satisfies*

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = -2X_2, \quad [X_2, X_3] = 2X_1. \tag{76}$$

Let ∇ be the Levi–Civita connection on $\mathbb{S}^3$. By (76) and the Koszul formula, we have

$$\begin{aligned} &\nabla_{X_1}X_2 = X_3, \quad \nabla_{X_2}X_1 = -X_3,, \quad \nabla_{X_1}X_1 = \nabla_{X_2}X_2 = \nabla_{X_3}X_3 = 0, \\ &\nabla_{X_1}X_3 = -X_2, \quad \nabla_{X_3}X_1 = X_2, \quad \nabla_{X_2}X_3 = -\nabla_{X_3}X_2 = X_1. \end{aligned} \tag{77}$$

Consider a non-integrable distribution $D_1 = span\{X_1, X_2\}$; then, we can get a metric of D_1. Let $U = X_1 + X_3$. By (77), we have

$$\nabla^{D_1}_{X_i}X_j = 0, \quad \forall i,j = 1,2, \; B(X_1,X_1) = B(X_2,X_2) = 0, \quad B(X_1,X_2) = X_3, \; B(X_2,X_1) = -X_3. \tag{78}$$

By (6), we obtain

$$\widetilde{\nabla}^{D_1}_X Y = \nabla^{D_1}_X Y + \lambda_1 g(X_1,Y)X - \lambda_2 g(X,Y)X_1, \; \widetilde{B}(X,Y) = B(X,Y) - \lambda_2 g(X,Y)X_3. \tag{79}$$

specially, let λ_1,λ_2 be constant.

Thus,

$$\begin{aligned} &\widetilde{\nabla}^{D_1}_{X_1}X_1 = (\lambda_1 - \lambda_2)X_1, \quad \widetilde{\nabla}^{D_1}_{X_1}X_2 = 0, \quad \widetilde{\nabla}^{D_1}_{X_2}X_1 = \lambda_1 X_2, \\ &\widetilde{\nabla}^{D_1}_{X_2}X_2 = -\lambda_2 X_1, \quad \widetilde{B}(X_1,X_1) = -\lambda_2 X_3, \quad \widetilde{B}(X_1,X_2) = X_3, \\ &\widetilde{B}(X_2,X_1) = -X_3, \quad \widetilde{B}(X_2,X_2) = -\lambda_2 X_3, \quad \widetilde{H} = -\lambda_2 X_3. \end{aligned} \tag{80}$$

By (13), (38), (39) and (80), we have

$$\begin{aligned} &\widetilde{R}^{D^1}(X_1,X_2)X_1 = [\lambda_1(\lambda_1 - \lambda_2) - 4]X_2, \widetilde{R}^{D^1}(X_1,X_2)X_2 = [\lambda_2(\lambda_1 - \lambda_2) + 4]X_1, \\ &\widetilde{K}^{D^1}(D_1) = 4 + \frac{(\lambda_1 - \lambda_2)^2}{2}, \widetilde{\tau}^{D_1} = 4 + \frac{(\lambda_1 - \lambda_2)^2}{2}. \end{aligned} \tag{81}$$

By (54), we have

$$\overline{\nabla}^{D_1}_X Y = \nabla^{D_1}_X Y + f_1 g(X_1,X)Y + f_2 g(X_1,Y)X, \; \overline{B}(X,Y) = B(X,Y). \tag{82}$$

where f_1,f_2 are constant.

Thus,

$$\begin{aligned} &\overline{\nabla}^{D_1}_{X_1}X_1 = (f_1 + f_2)X_1, \quad \overline{\nabla}^{D_1}_{X_1}X_2 = f_1 X_2, \quad \overline{\nabla}^{D_1}_{X_2}X_1 = f_2 X_2, \quad \widehat{\nabla}^{D_1}_{X_2}X_2 = 0 \\ &\overline{B}(X_1,X_1) = 0, \quad \overline{B}(X_1,X_2) = X_3, \\ &\overline{B}(X_2,X_1) = -X_3, \quad \overline{B}(X_2,X_2) = 0, \\ &\overline{R}^{D^1}(X_1,X_2)X_1 = (-4 - f_2^2)X_2, \quad \overline{R}^{D^1}(X_1,X_2)X_2 = 4X_1. \end{aligned} \tag{83}$$

Example 2. *Let $M = \mathbb{R} \times \mathbb{S}^3$ and $D^1 = span\{X_1, X_2\}$ and $T\mathbb{S}^3 = D^1 \oplus D^{1,\perp}$. Let $f(t) \in C^\infty(\mathbb{R})$ without zero points. Let $\pi_1 : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{R}; (t, x) \to t$ and $\pi_2 : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{S}^3; (t, x) \to x$. Let*

$$
\begin{aligned}
g_f^M &= \pi_1^* dt^2 \oplus f^2 \pi_2^* g^{D^1} \oplus \pi_2^* g^{D^{1,\perp}}; \\
D &= \pi_1^*(T\mathbb{R}) \oplus \pi_2^* D^1; \quad g^D = \pi_1^* dt^2 \oplus f^2 \pi_2^* g^{D^1},
\end{aligned}
\tag{84}
$$

where $\pi_1^* dt^2$, $\pi_2^* g^{D^1}$, $\pi_2^* g^{D^{1,\perp}}$ denote the pullback metrics of dt^2, g^{D^1}, $g^{D^{1,\perp}}$ and $\pi_1^*(T\mathbb{R})$, $\pi_2^* D^1$ denote the pullback bundles of $T\mathbb{R}$, D^1. We call (D, g^D) the warped product distribution on M and denote ∇^f as the Levi–Civita connection on (M, g_f^M); then, by the Koszul formula and (84), we get

$$
\begin{aligned}
&\nabla^f_{\partial_t}\partial_t = 0, \quad \nabla^f_{\partial_t} X_1 = \frac{f'}{f} X_1, \quad \nabla^f_{X_1}\partial_t = \frac{f'}{f} X_1, \quad \nabla^f_{\partial_t} X_2 = \frac{f'}{f} X_2, \\
&\nabla^f_{X_2}\partial_t = \frac{f'}{f} X_2, \quad \nabla^f_{\partial_t} X_3 = \nabla^f_{X_3}\partial_t = 0, \quad \nabla^f_{X_1} X_1 = \nabla^f_{X_2} X_2 = -ff'\partial_t, \\
&\nabla^f_{X_1} X_2 = X_3, \quad \nabla^f_{X_2} X_1 = -X_3, \quad \nabla^f_{X_1} X_3 = -\frac{X_2}{f^2}, \quad \nabla^f_{X_3} X_1 = (2 - \frac{1}{f^2}) X_2, \\
&\nabla^f_{X_2} X_3 = \frac{X_1}{f^2}, \quad \nabla^f_{X_3} X_2 = (\frac{1}{f^2} - 2) X_1, \quad \nabla^f_{X_3} X_3 = 0.
\end{aligned}
\tag{85}
$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_t(f) = f'$.

Let $D = span\{\partial_t, X_1, X_2\}$; by (85), we have

$$
\begin{aligned}
&\nabla^D_{\partial_t}\partial_t = 0, \quad \nabla^D_{\partial_t} X_1 = \frac{f'}{f} X_1, \quad \nabla^D_{X_1}\partial_t = \frac{f'}{f} X_1, \quad \nabla^D_{\partial_t} X_2 = \frac{f'}{f} X_2, \\
&\nabla^D_{X_2}\partial_t = \frac{f'}{f} X_2, \quad \nabla^D_{X_1} X_1 = \nabla^D_{X_2} X_2 = -ff'\partial_t, \\
&\nabla^D_{X_1} X_2 = 0, \quad \nabla^D_{X_2} X_1 = 0.
\end{aligned}
\tag{86}
$$

For $X, Y \in \Gamma(D)$, let E_1, E_2, E_3 are orthonormal bases of (D, g^D), and we define the Ricci tensor of D by $\mathrm{Ric}^D(X, Y) = \sum_{k=1}^3 g^D(R^D(X, E_k)Y, E_k)$. Then,

$$
\begin{aligned}
&\mathrm{Ric}^D(\partial_t, \partial_t) = \frac{2f''}{f}, \mathrm{Ric}^D(X_1, X_1) = \mathrm{Ric}^D(X_2, X_2) = ff'' + (f')^2 - 4, \\
&\mathrm{Ric}^D(\partial_t, X_1) = \mathrm{Ric}^D(\partial_t, X_2) = \mathrm{Ric}^D(X_1, \partial_t) = \mathrm{Ric}^D(X_2, \partial_t) = 0; \\
&\mathrm{Ric}^D(X_1, X_2) = \mathrm{Ric}^D(X_2, X_1) = 0.
\end{aligned}
\tag{87}
$$

For $X, Y \in \Gamma(D)$, if $\mathrm{Ric}^D(X, Y) = c_0 g^D(X, Y)$, we say that (D, g^D) is Einstein.

Theorem 11. *(D, g^D) is Einstein with the Einstein constant c_0 if and only if*
(1) $c_0 = 0, \quad f(t) = 2t + c_1 \quad \text{or} \quad f(t) = -2t + c_1,$
(2) $c_0 > 0, \quad f(t) = -\frac{2}{c_2 c_0} e^{\sqrt{\frac{c_0}{2}} t} + c_2 e^{-\sqrt{\frac{c_0}{2}} t},$
(3) $c_0 < 0, \quad f(t) = c_1 \cos(\sqrt{\frac{-c_0}{2}} t) + c_2 \sin(\sqrt{\frac{-c_0}{2}} t), \quad c_1^2 + c_2^2 = -\frac{8}{c_0}.$
where c_1, c_2 are constant.

Proof. By (87), (D, g^D) is Einstein with the Einstein constant c_0 if and only if

$$
f'' - \frac{c_0}{2} f = 0, \tag{88}
$$

$$
ff'' + (f')^2 - 4 = c_0 f^2. \tag{89}
$$

If $c_0 = 0$, by (88), then $f = c_2x + c_1$. Using (89), then $c_2 = 2, \text{or} - 2$, and so we get case (1).

If $c_0 > 0$, by (88), then $f = c_1e^{\sqrt{\frac{c_0}{2}}t} + c_2e^{-\sqrt{\frac{c_0}{2}}t}$. Using (89), then $(f')^2 = 4 + \frac{c_0}{2}f^2$, so $c_1 = \frac{-2}{c_2c_0}$ and we get case (2).

If $c_0 < 0$, by (88), then $f = c_1\cos(\sqrt{\frac{-c_0}{2}}t) + c_2\sin(\sqrt{\frac{-c_0}{2}}t)$. Using $(f')^2 = 4 + \frac{c_0}{2}f^2$, we get $c_1^2 + c_2^2 = -\frac{8}{c_0}$, and so case (3) holds. □

Let $U = \partial_t$, then

$$\widetilde{\nabla}^D_X Y = \nabla^D_X Y + \lambda_1 g(\partial_t, Y)X - \lambda_2 g(X, Y)\partial_t, \quad \widetilde{B}(X, Y) = B(X, Y). \tag{90}$$

where λ_1,λ_2 are constant.

By (90), we get

$$\begin{aligned}
&\widetilde{\nabla}^D_{\partial_t}\partial_t = (\lambda_1 - \lambda_2)\partial_t, \quad \widetilde{\nabla}^D_{\partial_t}X_1 = \frac{f'}{f}X_1, \quad \widetilde{\nabla}^D_{X_1}\partial_t = (\frac{f'}{f} + \lambda_1)X_1, \quad \widetilde{\nabla}^D_{\partial_t}X_2 = \frac{f'}{f}X_2, \\
&\widetilde{\nabla}^D_{X_2}\partial_t = (\frac{f'}{f} + \lambda_1)X_2, \quad \widetilde{\nabla}^D_{X_1}X_1 = \widetilde{\nabla}^D_{X_2}X_2 = (-ff' - \lambda_2 f^2)\partial_t, \\
&\widetilde{\nabla}^D_{X_1}X_2 = 0, \quad \widetilde{\nabla}^D_{X_2}X_1 = 0,
\end{aligned} \tag{91}$$

and

$$\begin{aligned}
&\widetilde{\mathrm{Ric}}^D(\partial_t, \partial_t) = 2[\frac{f'' + \lambda_2 f'}{f} + \lambda_1(\lambda_2 - \lambda_1)], \\
&\widetilde{\mathrm{Ric}}^D(X_1, X_1) = \widetilde{\mathrm{Ric}}^D(X_2, X_2), \\
&= ff'' + 2\lambda_1 ff' + 2\lambda_1\lambda_2 f^2 - \lambda_2 f^2 + (f')^2 + \lambda_2 ff' - 4, \\
&\widetilde{\mathrm{Ric}}^D(\partial_t, X_1) = \widetilde{\mathrm{Ric}}^D(\partial_t, X_2) = 0, \\
&\widetilde{\mathrm{Ric}}^D(X_1, \partial_t) = \widetilde{\mathrm{Ric}}^D(X_2, \partial_t) = 0; \\
&\widetilde{\mathrm{Ric}}^D(X_1, X_2) = \widetilde{\mathrm{Ric}}^D(X_2, X_1) = 0.
\end{aligned} \tag{92}$$

So $(D, g^D, \widetilde{\nabla}^D)$ is mixed Ricci flat.

By (55) and (86), we have

$$\begin{aligned}
&\overline{\nabla}^D_{\partial_t}\partial_t = (f_1 + f_2)\partial_t, \quad \overline{\nabla}^D_{\partial_t}X_1 = (\frac{f'}{f} + f_1)X_1, \quad \overline{\nabla}^D_{X_1}\partial_t = (\frac{f'}{f} + f_2)X_1, \quad \overline{\nabla}^D_{\partial_t}X_2 = (\frac{f'}{f} + f_1)X_2, \\
&\overline{\nabla}^D_{X_2}\partial_t = (\frac{f'}{f} + f_2)X_2, \quad \overline{\nabla}^D_{X_1}X_1 = \overline{\nabla}^D_{X_2}X_2 = -ff'\partial_t, \\
&\overline{\nabla}^D_{X_1}X_2 = 0, \quad \overline{\nabla}^D_{X_2}X_1 = 0.
\end{aligned} \tag{93}$$

According to the computation of $\widetilde{\nabla}^D$, we can obtain the Ricci tensor of $\overline{\nabla}^D$.

Example 3. *Let (H_3, g_{H_3}) be the Heisenberg group H_3 endowed with the Riemannian metric g; we choose an orthonormal basis $\{e_1.e_2.e_3\}$ of (H_3, g_{H_3}) which satisfies the commutation relations*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0. \tag{94}$$

By the Koszul formula, we can get the Levi–Civita connection ∇ of H_3:

$$\nabla_{e_j}e_j = 0, \quad 1 \le j \le 3, \quad \nabla_{e_1}e_2 = \frac{1}{2}e_3, \quad \nabla_{e_2}e_1 = -\frac{1}{2}e_3, \tag{95}$$
$$\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\frac{1}{2}e_2, \quad \nabla_{e_2}e_3 = \nabla_{e_3}e_2 = \frac{1}{2}e_1.$$

Let $D = span\{e_1, e_2\}$, by (95), then $\nabla^D_{e_i}e_j = 0, 1 \le i, j \le 2$. Let $U = e_1 + e_2 + e_3$, then

$$\widetilde{\nabla}^D_{e_1}e_1 = (\lambda_1 - \lambda_2)e_1 - \lambda_2 e_2, \quad \widetilde{\nabla}^D_{e_1}e_2 = \lambda_1 e_1, \quad \widetilde{\nabla}^D_{e_2}e_1 = \lambda_1 e_2, \quad \widetilde{\nabla}^D_{e_2}e_2 = (\lambda_1 - \lambda_2)e_2 - \lambda_2 e_1, \tag{96}$$
$$\widetilde{B}(e_1, e_1) = \widetilde{B}(e_2, e_2) = -\lambda_2 e_3, \quad \widetilde{B}(e_1, e_2) = \frac{1}{2}e_3, \quad \widetilde{B}(e_2, e_1) = -\frac{1}{2}e_3.$$
$$\widetilde{R}^D(e_1, e_2)e_1 = (\lambda_1^2 - \lambda_2^2)e_1 - (\lambda_1 - \lambda_2)^2 e_2,$$
$$\widetilde{R}^D(e_1, e_2)e_2 = (\lambda_1 - \lambda_2)^2 e_1 + (\lambda_1^2 - \lambda_2^2)e_2,$$

so $(D, g^D, \widetilde{\nabla}^D)$ is flat when $\lambda_1 = \lambda_2$. Similarly, we have

$$\widehat{\nabla}^D_{e_1}e_1 = (f_1 + f_2)e_1, \quad \widehat{\nabla}^D_{e_1}e_2 = f_2 e_1 + f_1 e_2, \quad \widehat{\nabla}^D_{e_2}e_1 = f_1 e_1 + f_2 e_2, \quad \widehat{\nabla}^D_{e_2}e_2 = (f_1 + f_2)e_2, \tag{97}$$
$$\widehat{R}^D(e_1, e_2)e_1 = \widehat{R}^D(e_1, e_2)e_2 = f_2^2(e_1 - e_2).$$

Example 4. *Let $M = \mathbb{R} \times H_3$ and $D^1 = span\{e_1, e_2\}$ and $TH_3 = D^1 \oplus D^{1,\perp}$, where H_3 is the Heisenberg group. Let $f(t) \neq 0 \in C^\infty(\mathbb{R})$ for any $t \in R$. Let $\pi_1 : \mathbb{R} \times H_3 \to \mathbb{R}; (t, x) \to t$ and $\pi_2 : \mathbb{R} \times H_3 \to H_3; (t, x) \to x$. Let*

$$g^M_f = \pi_1^* dt^2 \oplus f^2 \pi_2^* g^{D^1} \oplus \pi_2^* g^{D^{1,\perp}}; \tag{98}$$
$$D = \pi_1^*(T\mathbb{R}) \oplus \pi_2^* D^1; \quad g^D = \pi_1^* dt^2 \oplus f^2 \pi_2^* g^{D^1}.$$

The Levi–Civita connection ∇^f of (M, g^M_f) is given by

$$\nabla^f_{\partial_t}\partial_t = 0, \quad \nabla^f_{\partial_t}e_1 = \frac{f'}{f}e_1, \quad \nabla^f_{e_1}\partial_t = \frac{f'}{f}e_1, \quad \nabla^f_{\partial_t}e_2 = \frac{f'}{f}e_2, \tag{99}$$
$$\nabla^f_{e_2}\partial_t = \frac{f'}{f}e_2, \quad \nabla^f_{\partial_t}e_3 = \nabla^f_{e_3}\partial_t = 0, \quad \nabla^f_{e_1}e_1 = \nabla^f_{e_2}e_2 = -ff'\partial_t,$$
$$\nabla^f_{e_1}e_2 = \frac{1}{2}e_3, \quad \nabla^f_{e_2}e_1 = -\frac{1}{2}e_3, \quad \nabla^f_{e_1}e_3 = -\frac{e_2}{2f^2}, \quad \nabla^f_{e_3}e_1 = -\frac{1}{2f^2}e_2,$$
$$\nabla^f_{e_2}e_3 = \frac{e_1}{2f^2}, \quad \nabla^f_{e_3}e_2 = \frac{1}{2f^2}e_1, \quad \nabla^f_{e_3}e_3 = 0.$$

Let $D = span\{\partial_t, e_1, e_2\}$; by (99), we have

$$\nabla^D_{\partial_t}\partial_t = 0, \quad \nabla^D_{\partial_t}e_1 = \frac{f'}{f}e_1, \quad \nabla^D_{e_1}\partial_t = \frac{f'}{f}e_1, \quad \nabla^D_{\partial_t}e_2 = \frac{f'}{f}e_2, \tag{100}$$
$$\nabla^D_{e_2}\partial_t = \frac{f'}{f}e_2, \quad \nabla^D_{e_1}e_1 = \nabla^D_{e_2}e_2 = -ff'\partial_t,$$
$$\nabla^D_{e_1}e_2 = 0, \quad \nabla^D_{e_2}e_1 = 0.$$

The results of the Ricci tensor on D are as follows:

$$\begin{aligned}&\mathrm{Ric}^D(\partial_t,\partial_t)=\frac{2f''}{f},\mathrm{Ric}^D(e_1,e_1)=\mathrm{Ric}^D(e_2,e_2)=ff''+(f')^2,\\&\mathrm{Ric}^D(\partial_t,e_1)=\mathrm{Ric}^D(\partial_t,e_2)=\mathrm{Ric}^D(e_1,\partial_t)=\mathrm{Ric}^D(e_2,\partial_t)=0;\\&\mathrm{Ric}^D(e_1,e_2)=\mathrm{Ric}^D(e_2,e_1)=0.\end{aligned} \tag{101}$$

Theorem 12. *(D,g^D) is Einstein with the Einstein constant c_0 if and only if*
(1) $c_0=0,\quad f(t)=c_1,$
(2) $c_0>0,\quad f(t)=c_1e^{\sqrt{\frac{c_0}{2}}t}$ or $f(t)=c_2e^{-\sqrt{\frac{c_0}{2}}t},$
where c_1,c_2 are constant.

Proof. By (101), (D,g^D) is Einstein with the Einstein constant c_0 if and only if

$$f''-\frac{c_0}{2}f=0, \tag{102}$$

$$ff''+(f')^2=c_0f^2. \tag{103}$$

If $c_0=0$, by (102), then $f=c_2x+c_1$. Using (103), then $c_2=0$, and so we get case (1).

If $c_0>0$, by (102), then $f=c_1e^{\sqrt{\frac{c_0}{2}}t}+c_2e^{-\sqrt{\frac{c_0}{2}}t}$. Using (4.28), then $(f')^2=\frac{c_0}{2}f^2$, so $c_1=0$ or $c_2=0$, and we get case (2).

If $c_0<0$, by (102), then $f=c_1\cos(\sqrt{\frac{-c_0}{2}}t)+c_2\sin(\sqrt{\frac{-c_0}{2}}t)$. Using $(f')^2=\frac{c_0}{2}f^2$, we get $c_1=c_2=0$. However, $f\neq 0$; thus, in this case there is no solution. □

Theorem 13. *(D,g^D) is a distribution with a constant scalar curvature λ_0 if and only if*
(1) $\lambda_0=0,\quad f(t)=(c_2t+c_1)^{\frac{2}{3}},$
(2) $\lambda_0>0,\quad f(t)=(c_1e^{\sqrt{\frac{3\lambda_0}{8}}t}+c_2e^{-\sqrt{\frac{3\lambda_0}{8}}t})^{\frac{2}{3}},$
(3) $\lambda_0<0,\quad f(t)=(c_1\cos(\sqrt{-\frac{3\lambda_0}{8}}t)+c_2\sin(\sqrt{-\frac{3\lambda_0}{8}}t))^{\frac{2}{3}},$
where c_1,c_2 are constant.

Proof. By (101), we have

$$s^D=4\frac{f''}{f}+2\frac{(f')^2}{f^2}=\lambda_0. \tag{104}$$

Let $f(t)=w(t)^{\frac{2}{3}}$ and by (104), we get $w''(t)-\frac{3}{8}\lambda_0w(t)=0$. By the elementary methods for ordinary differential equations, we prove the above theorem. □

Let $U=\partial_t$, By (100), we get

$$\begin{aligned}&\widetilde{\nabla}^D_{\partial_t}\partial_t=(\lambda_1-\lambda_2)\partial_t,\quad \widetilde{\nabla}^D_{\partial_t}e_1=\frac{f'}{f}e_1,\quad \widetilde{\nabla}^D_{e_1}\partial_t=(\frac{f'}{f}+\lambda_1)e_1,\quad \widetilde{\nabla}^D_{\partial_t}e_2=\frac{f'}{f}e_2,\\&\widetilde{\nabla}^D_{e_2}\partial_t=(\frac{f'}{f}+\lambda_1)e_2,\quad \widetilde{\nabla}^D_{e_1}e_1=\widetilde{\nabla}^D_{e_2}e_2=(-ff'-\lambda_2f^2)\partial_t,\\&\widetilde{\nabla}^D_{e_1}e_2=\widetilde{\nabla}^D_{e_2}e_1=0.\end{aligned} \tag{105}$$

Theorem 14. *$(D,g^D,\widetilde{\nabla}^D)$ is a distribution with constant scalar curvature λ_0 for $U=\partial_t$ if and only if*
(1) $\lambda_0=\xi,\quad f(t)=(c_1e^{-\frac{\lambda_1+\lambda_2}{2}t}+c_2te^{-\frac{\lambda_1+\lambda_2}{2}t})^{-\xi},$
(2) $\lambda_0>\xi,\quad f(t)=(c_1e^{\frac{-(\lambda_1+\lambda_2)+\sqrt{\eta}}{2}t}+c_2e^{\frac{-(\lambda_1+\lambda_2)-\sqrt{\eta}}{2}t})^{-\xi},$

(3) $\lambda_0 < \xi$, $f(t) = (c_1 e^{-\frac{\lambda_1+\lambda_2}{2}t}\cos(\frac{\sqrt{-\eta}}{2}t) + c_2 e^{-\frac{\lambda_1+\lambda_2}{2}t}\sin(\frac{\sqrt{-\eta}}{2}t))^{-\xi}$,
where c_1, c_2 *are constant and* $\xi = -\frac{2}{3}(4\lambda_1^2 - 2\lambda_2^2 - 7\lambda_1\lambda_2)$, $\eta = 4\lambda_1^2 - 2\lambda_2^2 - 7\lambda_1\lambda_2 + \frac{3}{2}\lambda_0$.

Proof. By (105), we have

$$\widetilde{s}^D = 4\frac{f''}{f} + 4\frac{(\lambda_1+\lambda_2)f'}{f} + 2\frac{(f')^2}{f^2} + (6\lambda_1\lambda_2 - 2\lambda_1^2 - 2\lambda_2^2). \tag{106}$$

Let $f(t) = w(t)^{\frac{2}{3}}$ and by (106), we get $w''(t) + (\lambda_1+\lambda_2)w'(t) + \frac{3}{8}(6\lambda_1\lambda_2 - 2\lambda_1^2 - 2\lambda_2^2 - \lambda_0)w(t) = 0$. By the elementary methods for ordinary differential equations, we prove the above theorem. □

By (100), we have

$$\begin{aligned} &\overline{\nabla}^D_{\partial_t}\partial_t = (f_1+f_2)\partial_t, \quad \overline{\nabla}^D_{\partial_t}e_1 = (\frac{f'}{f}+f_1)e_1, \quad \overline{\nabla}^D_{e_1}\partial_t = (\frac{f'}{f}+f_2)e_1, \\ &\overline{\nabla}^D_{\partial_t}e_2 = (\frac{f'}{f}+f_1)e_2, \overline{\nabla}^D_{e_2}\partial_t = (\frac{f'}{f}+f_2)e_2, \quad \overline{\nabla}^D_{e_1}e_1 = \overline{\nabla}^D_{e_2}e_2 = -ff'\partial_t, \\ &\overline{\nabla}^D_{e_1}e_2 = \overline{\nabla}^D_{e_2}e_1 = 0. \end{aligned} \tag{107}$$

Then, we get

$$\overline{s}^D = 4\frac{f''}{f} + 2\frac{(f')^2}{f^2} + \frac{f_2 f'}{f} - 2f_2^2. \tag{108}$$

By Theorem 14, we have

Theorem 15. *$(D, g^D, \overline{\nabla}^D)$ is a distribution with a constant scalar curvature λ_0 for $U = \partial_t$ if and only if*
(1) $\lambda_0 = -\frac{8}{3}f_2^2$, $f(t) = (c_1 e^{-f_2 t} + c_2 t e^{-f_2 t})^{\frac{8}{3}f_2^2}$,
(2) $\lambda_0 > -\frac{8}{3}f_2^2$, $f(t) = (c_1 e^{-f_2 + \sqrt{4f_2^2 + \frac{3\lambda_0}{2}}t} + c_2 e^{-f_2 - \sqrt{4f_2^2 + \frac{3\lambda_0}{2}}t})^{\frac{8}{3}f_2^2}$,
(3) $\lambda_0 < -\frac{8}{3}f_2^2$, $f(t) = (c_1 e^{-f_2 t}\cos(\sqrt{-4f_2^2 - \frac{3\lambda_0}{2}}t) + c_2 e^{-f_2 t}\sin(\sqrt{-4f_2^2 - \frac{3\lambda_0}{2}}t))^{\frac{8}{3}f_2^2}$,
where c_1, c_2 *are constant.*

5. Conclusions and Future Research

For a Riemannian manifold with a semi-symmetric non-metric connection, the induced connection on a submanifold is also a semi-symmetric non-metric connection. The Gauss, Codazzi, and Ricci equations for distributions are a generalization of the case of submanifolds. Therefore, in this paper, we give the definition of the first generalized semi-symmetric non-metric connection and the second generalized semi-symmetric non-metric connection. The distribution can be viewed as a submanifold, so the corresponding metric of the Riemannian manifold distribution and orthogonal distribution are obtained. Then, by the definition of an non-integrable distribution, we define the curvature tensor $\widetilde{R}^D$(or $\overline{R}^D$) on D with respect to $\widetilde{\nabla}^D$(or $\overline{\nabla}$). By computation, we obtain the Gauss, Codazzi, and Ricci equations for non-integrable distributions in a Riemannian manifold with the first generalized semi-symmetric non-metric connection and the second generalized semi-symmetric non-metric connection, respectively. For a two-plane section $\Pi \subset D$, we define the sectional curvature $\widetilde{K}^D(\Pi)$(or $\overline{K}^D(\Pi)$ of D with the induced connection $\widetilde{\nabla}^D$(or $\overline{\nabla}^D$) and the scalar curvature $\widetilde{\tau}^D$(or $\overline{\tau}^D$)with respect to D and $\widetilde{\nabla}^D$(or $\overline{\nabla}^D$). Then, we obtain the Chen inequalities in both cases and give the equality case. We also give the results of the integrable distribution. Moreover, some properties of a totally geodesic and umbilical distribution are discussed in this paper.

In following research, we will focus on the Lorentzian metric of distributions.

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