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Explicit Identities for 3-Variable Degenerate Hermite Kampé de Fériet Polynomials and Differential Equation Derived from Generating Function

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Abstract: We get the 3-variable degenerate Hermite Kampé de Fériet polynomials and get symmetric identities for 3-variable degenerate Hermite Kampé de Fériet polynomials. We make differential equations coming from the generating functions of degenerate Hermite Kampé de Fériet polynomials to get some identities for 3-variable degenerate Hermite Kampé de Fériet polynomials,. Finally, we study the structure and symmetry of pattern about the zeros of the 3-variable degenerate Hermite Kampé de Fériet equations.

Keywords: differential equations; symmetric identities; 3-variable degenerate Hermite Kampé de Fériet polynomials; complex zeros

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1. Introduction

The classical Hermite numbers H_n and polynomials $H_n(x)$ are usually defined by the generating functions

$$e^{-t^2} = \sum_{n=0}^{\infty} H_n \frac{t^n}{n!}, \tag{1}$$

and

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \tag{2}$$

Clearly, $H_n = H_n(0)$.

These numbers and polynomials have been studied because of important roles in many areas of mathematics (see References [1,2]). The special polynomials of 3-variable give partial differential equations of physical phenomenon . Physical problems was expressed by the special functions of mathematical physics. We recall that the 3-variable Hermite polynomials $H_n(x, y, z)$ made by the generating function (see Reference [3])

$$\sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!} = e^{xt+yt^2+zt^3} \tag{3}$$

are solutions in the system of equations

$$\begin{aligned} &\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z), \\ &\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z), \\ &H_n(x, 0, 0) = x^n. \end{aligned}$$

In particular, one has

$$H_n(2x,-1,0)=H_n(x).$$

Many researchers studied special numbers and polynomials because of importance (see References [1–7]). The degenerate Bernoulli, Euler, Genocchi and tangent polynomials were studied in several papers (see References [8–12]). Recently, researchers have studied the differential equations which are related to generating functions of special polynomials (see References [13–18]).

We construct the 3-variable degenerate Hermite Kampé de Fériet polynomials and get symmetric identities for 3-variable degenerate Hermite Kampé de Fériet polynomials. Finally, we study the distribution and symmetry of pattern of the roots of the 3-variable degenerate Hermite Kampé de Fériet polynomials Hermite equations.

We define the 3-variable degenerate Hermite Kampé de Fériet polynomials $\mathbf{H}_n(x,y,z|\mu)$ made by the generating function

$$\mathfrak{F}(t,x,y,z|\mu)=\sum_{n=0}^{\infty}\mathbf{H}_n(x,y,z|\mu)\frac{t^n}{n!}=(1+\mu)^{\frac{xt}{\mu}}(1+\mu)^{\frac{yt^2}{\mu}}(1+\mu)^{\frac{zt^3}{\mu}}. \tag{4}$$

Since $(1+\mu)^{\frac{t}{\mu}}\to e^t$ as $\mu\to 0$, it is clear that (4) reduces to (3). If $\mu\to 0$ and $z=0$, Equation (4) is the generating function of the 2-variable Hermite polynomials $H_n(x,y,0)$. Observe that Hermite polynomials $H_n(x,y,0)$ with the 2-variable are the solution of the heat equation (see Reference [17])

$$\frac{\partial}{\partial y}H_n(x,y,0)=\frac{\partial^2}{\partial x^2}H_n(x,y,0),\quad H_n(x,0,0)=x^n.$$

Theorem 1. *For $n=0,1,\ldots$, the 3-variable degenerate Hermite Kampé de Fériet polynomials $\mathbf{H}_n(x,y,z|\mu)$ with the generating function (4) are the solution of the differential equation*

$$\begin{aligned}
&\left(3z\frac{\partial^2}{\partial x\partial y}+2y\frac{\partial^2}{\partial x^2}+x\frac{\log(1+\mu)}{\mu}\frac{\partial}{\partial x}-n\frac{\log(1+\mu)}{\mu}\right)\mathbf{H}_n(x,y,z|\mu)=0,\\
&\mathbf{H}_n(x,0,0|\mu)=\left(\frac{\log(1+\mu)}{\mu}\right)^n x^n.\\
&\mathbf{H}_n(0,y,0|\mu)=\begin{cases}\left(\frac{\log(1+\mu)}{\mu}\right)^k y^k\frac{(2k)!}{k!}, & \text{if } n=2k\\ 0, & \text{otherwise}\end{cases}\\
&\mathbf{H}_n(0,0,z|\mu)=\begin{cases}\left(\frac{\log(1+\mu)}{\mu}\right)^k z^k\frac{(3k)!}{k!}, & \text{if } n=3k\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

Proof. We see that

$$\mathfrak{F}(t,x,y,z|\mu)=(1+\mu)^{\frac{xt}{\mu}}(1+\mu)^{\frac{yt^2}{\mu}}(1+\mu)^{\frac{zt^3}{\mu}}$$

satisfies

$$\frac{\partial\mathfrak{F}(t,x,y,z|\mu)}{\partial t}-\frac{\log(1+\mu)}{\mu}\left(x+2yt+3zt^2\right)\mathfrak{F}(t,x,y,z|\mu)=0.$$

By substituting the series (4) for $\mathfrak{F}(t,x,y,z|\mu)$, one obtains

$$\begin{aligned}&\mathbf{H}_{n+1}(x,y,z|\mu)-x\frac{\log(1+\mu)}{\mu}\mathbf{H}_n(x,y,z|\mu)-2y\frac{\log(1+\mu)}{\mu}\mathbf{H}_{n-1}(x,y,z|\mu)\\&-n(n-1)3z\frac{\log(1+\mu)}{\mu}\mathbf{H}_{n-2}(x,y,z|\mu)=0,n=2,3,\ldots\end{aligned}\tag{5}$$

We get a recurrence relation for 3-variable degenerate Hermite Kampé de Fériet polynomials and another recurrence relation which comes from

$$\frac{\partial\mathfrak{F}(t,x,y,z|\mu)}{\partial x}-\frac{\log(1+\mu)}{\mu}t\mathfrak{F}(t,x,y,z|\mu)=0.$$

This implies

$$\frac{\partial\mathbf{H}_n(x,y,z|\mu)}{\partial x}-n\frac{\log(1+\mu)}{\mu}\mathbf{H}_{n-1}(x,y,z|\mu)=0,n=1,2,\ldots\tag{6}$$

On the other hand, since

$$\frac{\partial\mathfrak{F}(t,x,y,z|\mu)}{\partial y}-\frac{\log(1+\mu)}{\mu}t^2\mathfrak{F}(t,x,y,z|\mu)=0,$$

we get

$$\frac{\partial\mathbf{H}_n(x,y,z|\mu)}{\partial y}-n(n-1)\frac{\log(1+\mu)}{\mu}\mathbf{H}_{n-2}(x,y,z|\mu)=0,n=2,3,\ldots\tag{7}$$

Eliminate $\mathbf{H}_{n-1}(x,y,z|\mu)$ and $\mathbf{H}_{n-2}(x,y,z|\mu)$ from (5)–(7) to obtain

$$\mathbf{H}_{n+1}(x,y,z|\mu)-x\frac{\log(1+\mu)}{\mu}\mathbf{H}_n(x,y,z|\mu)-2y\frac{\partial\mathbf{H}_n(x,y,z|\mu)}{\partial x}-3z\frac{\partial\mathbf{H}_n(x,y,z|\mu)}{\partial y}=0.$$

Differentiate this equation and use (6) again to get

$$\begin{aligned}&3z\frac{\partial^2\mathbf{H}_n(x,y,z|\mu)}{\partial x\partial y}+2y\frac{\partial\mathbf{H}_n(x,y,z|\mu)}{\partial x}+x\frac{\log(1+\mu)}{\mu}\mathbf{H}_n(x,y,z|\mu)\\&-n\frac{\log(1+\mu)}{\mu}\mathbf{H}_n(x,y,z|\mu)=0,(n=0,1,\ldots),\end{aligned}\tag{8}$$

thus we have shown the theorem. □

Theorem 2. *The 3-variable degenerate Hermite Kampé de Fériet polynomials* $\mathbf{H}_n(x,y,z|\mu)$ *with the generating function (4) are the solution of the differential equation*

$$\left(3z\frac{\mu}{\log(1+\mu)}\frac{\partial^3}{\partial x^3}+2y\frac{\partial^2}{\partial x^2}+x\frac{\log(1+\mu)}{\mu}\frac{\partial}{\partial x}-n\frac{\log(1+\mu)}{\mu}\right)\mathbf{H}_n(x,y,z|\mu)=0,$$

$$\mathbf{H}_n(x,0,0|\mu)=\left(\frac{\log(1+\mu)}{\mu}\right)^n x^n, n=0,1,2,\ldots.$$

$$\mathbf{H}_n(0,y,0|\mu)=\begin{cases}\left(\frac{\log(1+\mu)}{\mu}\right)^k y^k\frac{(2k)!}{k!}, & \text{if } n=2k\\ 0, & \text{otherwise}\end{cases}$$

$$\mathbf{H}_n(0,0,z|\mu)=\begin{cases}\left(\frac{\log(1+\mu)}{\mu}\right)^k z^k\frac{(3k)!}{k!}, & \text{if } n=3k\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We get another recurrence relation which comes from

$$\frac{\partial^2 \mathfrak{F}(t,x,y,z|\mu)}{\partial x \partial y} - \left(\frac{\log(1+\mu)}{\mu}\right)^2 t^2 \mathfrak{F}(t,x,y,z|\mu) = 0.$$

This implies

$$\frac{\partial^2 \mathbf{H}_n(x,y,z|\mu)}{\partial x \partial y} = n(n-1)(n-2)\left(\frac{\log(1+\mu)}{\mu}\right)^2 \mathbf{H}_{n-3}(x,y,z|\mu) = 0, n = 3,4,\ldots \quad (9)$$

Again, we also have

$$\frac{\partial^3 \mathfrak{F}(t,x,y,z|\mu)}{\partial x^3} - \left(\frac{\log(1+\mu)}{\mu}\right)^3 t^3 \mathfrak{F}(t,x,y,z|\mu) = 0.$$

This implies

$$\frac{\partial^3 \mathbf{H}_n(x,y,z|\mu)}{\partial x^3} = n(n-1)(n-2)\left(\frac{\log(1+\mu)}{\mu}\right)^3 \mathbf{H}_{n-3}(x,y,z|\mu) = 0, n = 3,4,\ldots \quad (10)$$

Thus, from (8)–(10), the degenerate Hermite Kampé de Fériet polynomials $\mathbf{H}_n(x,y,z|\mu)$ of 3-variable with the generating function (4) are the solution of the differential equation

$$\begin{aligned} &\frac{3z\mu}{\log(1+\mu)} \frac{\partial^3 \mathbf{H}_n(x,y,z|\mu)}{\partial x^3} + 2y \frac{\partial^2 \mathbf{H}_n(x,y,z|\mu)}{\partial x^2} \\ &\quad + x \frac{\log(1+\mu)}{\mu} \frac{\partial \mathbf{H}_n(x,y,z|\mu)}{\partial x} - n \frac{\log(1+\mu)}{\mu} \mathbf{H}_n(x,y,z|\mu) = 0. \end{aligned}$$

Therefore, we are done. □

We see another application of the differential equation for $\mathbf{H}_n(x,y,z|\mu)$. The polynomials $\mathbf{H}_n(x,y,z|\mu)$ have this relations

$$\begin{aligned} &\frac{\partial \mathbf{H}_n(x,y,z|\mu)}{\partial y} - \frac{\mu}{\log(1+\mu)} \frac{\partial^2 \mathbf{H}_n(x,y,z|\mu)}{\partial x^2} = 0, \\ &\frac{\partial \mathbf{H}_n(x,y,z|\mu)}{\partial z} - \left(\frac{\mu}{\log(1+\mu)}\right)^2 \frac{\partial^2 \mathbf{H}_n(x,y,z|\mu)}{\partial x^2} = 0, \end{aligned}$$

which in view of the initial condition are solved by

$$\mathbf{H}_n(x,0,0|\mu) = \left(\frac{\log(1+\mu)}{\mu}\right)^n x^n, n = 0,1,2,\ldots.$$

$$\mathbf{H}_n(0,y,0|\mu) = \begin{cases} \left(\frac{\log(1+\mu)}{\mu}\right)^k y^k \frac{(2k)!}{k!}, & \text{if } n = 2k \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{H}_n(0,0,z|\mu) = \begin{cases} \left(\frac{\log(1+\mu)}{\mu}\right)^k z^k \frac{(3k)!}{k!}, & \text{if } n = 3k \\ 0, & \text{otherwise} \end{cases}$$

The Stirling numbers of the first kind, $S_1(n,k)$, were defined by (see References [8–10])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k.$$

$S_1(n,k)$ is

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{1}{m!}(\log(1+t))^m,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$. We see the binomial theorem: for a variable x,

$$\begin{aligned}(1+\mu)^{\frac{xt}{\mu}} &= \sum_{m=0}^{\infty}\left(\frac{tx}{\mu}\right)_m \frac{\mu^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m} S_1(m,l)\left(\frac{tx}{\mu}\right)^l \frac{\mu^m}{m!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty} S_1(m,l)x^l\mu^{m-l}\frac{l!}{m!}\right)\frac{t^l}{l!}.\end{aligned} \tag{11}$$

By (4), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathbf{H}_n(x,y,z|\mu)\frac{t^n}{n!} = (1+\mu)^{\frac{xt}{\mu}}(1+\mu)^{\frac{yt^2}{\mu}}(1+\mu)^{\frac{zt^2}{\mu}} \\ &= \sum_{k=0}^{\infty}\left(\frac{y\log(1+\mu)}{\mu}\right)^k\frac{t^{2k}}{k!}\sum_{l=0}^{\infty}\left(\frac{x\log(1+\mu)}{\mu}\right)^l\frac{t^l}{l!}\sum_{j=0}^{\infty}\left(\frac{x\log(1+\mu)}{\mu}\right)^j\frac{t^{3j}}{j!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\left(\frac{\log(1+\mu)}{\mu}\right)^{n-k} y^k x^{n-2k}\frac{n!}{k!(n-2k)!}\right)\frac{t^n}{n!}\sum_{j=0}^{\infty}\left(\frac{x\log(1+\mu)}{\mu}\right)^j\frac{t^{3j}}{j!} \\ &\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{3}]}\sum_{l=0}^{[\frac{n-3k}{2}]}\left(\frac{\log(1+\mu)}{\mu}\right)^{n-2k-l}\frac{z^k y^l x^{n-3k-2l}n!(n-3k)!}{k!l!(n-3k)!(n-3k-2l)!}\right)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

If we compare the coefficients $\frac{t^n}{n!}$ on the both sides of (12), we have representation of $\mathbf{H}_n(x,y,z|\mu)$.

$$\mathbf{H}_n(x,y,z|\mu) = \sum_{k=0}^{[\frac{n}{3}]}\sum_{l=0}^{[\frac{n-3k}{2}]}\left(\frac{\log(1+\mu)}{\mu}\right)^{n-2k-l}\frac{z^k y^l x^{n-3k-2l}n!(n-3k)!}{k!l!(n-3k)!(n-3k-2l)!},$$

and $[\quad]$ denotes taking the integer part.

The following elementary properties of $\mathbf{H}_n(x,y,z|\mu)$ are deduced form (4). We delete the details.

Theorem 3. *For any positive n, we have*

1. $\mathbf{H}_n(x,0,0|\mu) = \sum\limits_{m=n}^{\infty} S_1(m,n) x^n \mu^{m-n} \dfrac{n!}{m!}$.
2. $\mathbf{H}_n(x_1+x_2,y,z|\mu) = \sum\limits_{l=0}^{n} \sum\limits_{m=l}^{\infty} \dbinom{n}{l} \mathbf{H}_{n-l}(x_1,y,z|\mu) S_1(m,l) x_2^l \mu^{m-l} \dfrac{l1}{m!}$.
3. $\mathbf{H}_n(x,y_1+y_2,z|\mu) = \sum\limits_{k=0}^{[\frac{n}{2}]} \mathbf{H}_{n-2k}(x,y_1,z|\mu) \left(\dfrac{\log(1+\mu)}{\mu} \right)^k y_2^k \dfrac{n!}{k!(n-2k)!}$.
4. $\mathbf{H}_n(x,y_1+y_2,z|\mu) = \sum\limits_{k=0}^{[\frac{n}{2}]} \sum\limits_{m=k}^{\infty} \dfrac{S_1(m,k) y_2^k \mu^{m-k} n!}{m!(n-2k)!} \mathbf{H}_{n-2k}(x,y_1,z|\mu)$.
5. $\mathbf{H}_n(x,y,z_1+z_2|\mu) = \sum\limits_{k=0}^{[\frac{n}{3}]} \mathbf{H}_{n-3k}(x,y,z_1|\mu) \left(\dfrac{\log(1+\mu)}{\mu} \right)^k z_2^k \dfrac{n!}{k!(n-3k)!}$.
6. $\mathbf{H}_n(x,y,z_1+z_2|\mu) = \sum\limits_{k=0}^{[\frac{n}{3}]} \sum\limits_{m=k}^{\infty} \dfrac{S_1(m,k) z_2^k \mu^{m-k} n!}{m!(n-3k)!} \mathbf{H}_{n-3k}(x,y,z_1|\mu)$.
7. $\mathbf{H}_n(x_1+x_2,y_1+y_2,z_1+z_2|\mu) = \sum\limits_{l=0}^{n} \dbinom{n}{l} \mathbf{H}_l(x_1,y_1,z_1|\mu) \mathbf{H}_{n-l}(x_2,y_2,z_2|\mu)$.

The paper is written by this process: We make symmetric identities about 3-variable degenerate Hermite Kampé de Fériet polynomials in Section 2. We also get formulas of 3-variable degenerate Hermite Kampé de Fériet polynomials. We induce the differential equations getting from the generating function of 3-variable degenerate Hermite Kampé de Fériet polynomials in Section 3:

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)^N \mathfrak{F}(t,x,y,z|\mu) - a_0(N,x,y,z|\mu)\mathfrak{F}(t,x,y,z|\mu) - a_1(N,x,y,z|\mu) t \mathfrak{F}(t,x,y,z|\mu) - \cdots \\ - a_{N-1}(N,x,y,z|\mu) t^{N-1} \mathfrak{F}(t,x,y,z|\mu) - a_N(N,x,y,z|\mu) t^N \mathfrak{F}(t,x,y,z|\mu) = 0. \end{aligned}$$

In Section 4, we study distribution of computer graphic about the roots of the 3-variable degenerate Hermite Kampé de Fériet equation $\mathbf{H}_n(x,y,z|\mu) = 0$. Finally, we see the symmetric pattern of the roots of polynomials $\mathbf{H}_n(x,y,z|\mu) = 0$ and indicate some open problems.

2. Symmetric Identities for the 3-Variable Degenerate Hermite Kampé de Fériet Polynomials

In this section, we give symmetric identities for the 3-variable degenerate Hermite Kampé de Fériet polynomials. We also get some formulas and properties of the 3-variable degenerate Hermite Kampé de Fériet polynomials.

Theorem 4. *Let $a, b > 0$ and $a \neq b$. Then*

$$a^m \mathbf{H}_m(bx, b^2 y, b^3 z|\mu) = b^m \mathbf{H}_m(ax, a^2 y, a^3 z|\mu).$$

Proof. Let $a, b > 0$ and $a \neq b$. We start with

$$\mathcal{F}(t,x,y,z,\mu) = (1+\mu)^{\frac{abxt}{\mu}} (1+\mu)^{\frac{a^2 b^2 y t^2}{\mu}} (1+\mu)^{\frac{a^3 b^3 z t^3}{\mu}}.$$

Then the expression for $\mathcal{F}(t,x,y,z,\mu)$ is symmetric in a and b

$$\mathcal{F}(t,x,y,z,\mu) = \sum_{m=0}^{\infty} \mathbf{H}_m(ax, a^2 y, a^3 z|\mu) \frac{(bt)^m}{m!} = \sum_{m=0}^{\infty} b^m \mathbf{H}_m(ax, a^2 y, a^3 z|\mu) \frac{t^m}{m!}.$$

We can get that

$$\mathcal{F}(t,x,y,z,\mu) = \sum_{m=0}^{\infty} \mathbf{H}_m(bx, b^2y, b^3z|\mu)\frac{(at)^m}{m!} = \sum_{m=0}^{\infty} a^m \mathbf{H}_m(bx, b^2y, b^3z|\mu)\frac{t^m}{m!}.$$

When we compare the coefficients of $\frac{t^m}{m!}$ on the right hand sides of the last two equations, the proof is completed. $\square$

When we let $\mu \to 0$ in Theorem 4, we have the corollary

Corollary 1. *Let $a, b > 0$ and $a \neq b$. Then*

$$a^m H_m(bx, b^2y, b^3z) = b^m H_m(ax, a^2y, a^3z).$$

Again, we now use

$$\mathcal{G}(t,x,y,z,\mu) = \frac{abt(1+\mu)^{\frac{abxt}{\mu}}(1+\mu)^{\frac{a^2b^2yt^2}{\mu}}(1+\mu)^{\frac{a^3b^3zt^3}{\mu}}\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}.$$

For $\mu \in \mathbb{C}$, we define the degenerate Bernoulli polynomials related to the generating function

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x|\mu)\frac{t^n}{n!} = \frac{t}{(1+\mu)^{\frac{t}{\mu}}-1}(1+\mu)^{\frac{xt}{\mu}}.$$

If we give $x = 0$, $\mathcal{B}_n(\mu) = \mathcal{B}_n(0,\mu)$ are called the degenerate Bernoulli numbers. Let us look at few terms:

$$\begin{aligned}
\mathcal{B}_0(x|\mu) &= \frac{\mu}{\log(1+\mu)},\\
\mathcal{B}_1(x|\mu) &= x - \frac{1}{2},\\
\mathcal{B}_2(x|\mu) &= \frac{x^2\log(1+\mu)}{\mu} - \frac{x\log(1+\mu)}{\mu} + \frac{\log(1+\mu)}{6\mu},\\
\mathcal{B}_3(x|\mu) &= \frac{x^3\log(1+\mu)^2}{\mu^2} - \frac{3x^2\log(1+\mu)^2}{2\mu^2} + \frac{x\log(1+\mu)^2}{2\mu^2},\\
\mathcal{B}_4(x|\mu) &= \frac{x^4\log(1+\mu)^3}{\mu^3} - \frac{2x^3\log(1+\mu)^3}{\mu^3} + \frac{x^2\log(1+\mu)^3}{\mu^3} - \frac{\log(1+\mu)^3}{30\mu^3}.
\end{aligned}$$

Let each integer $k \geq 0$. $S_k(n) = 0^k + 1^k + 2^k + \cdots + (n-1)^k$ is called sum of integers. A generalized falling factorial sum $\sigma_k(n,\mu)$ can be defined by the generation function

$$\sum_{k=0}^{\infty} \sigma_k(n,\mu)\frac{t^k}{k!} = \frac{(1+\mu)^{\frac{(n+1)t}{\mu}}-1}{(1+\mu)^{\frac{t}{\mu}}-1}.$$

We look at this $\lim_{\mu\to 0}\sigma_k(n,\mu) = S_k(n)$.

Theorem 5. *Let $a, b > 0$ and $a \neq b$. Then*

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^{i}b^{n+1-i}\mathcal{B}_m(\mu)\mathbf{H}_{i-m}(bx,b^2y,b^3z|\mu)\sigma_{n-i}(a-1,\mu)\\
&=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^{i}a^{n+1-i}\mathcal{B}_m(\mu)\mathbf{H}_{i-m}(ax,a^2y,a^3z|\mu)\sigma_{n-i}(b-1,\mu).
\end{aligned}$$

Proof. From $\mathcal{G}(t,x,y,z,\mu)$, we get the following result:

$$\begin{aligned}
&\mathcal{G}(t,x,y,z,\mu)\\
&=\frac{abt(1+\mu)^{\frac{abxt}{\mu}}(1+\mu)^{\frac{a^2b^2yt^2}{\mu}}(1+\mu)^{\frac{a^3b^3zt^3}{\mu}}\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}\\
&=\frac{abt}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)}(1+\mu)^{\frac{abxt}{\mu}}(1+\mu)^{\frac{a^2b^2yt^2}{\mu}}(1+\mu)^{\frac{a^3b^3zt^3}{\mu}}\frac{\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}\\
&=b\sum_{n=0}^{\infty}\mathcal{B}_n(\mu)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\mathbf{H}_n(bx,b^2y,b^3z|\mu)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\sigma_k(a-1,\mu)\frac{(bt)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^{i}b^{n+1-i}\mathcal{B}_m(\mu)\mathbf{H}_{i-m}(bx,b^2y,b^3z|\mu)\sigma_{n-i}(a-1,\mu)\right)\frac{t^n}{n!}.
\end{aligned}$$

If we follow a similar way, we have

$$\begin{aligned}
&\mathcal{G}(t,x,y,z,\mu)\\
&=\frac{abt}{\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}(1+\mu)^{\frac{abxt}{\mu}}(1+\mu)^{\frac{a^2b^2yt^2}{\mu}}(1+\mu)^{\frac{a^3b^3zt^3}{\mu}}\frac{\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)}\\
&=a\sum_{n=0}^{\infty}\mathcal{B}_n(\mu)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\mathbf{H}_n(ax,a^2y,a^3z|\mu)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\sigma_k(a-1,\mu)\frac{(at)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^{i}a^{n+1-i}\mathcal{B}_m(\mu)\mathbf{H}_{i-m}(ax,a^2y,a^3z|\mu)\sigma_{n-i}(b-1,\mu)\right)\frac{t^n}{n!}.
\end{aligned}$$

If we compare the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, then the proof is completed. □

If we give $\mu \to 0$ in Theorem 5, we have the corollary

Corollary 2. *Let $a, b > 0$ and $a \neq b$. Then*

$$\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^i b^{n+1-i}B_m H_{i-m}(bx, b^2y, b^3z)S_{n-i}(a-1)$$
$$=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^i a^{n+1-i}B_m H_{i-m}(ax, a^2y, a^3z)S_{n-i}(b-1),$$

where B_m are Bernoulli numbers (see References [8–10]).

Theorem 6. *Let m, n, N be nonnegative integers. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}\left(\frac{\log(1+\mu)}{\mu}\right)^{m-k}\mathbf{H}_{N+k}(x,y,z|\mu)$$
$$=\sum_{k=0}^{N}\binom{N}{k}n^{N-k}\left(\frac{\log(1+\mu)}{\mu}\right)^{N-k}\mathbf{H}_{m+k}\left(x-n\left(\frac{\log(1+\mu)}{\mu}\right), y, z|\mu\right).$$

Proof. If we take N-th many derivative with respect to t in (4), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) &= \left(\frac{\partial}{\partial t}\right)^N (1+\mu)^{\frac{xt}{\mu}}(1+\mu)^{\frac{yt^2}{\mu}}(1+\mu)^{\frac{zt^3}{\mu}} \\ &= \sum_{m=0}^{\infty}\mathbf{H}_{m+N}(x,y,z|\mu)\frac{t^m}{m!}.\end{aligned} \tag{13}$$

If we use the Cauchy product and multiplying the exponential series $e^{xt} = \sum_{m=0}^{\infty} x^m \frac{t^m}{m!}$ on both sides of (13), we get

$$\begin{aligned}& e^{-n\left(\frac{\log(1+\mu)}{\mu}\right)t}\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) \\ &= \left(\sum_{m=0}^{\infty}(-n)^m\left(\frac{\log(1+\mu)}{\mu}\right)^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}\mathbf{H}_{m+N}(x,y,z|\mu)\frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}\left(\frac{\log(1+\mu)}{\mu}\right)^{m-k}\mathbf{H}_{N+k}(x,y,z|\mu)\right)\frac{t^m}{m!}.\end{aligned} \tag{14}$$

When we use (14) and the Leibniz rule, we have

$$\begin{aligned}& e^{-n\left(\frac{\log(1+\mu)}{\mu}\right)t}\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) \\ &= \sum_{k=0}^{N}\binom{N}{k}n^{N-k}\left(\frac{\log(1+\mu)}{\mu}\right)^{N-k}\left(\frac{\partial}{\partial t}\right)^k\left(e^{-n\left(\frac{\log(1+\mu)}{\mu}\right)t}\mathfrak{F}(t,x,y,z|\mu)\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}n^{N-k}\left(\frac{\log(1+\mu)}{\mu}\right)^{N-k}\mathbf{H}_{m+k}\left(x-n\left(\frac{\log(1+\mu)}{\mu}\right), y, z|\mu\right)\right)\frac{t^m}{m!}.\end{aligned} \tag{15}$$

If we use (14) and (15), and compare the coefficients of $\frac{t^m}{m!}$, then the proof is completed. □

If we plug in $m = 0$ in (15), then we obtain the following theorem

Theorem 7. *For $N = 0, 1, 2, \dots$, we have*

$$\mathbf{H}_N(x,y,z|\mu) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} \left(\frac{\log(1+\mu)}{\mu}\right)^{N-k} \mathbf{H}_k\left(x - n\left(\frac{\log(1+\mu)}{\mu}\right), y, z|\mu\right).$$

3. Differential Equations Related to 3-Variable Degenerate Hermite Kampé de Fériet Polynomials

Many researchers have studied differential equations which are related to the generating functions of special numbers and polynomials in References [13–18] in order to make formulas for special numbers and polynomials. Recall that

$$\begin{aligned} \mathfrak{F} = \mathfrak{F}(t,x,y,z|\mu) &= (1+\mu)^{\frac{xt}{\mu}} (1+\mu)^{\frac{yt^2}{\mu}} (1+\mu)^{\frac{zt^3}{\mu}} \\ &= \sum_{n=0}^{\infty} \mathbf{H}_n(x,y,z|\mu) \frac{t^n}{n!}, \quad \mu, x, t \in \mathbb{C}. \end{aligned} \tag{16}$$

In this section, we study the differential equations with coefficients $a_i(N,x,y,z|\mu)$ coming from the generating functions of the 3-variable degenerate Hermite Kampé de Fériet polynomials:

$$\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) - a_0(N,x,y,z|\mu)\mathfrak{F}(t,x,y,z|\mu) - \cdots - a_N(N,x,y,z|\mu)t^N \mathfrak{F}(t,x,y,z|\mu) = 0.$$

From (16), it follows

$$\begin{aligned} \mathfrak{F}^{(1)} &= \frac{\partial}{\partial t}\mathfrak{F}(t,x,y,\mu) \\ &= \frac{\partial}{\partial t}\left((1+\mu)^{\frac{xt}{\mu}} (1+\mu)^{\frac{yt^2}{\mu}} (1+\mu)^{\frac{zt^3}{\mu}}\right) \\ &= \left(\frac{(x+2yt+3zt^2)\log(1+\mu)}{\mu}\right)(1+\mu)^{\frac{xt}{\mu}} (1+\mu)^{\frac{yt^2}{\mu}} (1+\mu)^{\frac{zt^3}{\mu}} \\ &= \left(\frac{(x+2yt+3zt^2)\log(1+\mu)}{\mu}\right)\mathfrak{F}(t,x,y,z|\mu), \end{aligned} \tag{17}$$

$$\begin{aligned} \mathfrak{F}^{(2)} &= \frac{\partial}{\partial t}\mathfrak{F}^{(1)}(t,x,y,z|\mu) \\ &= \left(\frac{(2y+6zt)\log(1+\mu)}{\mu}\right)\mathfrak{F}(t,x,y,z|\mu) + \left(\frac{(x+2yt+3zt^2)\log(1+\mu)}{\mu}\right)\mathfrak{F}^{(1)}(t,x,y,z|\mu) \\ &= \left(\left(\frac{\log(1+\mu)}{\mu}\right)2y + \left(\frac{\log(1+\mu)}{\mu}\right)^2 x^2\right)\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \left(\left(\frac{\log(1+\mu)}{\mu}\right)6z + \left(\frac{\log(1+\mu)}{\mu}\right)^2 4xy\right)t\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \left(\left(\frac{\log(1+\mu)}{\mu}\right)^2 (4y^2+6xz)\right)t^2\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \left(\left(\frac{\log(1+\mu)}{\mu}\right)^2 12yz\right)t^3\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \left(\left(\frac{\log(1+\mu)}{\mu}\right)^2 (3z)^2\right)t^2\mathfrak{F}(t,x,y,z|\mu). \end{aligned} \tag{18}$$

When we continue this process, we can guess that

$$\begin{aligned}\mathfrak{F}^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) \\ &= \sum_{i=0}^{2N} a_i(N,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu), (N=0,1,2,\ldots).\end{aligned} \tag{19}$$

If we differentiate (19) with respect to t, we have

$$\begin{aligned}\mathfrak{F}^{(N+1)} = \frac{\partial \mathfrak{F}^{(N)}}{\partial t} &= \sum_{i=0}^{2N} a_i(N,x,y,z|\mu)it^{i-1}\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=0}^{2N} a_i(N,x,y,z|\mu)t^i\mathfrak{F}^{(1)}(t,x,y,z|\mu) \\ &= \sum_{i=0}^{2N}(i)a_i(N,x,y,z|\mu)t^{i-1}\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=0}^{2N}\frac{x\log(1+\mu)}{\mu}a_i(N,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=0}^{2N}\frac{2y\log(1+\mu)}{\mu}a_i(N,x,y,z|\mu)t^{i+1}\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=0}^{2N}\frac{3z\log(1+\mu)}{\mu}a_{i-1}(N,x,y,z|\mu)t^{i+2}\mathfrak{F}(t,x,y,z|\mu) \\ &= \sum_{i=0}^{2N-1}(i+1)a_{i+1}(N,x,y,z|\mu)t^iF(t,x,y,z|\mu) \\ &\quad + \sum_{i=0}^{2N}\frac{x\log(1+\mu)}{\mu}a_i(N,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=1}^{2N+1}\frac{2y\log(1+\mu)}{\mu}a_{i-1}(N,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu) \\ &\quad + \sum_{i=2}^{2N+2}\frac{3z\log(1+\mu)}{\mu}a_{i-2}(N,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu).\end{aligned} \tag{20}$$

Now we plug in $N+1$ instead of N in (19) to find

$$\mathfrak{F}^{(2N+2)} = \sum_{i=0}^{N+1} a_i(N+1,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu). \tag{21}$$

Comparing the coefficients from (20) and (21), we get

$$\begin{aligned}
a_0(N+1,x,y,z|\mu) &= a_1(N,x,y,z|\mu) + \frac{x\log(1+\mu)}{\mu}a_0(N,x,y,z|\mu),\\
a_1(N+1,x,y,z|\mu) &= 2a_2(N,x,y,z|\mu) + \frac{x\log(1+\mu)}{\mu}a_1(N,x,y,z|\mu)\\
&\quad + \frac{2y\log(1+\mu)}{\mu}a_0(N,x,y,z|\mu),\\
a_{2N}(N+1,x,y,z|\mu) &= \frac{x\log(1+\mu)}{\mu}a_{2N}(N,x,y,z|\mu)\\
&\quad + \frac{2y\log(1+\mu)}{\mu}a_{2N-1}(N,x,y,z|\mu) + \frac{3z\log(1+\mu)}{\mu}a_{2N-2}(N,x,y,z|\mu),\\
a_{2N+1}(N+1,x,y,z|\mu) &= \frac{2y\log(1+\mu)}{\mu}a_N(N,x,y,z|\mu)\\
&\quad + \frac{3z\log(1+\mu)}{\mu}a_{2N-1}(N,x,y,z|\mu),\\
a_{2N+2}(N+1,x,y,z|\mu) &= \frac{3z\log(1+\mu)}{\mu}a_{2N}(N,x,y,z|\mu),
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
a_i(N+1,x,y,z|\mu) &= (i+1)a_{i+1}(N,x,y,z|\mu)\\
&\quad + \frac{x\log(1+\mu)}{\mu}a_i(N,x,y,z|\mu)\\
&\quad + \frac{2y\log(1+\mu)}{\mu}a_{i-1}(N,x,y,z|\mu)\\
&\quad + \frac{3z\log(1+\mu)}{\mu}a_{i-2}(N,x,y,z|\mu), (2 \le i \le 2N-1).
\end{aligned} \tag{23}$$

In addition, by (16), we have

$$\mathfrak{F}(t,x,y,z|\mu) = \mathfrak{F}^{(0)}(t,x,y,z|\mu) = a_0(0,x,y,z|\mu)\mathfrak{F}(t,x,y,z|\mu). \tag{24}$$

By (24), we get

$$a_0(0,x,y,z|\mu) = 1. \tag{25}$$

It is easy to show that

$$\begin{aligned}
&\frac{x\log(1+\mu)}{\mu}\mathfrak{F}(t,x,y,z|\mu) + \frac{2y\log(1+\mu)}{\mu}t\mathfrak{F}(t,x,y,z|\mu)\\
&\qquad + \frac{2z\log(1+\mu)}{\mu}t^2\mathfrak{F}(t,x,y,z|\mu)\\
&= \mathfrak{F}^{(1)}(t,x,y,z|\mu)\\
&= \sum_{i=0}^{2} a_i(1,x,y,z|\mu)t^i\mathfrak{F}(t,x,y,z|\mu)\\
&= a_0(1,x,y,z|\mu)\mathfrak{F}(t,x,y,z|\mu) + a_1(1,x,y,z|\mu)t\mathfrak{F}(t,x,y,z|\mu)\\
&\qquad + a_2(1,x,y,z|\mu)t^2\mathfrak{F}(t,x,y,z|\mu).
\end{aligned} \tag{26}$$

Thus, by (26), we also get

$$\begin{aligned} a_0(1, x, y, \mu) &= \frac{x \log(1+\mu)}{\mu}, \\ a_1(1, x, y, z|\mu) &= \frac{2y \log(1+\mu)}{\mu}, \\ a_2(1, x, y, z|\mu) &= \frac{3z \log(1+\mu)}{\mu}. \end{aligned} \tag{27}$$

From (22), we note that

$$\begin{aligned} & a_0(N+1, x, y, z|\mu) = a_1(N, x, y, z|\mu) + \frac{x \log(1+\mu)}{\mu} a_0(N, x, y, z|\mu), \\ & a_0(N, x, y, z|\mu) = a_1(N-1, x, y, z|\mu) + \frac{x \log(1+\mu)}{\mu} a_0(N-1, x, y, z|\mu), \\ & \cdots \\ & a_0(N+1, x, y, z|\mu) = \sum_{i=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^i a_1(N-i, x, y, z|\mu) \\ & \qquad + \left(\frac{\log(1+\mu)}{\mu} \right)^{N+1} x^{N+1}, \end{aligned} \tag{28}$$

For $i = 1$, we have

$$\begin{aligned} a_1(N+1, x, y, z|\mu) = {} & 2 \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^k a_2(N-k, x, y, z|\mu) \\ & + \frac{2y \log(1+\mu)}{\mu} \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^k a_0(N-k, x, y, z|\mu), \end{aligned} \tag{29}$$

Continuing this process, we can deduce that, for $2 \leq i \leq 2N-1$,

$$\begin{aligned} a_i(N+1, x, y, z|\mu) = {} & (i+1) \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^k a_{i+1}(N-k, x, y, z|\mu) \\ & + \frac{2y \log(1+\mu)}{\mu} \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^k a_{i-1}(N-k, x, y, z|\mu) \\ & + \frac{3z \log(1+\mu)}{\mu} \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^k a_{i-2}(N-k, x, y, z|\mu). \end{aligned} \tag{30}$$

For $i = 2N$, we get

$$
\begin{aligned}
a_{2N}(N+1,x,y,z|\mu) &= \frac{x\log(1+\mu)}{\mu}a_{2N}(N,x,y,z|\mu) \\
&+ \frac{2y\log(1+\mu)}{\mu}a_{2N-1}(N,x,y,z|\mu) + \frac{3z\log(1+\mu)}{\mu}a_{2N-2}(N,x,y,z|\mu), \\
a_{2N-2}(N,x,y,z|\mu) &= \frac{x\log(1+\mu)}{\mu}a_{2N-2}(N-1,x,y,z|\mu) \\
&+ \frac{2y\log(1+\mu)}{\mu}a_{2N-1}(N-1,x,y,z|\mu) + \frac{3z\log(1+\mu)}{\mu}a_{2N-4}(N-1,x,y,z|\mu), \\
\dots, \\
a_{2N}(N+1,x,y,z|\mu) &= \frac{x\log(1+\mu)}{\mu}\sum_{k=0}^{N}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k}(N-k,x,y,z|\mu) \\
&+ \frac{2y\log(1+\mu)}{\mu}\sum_{k=0}^{N-1}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k-1}(N-k,x,y,z|\mu).
\end{aligned} \tag{31}
$$

For $i = 2N+1$, we obtain

$$
\begin{aligned}
&a_{2N+1}(N+1,x,y,z|\mu) \\
&\quad = \left(\frac{2y\log(1+\mu)}{\mu}\right)^k \sum_{k=0}^{N}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k}(N-k,x,y,z|\mu).
\end{aligned} \tag{32}
$$

For $i = 2N+2$, we have

$$
a_{2N+2}(N+1,x,y,z|\mu) = \left(\frac{3z\log(1+\mu)}{\mu}\right)^{N+1}. \tag{33}
$$

As a matrix, $a_i(j,x,y,z|\mu)_{0\le i\le 2N+2, 0\le j\le N+1}$ is given by

$$
\begin{pmatrix}
1 & \frac{x\log(1+\mu)}{\mu} & \left(\frac{x\log(1+\mu)}{\mu}\right)^2 + \frac{2y\log(1+\mu)}{\mu} & \cdots & \cdot \\
0 & \frac{2y\log(1+\mu)}{\mu} & \frac{6z\log(1+\mu)}{\mu} + \left(\frac{\log(1+\mu)}{\mu}\right)^2 4xy & \cdots & \cdot \\
0 & \frac{3z\log(1+\mu)}{\mu} & \left(\frac{2y\log(1+\mu)}{\mu}\right)^2 + \left(\frac{\log(1+\mu)}{\mu}\right)^2 6xz & \cdots & \cdot \\
0 & 0 & \left(\frac{\log(1+\mu)}{\mu}\right)^2 12yz & \cdots & \cdot \\
0 & 0 & \left(\frac{3z\log(1+\mu)}{\mu}\right)^2 & \cdots & \cdot \\
\vdots & \vdots & \vdots & \ddots & \cdot \\
0 & 0 & 0 & \cdots & \left(\frac{3z\log(1+\mu)}{\mu}\right)^{N+1}
\end{pmatrix}
$$

Therefore, by (28)-(33), we get the following theorem:

Theorem 8. *Let $N = 0, 1, 2, \ldots$. The differential equation*

$$\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) - \left(\sum_{i=0}^{N} a_i(N,x,y,z|\mu)t^i\right)\mathfrak{F}(t,x,y,z|\mu) = 0$$

has a solution

$$\mathfrak{F} = \mathfrak{F}(t,x,y,z|\mu) = (1+\mu)^{\frac{xt}{\mu}}(1+\mu)^{\frac{yt^2}{\mu}}(1+\mu)^{\frac{zt^3}{\mu}},$$

where

$$\begin{aligned}
a_0(N+1,x,y,z|\mu) &= \sum_{i=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^i a_1(N-i,x,y,z|\mu) \\
&\quad + \left(\frac{\log(1+\mu)}{\mu}\right)^{N+1} x^{N+1}, \\
a_1(N+1,x,y,z|\mu) &= 2\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_2(N-k,x,y,z|\mu) \\
&\quad + \frac{2y\log(1+\mu)}{\mu}\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_0(N-k,x,y,z|\mu) \\
a_{2N}(N+1,x,y,z|\mu) &= \frac{x\log(1+\mu)}{\mu}\sum_{k=0}^{N}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k}(N-k,x,y,z|\mu) \\
&\quad + \frac{2y\log(1+\mu)}{\mu}\sum_{k=0}^{N-1}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k-1}(N-k,x,y,z|\mu) \\
a_{2N+1}&(N+1,x,y,z|\mu) \\
&= \left(\frac{2y\log(1+\mu)}{\mu}\right)^k \sum_{k=0}^{N}\left(\frac{3z\log(1+\mu)}{\mu}\right)^k a_{2N-2k}(N-k,x,y,z|\mu), \\
a_{2N+2}(N+1,x,y,z|\mu) &= \left(\frac{3z\log(1+\mu)}{\mu}\right)^{N+1}, \\
a_i(N+1,x,y,z|\mu) &= (i+1)\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_{i+1}(N-k,x,y,z|\mu) \\
&+ \frac{2y\log(1+\mu)}{\mu}\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_{i-1}(N-k,x,y,z|\mu) \\
&+ \frac{3z\log(1+\mu)}{\mu}\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_{i-2}(N-k,x,y,z|\mu), (2 \le i \le 2N-1).
\end{aligned}$$

We have a picture of the surface for this solution.

In Figure 1a , we choose $-1 \le x \le 1, -1 \le t \le 1, \mu = 1/3$, and $y = 2, z = 1$. In Figure 1b, we choose $-2 \le y \le 2, -1 \le t \le 1, \mu = 1/3$, and $x = 5, z = 3$.

When we take N-th many derivative with respect to t for (4) , we have

$$\left(\frac{\partial}{\partial t}\right)^N \mathfrak{F}(t,x,y,z|\mu) = \sum_{m=0}^{\infty} \mathbf{H}_{m+N}(x,y,z|\mu)\frac{t^m}{m!}. \tag{34}$$

From (19) and (34), we have the following theorem:

Theorem 9. *For $N = 0, 1, 2, \ldots$, one obtains*

$$\mathbf{H}_{m+N}(x,y,z|\mu) = \sum_{i=0}^{m} \frac{\mathbf{H}_{m-i}(x,y,z|\mu) a_i(N,x,y,z|\mu) m!}{(m-i)!}. \tag{35}$$

From (35) with $m = 0$ one obtains the following corollary:

Corollary 3. *For $N = 0, 1, 2, \ldots$, one obtains*

$$\mathbf{H}_N(x,y,z|\mu) = a_0(N,x,y,z|\mu),$$

where

$$\begin{aligned} a_0(N+1,x,y,z|\mu) &= 1, \\ a_0(N+1,x,y,z|\mu) &= \sum_{i=0}^{N} \left(\frac{x \log(1+\mu)}{\mu} \right)^i a_1(N-i,x,y,z|\mu) \\ &\quad + \left(\frac{\log(1+\mu)}{\mu} \right)^{N+1} x^{N+1}. \end{aligned}$$

The first 3-variable degenerate Hermite Kampé de Feŕiet polynomials read

$$\mathbf{H}_0(x,y,z|\mu) = 1,$$

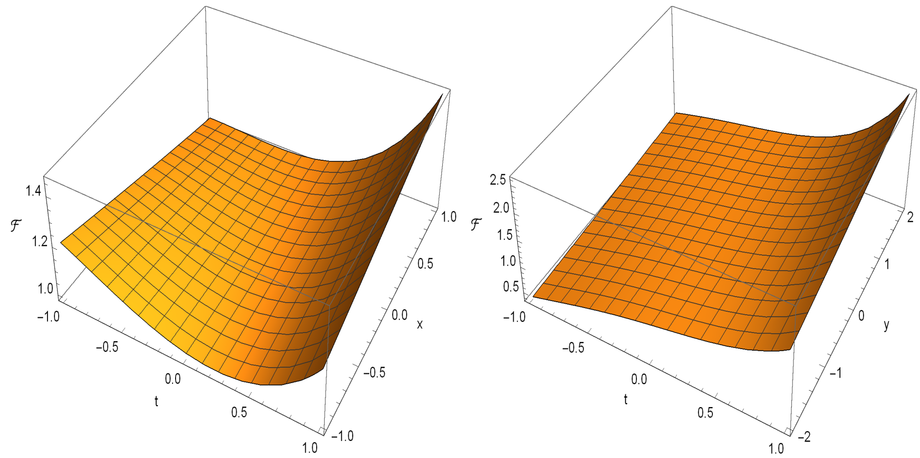
$$\mathbf{H}_1(x,y,z|\mu) = \frac{x \log(1+\mu)}{\mu},$$

$$\mathbf{H}_2(x,y,z|\mu) = \frac{x^2 (\log(1+\mu))^2}{\mu^2} + \frac{2y \log(1+\mu)}{\mu},$$

$$\mathbf{H}_3(x,y,z|\mu) = \frac{x^3 (\log(1+\mu))^3}{\mu^3} + \frac{6xy (\log(1+\mu))^2}{\mu^2} + \frac{6z (\log(1+\mu))^2}{\mu^2},$$

$$\begin{aligned} \mathbf{H}_4(x,y,z|\mu) = {}& \frac{x^4 (\log(1+\mu))^4}{\mu^4} + \frac{12x^2 y (\log(1+\mu))^3}{\mu^3} + \frac{24xz (\log(1+\mu))^2}{\mu^2} \\ &+ \frac{12y^2 (\log(1+\mu))^2}{\mu^2}, \end{aligned}$$

$$\begin{aligned} \mathbf{H}_5(x,y,z|\mu) = {}& \frac{x^5 (\log(1+\mu))^5}{\mu^5} + \frac{20x^3 y (\log(1+\mu))^4}{\mu^4} + \frac{60x^2 z (\log(1+\mu))^3}{\mu^3} + \\ &+ \frac{60xy^2 (\log(1+\mu))^3}{\mu^3} + \frac{120yz (\log(1+\mu))^2}{\mu^2}. \end{aligned}$$



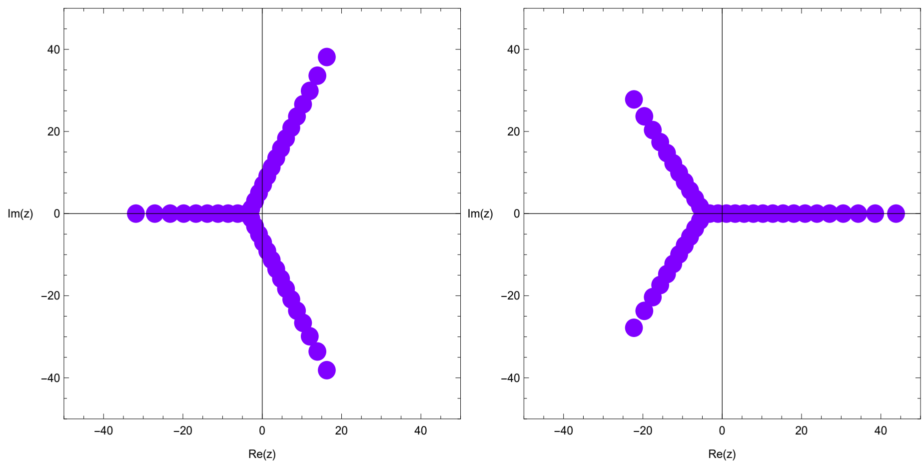
(a) $\mu = 1/3, y = 2,$ and $z = 1$ (b) $\mu = 1/3, x = 5,$ and $z = 3$

Figure 1. The surface for the solution $\mathfrak{F}(t, x, y, z|\mu)$.

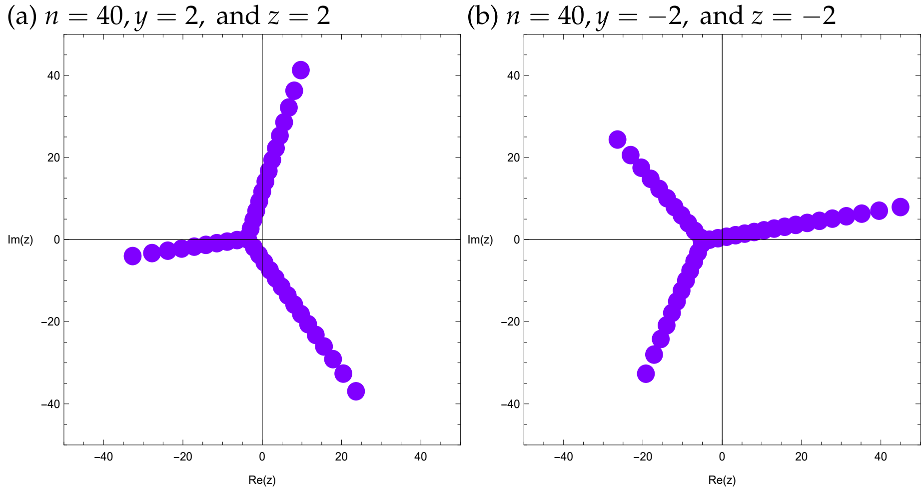
4. Roots of the 3-Variable Degenerate Hermite Kampé de Fériet Equations

In this section we give a theoretical prediction via numerical experiments by finding a regular pattern for the roots of the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$. To do this, we examine examples of several cases.

We look for the roots of $\mathbf{H}_n(x, y, z|\mu) = 0$ for $n = 40, y = 2, -2, 2+i, -2-i, z = 2, -2, 2+i, -2-i, \mu = 1/3,$ and $x \in \mathbb{C}$ (Figure 2).



(a) $n = 40, y = 2,$ and $z = 2$ (b) $n = 40, y = -2,$ and $z = -2$



(c) $n = 40, y = 2+i,$ and $z = 2+i$ (d) $n = 40, y = -2-i,$ and $z = -2-i$

Figure 2. Zeros of $\mathbf{H}_n(x, y, z|\mu) = 0$.

In Figure 2a, we select $n = 40$, $y = 2$, and $z = 2$. In Figure 2b, we select $n = 40$, $y = -2$, and $z = -2$. In Figure 2c, we select $n = 40$, $y = 2 + i$, and $z = 2 + i$. In Figure 2d, we select $n = 40$, $y = -2 - i$, and $z = -2 - i$. A picture of the roots of the 3-variable degenerate Hermite Kampé de Fériet equation $\mathbf{H}_n(x, y, z|\mu) = 0$ for $1 \leq n \leq 40, \mu = 1/3$ from a 3-D structure are shown in Figure 3.

In Figure 3a, we select $y = 2$ and $z = 2$. In Figure 3b, we select $y = -2$ and $z = -2$. In Figure 3c, we select $y = 2 + i$ and $z = 2 + i$. In Figure 3d, we select $y = -2 - i$ and $z = -2 - i$. Our distributions for approximated solutions of real roots of equation $\mathbf{H}_n(x, y, z|\mu) = 0$ are shown in Tables 1 and 2.

We can observe a regular pattern of the complex roots of the 3-variable degenerate Hermite Kampé de Fériet equation $\mathbf{H}_n(x, y, z|\mu) = 0$. We hope to prove regular pattern of the complex roots of the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ (Table 1).

A picture of real roots of equations $\mathbf{H}_n(x, y, z|\mu) = 0$ for $1 \leq n \leq 40, \mu = 1/3$ are displayed in Figure 4.

Table 1. Numbers of real and complex zeros of $\mathbf{H}_n(x, y, z|\mu) = 0$.

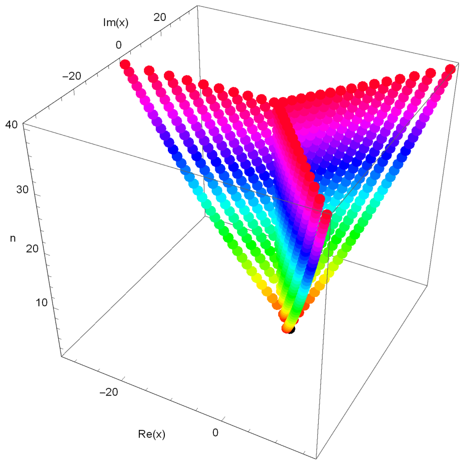
Degree n	$y = 2, z = 2, \mu = 1/3$		$y = -2 - i, z = -2 - i, \mu = 1/3$	
	Real Zeros	**Complex Zeros**	**Real Zeros**	**Complex Zeros**
1	1	0	1	0
2	0	2	0	2
3	1	2	0	3
4	0	4	0	4
5	1	4	0	5
6	2	4	0	6
7	1	6	0	7
8	2	6	0	8
9	1	8	0	9
10	2	8	0	10

In Figure 4a, we select $y = 2$ and $z = 2$. In Figure 4b, we select $y = 2$ and $z = -2$. In Figure 4c, we select $y = -2$ and $z = 2$. In Figure 4d, we select $y = -2$ and $z = -2$.

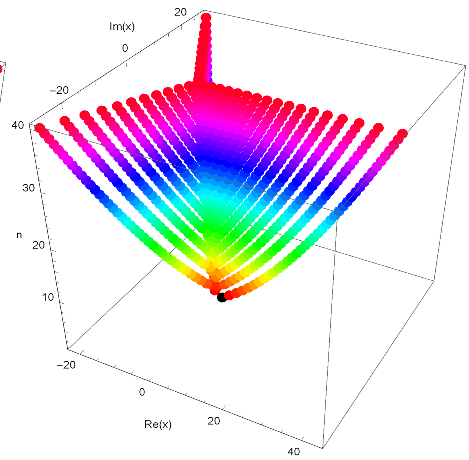
Next, we obtain an approximate solution satisfying $\mathbf{H}_n(x, y, z|\mu) = 0, x \in \mathbb{C}$ for given $n, y = 2, z = 2$, and $\mu = 1/3$ in the Table 2.

Table 2. Approximate roots of $\mathbf{H}_n(x,y,z|\mu)=0, x\in\mathbb{C}, y=2, z=2,$ and $\mu=1/3$.

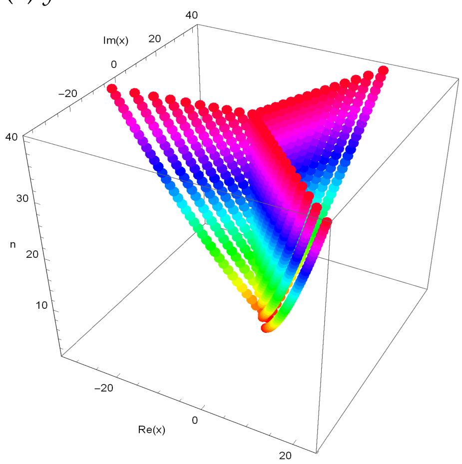
Degree n	x
1	0
2	$-2.1528\,i$, $2.1528\,i$
3	-1.0705, $0.5352+3.8424\,i$, $0.5352-3.8424\,i$
4	$-1.1238-0.9119\,i$, $-1.1238+0.9119\,i$, $1.1238+5.4316\,i$ $1.1238-5.4316\,i$
5	-2.6937, $-0.3500-2.3168\,i$, $-0.3500+2.3168\,i$ $1.6969+6.9000\,i$, $1.6969-6.9000\,i$
6	-3.6974, -1.1016, $0.1447-3.6924\,i$ $0.1447+3.6924\,i$, $2.2548+8.2725\,i$, $2.2548-8.2725\,i$
7	-4.8260, $-1.0340+1.3912\,i$, $-1.0340-1.3912\,i$, $0.6498+4.9794\,i$ $0.6498-4.9794\,i$, $2.7972-9.5687\,i$, $2.7972+9.5687\,i$



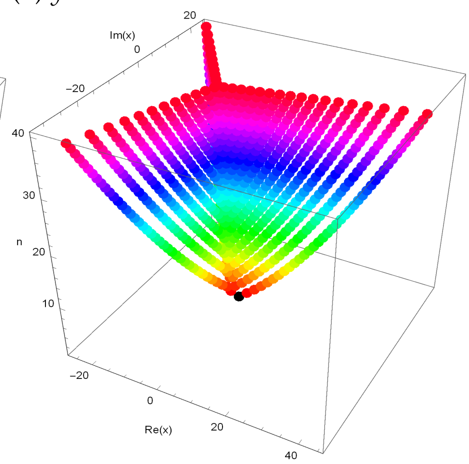
(a) $y=2$ and $z=2$



(b) $y=-2$ and $z=-2$

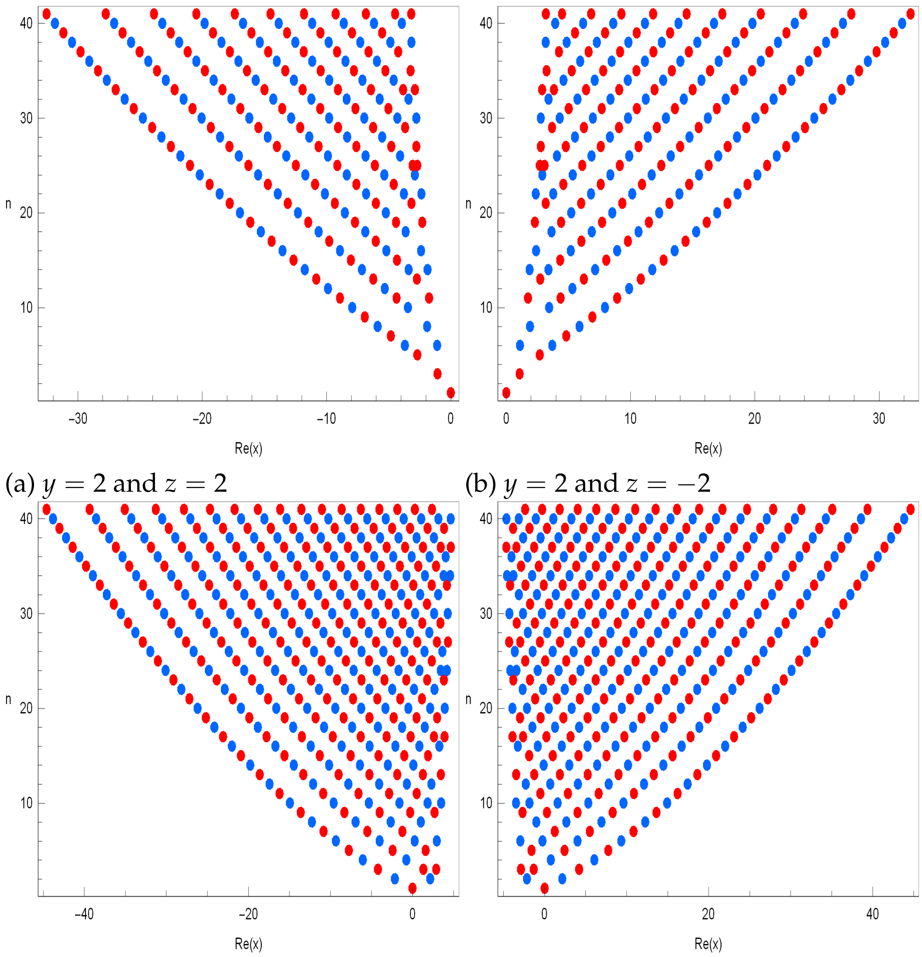


(c) $y=2+i$ and $z=2+i$



(d) $y=-2-i$, and $z=-2-i$

Figure 3. Stacks of zeros of $\mathbf{H}_n(x,y,z|\mu)=0, 1\le n\le 40$.



(a) $y = 2$ and $z = 2$ (b) $y = 2$ and $z = -2$

(c) $y = -2$ and $z = 2$ (d) $y = -2$ and $z = -2$

Figure 4. Real zeros of $\mathbf{H}_n(x, y, z|\mu) = 0$ for $1 \leq n \leq 40$.

5. Conclusions and Future Directions

In this article, we constructed the 3-variable degenerate Hermite Kampé de Fériet polynomials and got symmetric identities for 3-variable degenerate Hermite Kampé de Fériet polynomials. We also made the differential equations which are related to the generating function of $\mathbf{H}_n(x, y, z|\mu)$. We also studied the symmetry of pattern of the roots of the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ for various variables x, y, and z. As a result, we found that the distribution of the roots of 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ has regular pattern. So, we make the following series of conjectures with numerical experiments:

We use some notations. $R_{\mathbf{H}_n(x,y,z|\mu)}$ denotes the number of real zeros of $\mathbf{H}_n(x, y, z|\mu) = 0$ on the real plane, that is, $Im(x) = 0$, and $C_{\mathbf{H}_n(x,y,z|\mu)}$ denotes the number of complex zeros of $\mathbf{H}_n(x, y, z|\mu) = 0$. Since n is the degree of the polynomial $\mathbf{H}_n(x, y, z|\mu)$, we obtaine $R_{\mathbf{H}_n(x,y,z|\mu)} = n - C_{\mathbf{H}_n(x,y,z|\mu)}$.

We can see a regular pattern of the complex roots of the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ for y, z, and μ. Therefore, we can make thebelow conjecture.

Conjecture 1. *Let $n = 2, 3, \ldots$, and $y \in \mathbb{C}$. Prove or disprove that*

$$R_{\mathbf{H}_n(x,y,z|\mu)} = 0, \quad C_{\mathbf{H}_n(x,y,z|\mu)} = n,$$

where $\mathbb{C}$ is the set of complex numbers.

Conjecture 2. *For $n = 2, 3, \ldots,$ and $z \in \mathbb{C}$, prove or disprove that*

$$R_{\mathbf{H}_n(x,y,z|\mu)} = 0, \quad C_{\mathbf{H}_n(x,y,z|\mu)} = n.$$

The Conjectures 1 and 2 are unsolved problems for all variables y, z and μ.

We see that the solutions of the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ does not show reflection symmetry about $Re(x) = a$ for $a \in \mathbb{R}$ (see Figures 2–4).

Conjecture 3. *Prove that $\mathbf{H}_n(x, y, z|\mu), x \in \mathbb{C}, y, z \in \mathbb{R}$ as an analytic complex function has reflection symmetry $Im(x) = 0$.*

Finally, we consider the more general problems. How many roots does $\mathbf{H}_n(x, y, z|\mu) = 0$ have? We are not able to decide whether $\mathbf{H}_n(x, y, z|\mu) = 0$ has n distinct solutions. We would like to know the number of complex roots $C_{\mathbf{H}_n(x,y,z|\mu)}$ of $\mathbf{H}_n(x, y, z|\mu) = 0$.

Conjecture 4. *Prove or disprove that $\mathbf{H}_n(x, y, z|\mu) = 0$ has n distinct solutions.*

The conjecture 4 is unsolved problem for all variables n (see Tables 1 and 2).

If we can theoretically prove the above problems by drawing new ideas from various numerical results, we look forward to contributing to research related to the 3-variable degenerate Hermite Kampé de Fériet equations $\mathbf{H}_n(x, y, z|\mu) = 0$ in applied mathematics, mathematical physics, and engineering.

Author Contributions: C.-S.R. and K-W.H. did typing together; C.-S.R. and Y.-S.S. draw pictures by using computor. K.-W.H., C.-S.R., and Y.-S.S. made Section 1. Y.-S.S. made Section 2. K.-W.H. made Section 3. C.-S.R. made Section 4. These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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