

## Article

# A New Proof for Lyapunov-Type Inequality on the Fractional Boundary Value Problem

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**Abstract:** In this article, some new Lyapunov-type inequalities for a class of fractional boundary value problems are established by use of the nonsymmetry property of Green's function corresponding to appropriate boundary conditions.

**Keywords:** Lyapunov-type inequality; Green's function; boundary value problem

## 1. Introduction

The study of the Lyapunov inequalities and its various generalizations has become a very popular topic in mathematics research. The recent growing interest in Lyapunov inequalities is mainly due to its close relation to the applications of differential operators such as asymptotic theory, eigenvalue problems, disconjugacy, stability criteria, and oscillation theory [1–8]. The study of Lyapunov inequalities initiated from the second order ordinary differential equations with Dirichlet conditions, and now great progress on the generalizations of the classical Lyapunov inequality has been made for various types of differential equations involving integer differential operator or fractional differential operator (see [9–26] for details). For example, the first Lyapunov-type inequality for a fractional differential operator was obtained by Ferreira [18] that if the Riemann–Liouville fractional boundary value problem

$$\begin{cases} D_{c+}^{\beta} v(t) + p(t)v(t) = 0, & c < t < d, \quad 1 < \beta < 2, \\ v(c) = v(d) = 0 \end{cases} \quad (1)$$

has a nontrivial solution, where  $p \in C[c, d]$ , then

$$\int_c^d |p(s)| ds > \Gamma(\beta) \left( \frac{4}{d-c} \right)^{\beta-1}. \quad (2)$$

In 2014, Ferreira [19] also derived a Lyapunov-type inequality

$$\int_c^d |p(s)| ds > \frac{\Gamma(\beta)\beta^{\beta}}{[(\beta-1)(d-c)]^{\beta-1}}$$

for a boundary value problem depending on the Caputo operator as follows:

$$\begin{cases} {}^C D_{c+}^{\beta} v(t) + p(t)v(t) = 0, & c < t < d, \quad 1 < \beta < 2, \\ v(c) = v(d) = 0. \end{cases}$$

Following Ferreira [18,19], some Lyapunov-type inequalities for Riemann–Liouville, Caputo, focal  $q$ -difference, and nested type fractional fractional boundary value problems were obtained, see [6,14–24] and references therein.



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The various analysis methods are devoted to the proofs of Lyapunov inequalities and Lyapunov-type inequalities. However, as Ref. [20] points out, the Green's function approach seems to be the only way to investigate Lyapunov-type inequalities for fractional boundary value problems. This method dates back to Nehari [8]. It depends on the construction of the Green's function of the fractional boundary value problem being considered and then finding the maximum of the corresponding Green's function. However, for higher-order fractional boundary value problems, the Green's function becomes complex, and it is difficult to find its maximum value.

In 2019, Dhar and Kelly [23] provided a non Green's function approach to show a Lyapunov-type inequality

$$\int_c^d |p(s)| ds > \frac{2(2\beta - 3)^{\frac{1}{2}} \Gamma(\beta - 1)}{(d - c)^{\beta - 1}} \quad (3)$$

if Riemann–Liouville fractional boundary value problems

$$\begin{cases} D_{c+}^{\beta} v(t) + p(t)v(t) = 0, & c < t < d, \quad 2 < \beta \leq 3, \\ v(c) = v'(c) = v(d) = 0, \end{cases} \quad (4)$$

has a nontrivial solution  $v(t)$  which satisfies

$$D_{c+}^{\beta-1} v(\xi) = 0 \text{ for some } \xi \in [c, d]. \quad (5)$$

Recently, Dhar and Kong [24] studied the Riemann–Liouville fractional boundary value problem

$$\begin{cases} D_{c+}^{\beta} v(t) + p(t)v(t) = 0, & c < t < d, \quad 2 < \beta \leq 3, \\ v(c) = v'(c) = v'(d) = 0, \end{cases} \quad (6)$$

Let  $p_+(t) = \max\{p(t), 0\}$ . They proved the following Lyapunov-type inequalities:

$$\int_c^d p_+(s) ds > \frac{(\beta - 1)^{\beta-1} \Gamma(\beta)}{(\beta - 2)^{\beta-2} (d - c)^{\beta-1}} \quad (7)$$

if (6) has a nontrivial solution  $v(t)$  and

$$v(t) \text{ does not have any zeros in } (c, d). \quad (8)$$

In particular, Aktaş and Çakmak [9] considered third order differential equations (6) with  $\beta = 3$  and proved

$$\frac{8}{(d - c)^2} \leq \int_c^d |p(s)| ds \quad (9)$$

if (6) has a nontrivial solution with  $\beta = 3$ .

In 2016, Cabrera, Sadarangani, and Samet [26] investigated a class of nonlocal fractional boundary value problems

$$\begin{cases} D_{c+}^{\beta} v(t) + p(t)v(t) = 0, & c < t < d, \quad 2 < \beta \leq 3, \\ v(c) = v'(c) = 0, \quad v'(d) = \alpha v(\xi), \end{cases} \quad (10)$$

where  $D_{c+}^{\beta}$  denotes the standard Riemann–Liouville fractional derivative of order  $\beta$ ,  $c < \xi < d$ ,  $0 \leq \alpha(\xi - c)^{\beta-1} < (\beta - 1)(d - c)^{\beta-2}$  and obtained the following Hartman–Wintner-type inequalities:

$$\int_c^d (s - c)(d - s)^{\beta-2} |p(s)| ds \geq \left( 1 + \frac{\alpha(d - c)^{\beta-1}}{(\beta - 1)(d - c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}} \right)^{-1} \Gamma(\beta) \quad (11)$$

and Lyapunov-type inequality

$$\int_c^d |p(s)| ds \geq \frac{\Gamma(\beta)(\beta - 1)^{\beta-1}}{(d - c)^{\beta-1}(\beta - 2)^{\beta-2}} \left( 1 + \frac{\alpha(d - c)^{\beta-1}}{(\beta - 1)(d - c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}} \right)^{-1}. \quad (12)$$

We note that the two results (4) and (7) above required additional conditions (5) and (8), respectively. Motivated by the above works, we attempt to remove restriction (8) and develop a new method for proving Lyapunov-type inequality for Riemann–Liouville fractional boundary value problems (6) and (10). More precisely, we utilize the properties of Green's function to obtain some Lyapunov-type inequality for fractional boundary value problem (6) without restriction (8) and obtain a generalization of Lyapunov-type inequality for fractional boundary value problem (10). It should be noted that the Lyapunov-type inequality (2) for a fractional boundary value problem (1) can be obtained by a similar discussion used in this article. Hence, we provide another way to study the Lyapunov-type inequality for a variety of fractional boundary value problems.

## 2. Preliminaries

In this section, we introduce the basic results on fractional calculus theory.

**Definition 1** ([25]). The Riemann–Liouville fractional integral of order  $\beta > 0$  for integrable function  $g$  defined on the interval  $[c, d]$  is defined by

$$I_{c+}^{\beta} g(t) = \frac{1}{\Gamma(\beta)} \int_c^t (t - s)^{\beta-1} g(s) ds, \quad t \in [c, d],$$

where  $\Gamma(\beta)$  denotes the Gamma function.

**Definition 2** ([25]). The Riemann–Liouville fractional derivative of order  $\beta > 0$  for integrable function  $g$  defined on the interval  $[c, d]$  is given by

$$D_{c+}^{\beta} g(t) = \frac{1}{\Gamma(n - \beta)} \left( \frac{d}{dt} \right)^n \int_c^t \frac{g(s)}{(t - s)^{\beta-n+1}} ds, \quad t \in [c, d]$$

where  $n - 1 \leq \beta < n$ , provided that the right-hand side is pointwise defined on  $[c, d]$ .

**Lemma 1** ([25]). Assume that  $v \in C(c, d) \cap L(c, d)$  with Riemann–Liouville fractional derivative of order  $\beta > 0$ . Then,

$$I_{c+}^{\beta} D_{c+}^{\beta} v(t) = v(t) + c_1(t - c)^{\beta-1} + c_2(t - c)^{\beta-2} + \dots + c_N(t - c)^{\beta-N}$$

for some  $c_i \in \mathbb{R}$ ,  $i = 1, \dots, N$ ,  $N = [\beta]$ .

In the following, we need an integral representation of the solution to non-homogeneous fractional differential equations which appear in [26].

**Lemma 2** ([26]). Given  $\sigma \in C(c, d) \cap L(c, d)$  for some  $c < d$ ,  $\xi \in (c, d)$ ,  $2 < \beta \leq 3$ , and  $0 \leq \alpha(\xi - c)^{\beta-1} < (\beta - 1)(d - c)^{\beta-2}$ . Then,  $v \in C[c, d]$  is a solution to the problem, the unique solution of

$$\begin{cases} D_{c+}^{\beta} v(t) + \sigma(t) = 0, & c < t < d, \\ v(c) = v'(c) = 0, & v'(d) = \alpha v(\xi), \end{cases} \quad (13)$$

if and only if  $u$  satisfies the integral equation

$$v(t) = \int_c^d H(t, s) \sigma(s) ds,$$

where

$$H(t, s) = G(t, s) + \frac{\alpha(t - c)^{\beta-1}}{(\beta - 1)(d - c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}} G(\xi, s)$$

$$G(t, s) = \begin{cases} \frac{(t - c)^{\beta-1}(d - s)^{\beta-2} - (t - s)^{\beta-1}(d - c)^{\beta-2}}{\Gamma(\beta)(d - c)^{\beta-2}}, & c \leq s \leq t \leq d, \\ \frac{(t - c)^{\beta-1}(d - s)^{\beta-2}}{\Gamma(\beta)(d - c)^{\beta-2}}, & c \leq t \leq s \leq d. \end{cases} \quad (14)$$

**Lemma 3.** The function  $G(t, s)$  defined above satisfies the following conditions:

- (i)  $0 \leq G(t, s) \leq G(d, s) = \frac{(s - c)(d - s)^{\beta-2}}{\Gamma(\beta)}, c \leq t, s \leq d,$
- (ii)  $\max_{s \in [c, d]} G(d, s) = G\left(d, \frac{d + c(\beta - 2)}{\beta - 1}\right) = \frac{(d - c)^{\beta-1}(\beta - 2)^{\beta-2}}{\Gamma(\beta)(\beta - 1)^{\beta-1}},$
- (iii)  $G(t, s) \leq \frac{(t - c)^{\beta-1}(d - s)^{\beta-2}}{\Gamma(\beta)(d - c)^{\beta-2}}, c \leq t, s \leq d,$
- (iv)  $H(s) \leq G(d, s)$  for  $s \in [c, d]$ , where  $H(s) = G(s, s) = \frac{(s - c)^{\beta-1}(d - s)^{\beta-2}}{\Gamma(\beta)(d - c)^{\beta-2}},$
- (v)  $\max_{s \in [c, d]} H(s) = H\left(\frac{d(\beta - 1) + c(\beta - 2)}{2\beta - 3}\right) = \frac{(d - c)^{\beta-1}(\beta - 1)^{\beta-1}(\beta - 2)^{\beta-2}}{\Gamma(\beta)(2\beta - 3)^{2\beta-3}}.$

**Proof.** (i) and (ii) are taken from [26].

(iii) From the expression of function  $G$ , we obtain that, for  $c \leq t, s \leq d$ ,  $G(t, s) \leq \frac{(t - c)^{\beta-1}(d - s)^{\beta-2}}{\Gamma(\beta)(d - c)^{\beta-2}}.$

(iv) Note that  $H(s) = G(s, s)$ . Then, (iv) holds from part (i).

(v) It is sufficient to show that  $H'(s) = 0$  if  $s = \frac{d(\beta - 1) + c(\beta - 2)}{2\beta - 3}$ , and hence the proof is completed.  $\square$

### 3. Main Results

**Theorem 1.** Let  $c < \xi < d$ ,  $0 \leq \alpha(\xi - c)^{\beta-1} < (\beta - 1)(d - c)^{\beta-2}$ , and  $p \in C[c, d]$ . Suppose that problem (10) has a nontrivial continuous solution, then

$$\left(1 + \frac{\alpha(\xi - c)^{\beta-1}}{(\beta - 1)(d - c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}}\right)^{-1} \Gamma(\beta)(d - c)^{\beta-2} \leq \int_c^d (s - c)^{\beta-1}(d - s)^{\beta-2} |p(s)| ds. \quad (15)$$

**Proof.** Let  $X = C[c, d]$  be the Banach space equipped with norm  $\|v\| = \max_{t \in [c, d]} |v(t)|.$

From Lemma 2, it follows that the solutions to fractional boundary value problem coincide with the fixed points of operator  $S$ , where the linear operator  $S : X \rightarrow X$  is given by

$$(Sv)(t) = \int_c^d H(t,s)p(s)v(s)ds, \quad v \in X.$$

By Lemma 3 (iii), for  $u \in X$ , we have

$$\begin{aligned} & |(Sv)(t)| \\ & \leq \int_c^d H(t,s)|p(s)v(s)|ds \\ & \leq \int_c^d G(t,s)|p(s)v(s)|ds + \frac{\alpha(t-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \int_c^d G(\xi,s)|p(s)v(s)|ds \\ & \leq (t-c)^{\beta-1} \int_c^d \frac{(d-s)^{\beta-2}|p(s)|}{\Gamma(\beta)(d-c)^{\beta-2}}ds \cdot \|v\| \\ & \quad + \frac{\alpha(t-c)^{\beta-1}(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \int_c^d \frac{(d-s)^{\beta-2}|p(s)|}{\Gamma(\beta)(d-c)^{\beta-2}}ds \cdot \|v\| \\ & = (t-c)^{\beta-1} \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d \frac{(d-s)^{\beta-2}|p(s)|}{\Gamma(\beta)(d-c)^{\beta-2}}ds \cdot \|v\|, \quad t \in [c,d]. \end{aligned}$$

The above inequality yields that the linear operator  $S$  maps all elements in  $X$  into the following vector subspace:

$$X_1 = \left\{ v \in X : \frac{|v(t)|}{(t-c)^{\beta-1}} \text{ is bounded for } t \in [c,d] \right\}.$$

Obviously,  $X_1$  is a subspace of  $X$ , and  $X_1$  is an Banach space with the norm

$$\|v\|_1 = \sup_{t \in [c,d]} \frac{|v(t)|}{(t-c)^{\beta-1}}.$$

Thus, we suppose that  $v \in X_1$  is a nontrivial solution of problem (10). Using Lemma 2,  $u$  must satisfy

$$v(t) = \int_c^d H(t,s)p(s)v(s)ds, \quad t \in [c,d].$$

Making use of Lemma 3, for any  $t \in [c,d]$ , we have

$$\begin{aligned} & |v(t)| \leq (t-c)^{\beta-1} \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d \frac{(d-s)^{\beta-2}}{\Gamma(\beta)(d-c)^{\beta-2}}|p(s)| \cdot |v(s)|ds \\ & \leq (t-c)^{\beta-1} \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d \frac{(d-s)^{\beta-2}}{\Gamma(\beta)(d-c)^{\beta-2}}|p(s)| \cdot \|v\|_1(s-c)^{\beta-1}ds \\ & = (t-c)^{\beta-1} \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d H(s)|p(s)|ds \cdot \|v\|_1 \\ & = (t-c)^{\beta-1} \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d \frac{(s-c)^{\beta-1}(d-s)^{\beta-2}}{\Gamma(\beta)(d-c)^{\beta-2}}|p(s)|ds \cdot \|v\|_1, \quad t \in [c,d]. \end{aligned}$$

Taking into account the definition of  $\|\cdot\|_1$ , we conclude that

$$\|v\|_1 \leq \left( 1 + \frac{\alpha(\xi-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi-c)^{\beta-1}} \right) \int_c^d \frac{(s-c)^{\beta-1}(d-s)^{\beta-2}}{\Gamma(\beta)(d-c)^{\beta-2}}|p(s)|ds \cdot \|v\|_1,$$

or, equivalently,

$$\left(1 + \frac{\alpha(\xi - c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}}\right)^{-1} \Gamma(\beta)(d-c)^{\beta-2} \leq \int_c^d (s-c)^{\beta-1}(d-s)^{\beta-2}|p(s)|ds.$$

This finishes the proof.  $\square$

From the above argument, we know that the foremost Banach space used in Theorem 1 is  $X_1$ , not  $X$ . If we consider fractional boundary value problem (10) in  $X$  by the use of the property of function  $G(t, s)$  (Lemma 3i), the following result holds as stated in [26].

**Theorem 2.** *Under the assumptions of Theorem 1, we have*

$$\int_c^d (s-c)(d-s)^{\beta-2}|p(s)|ds \geq \left(1 + \frac{\alpha(d-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}}\right)^{-1} \Gamma(\beta). \quad (16)$$

In the following, we will show that inequality (15) improves inequality (16). With regard to this, we need to show that

$$(d-c)^{\beta-2} \int_c^d (s-c)(d-s)^{\beta-2}|p(s)|ds \geq \int_c^d (s-c)^{\beta-1}(d-s)^{\beta-2}|p(s)|ds. \quad (17)$$

To evaluate this relation, it is convenient to adopt variable substitution. Therefore, we set  $s = c + t(d-c)$ . Then, we have  $ds = (d-c)dt$ , and the substitution rule gives

$$(d-c)^{\beta-2} \int_c^d (s-c)(d-s)^{\beta-2}|p(s)|ds = (d-c)^{2\beta-2} \int_0^1 t(1-t)^{\beta-2}|p(c+t(d-c))|dt,$$

and

$$\int_c^d (s-c)^{\beta-1}(d-s)^{\beta-2}|p(s)|ds = (d-c)^{2\beta-2} \int_0^1 t^{\beta-1}(1-t)^{\beta-2}|p(c+t(d-c))|dt.$$

Thus, (17) holds, following from the fact that  $\beta \in (2, 3]$  and  $t \in [0, 1]$ .

With the help of Lemma 3, we can obtain the following two Lyapunov-type inequalities. The first inequality follows from Theorem 1, while the second one follows from Theorem 2.

**Corollary 1.** *Under assumptions of Theorem 1, we have*

$$\left(1 + \frac{\alpha(\xi - c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}}\right)^{-1} \frac{\Gamma(\beta)(2\beta-3)^{2\beta-3}}{(d-c)^{\beta-1}(\beta-1)^{\beta-1}(\beta-2)^{\beta-2}} \leq \int_c^d |p(s)|ds \quad (18)$$

and

$$\left(1 + \frac{\alpha(d-c)^{\beta-1}}{(\beta-1)(d-c)^{\beta-2} - \alpha(\xi - c)^{\beta-1}}\right)^{-1} \frac{\Gamma(\beta)(\beta-1)^{\beta-1}}{(d-c)^{\beta-1}(\beta-2)^{\beta-2}} \leq \int_c^d |p(s)|ds. \quad (19)$$

In particular, if  $\alpha = 0$ , problem (10) is reduced to problem (6), inequalities (15) and (16) become

$$\Gamma(\beta)(d-c)^{\beta-2} \leq \int_c^d (s-c)^{\beta-1}(d-s)^{\beta-2}|p(s)|ds \quad (20)$$

and

$$\Gamma(\beta) \leq \int_c^d (s-c)(d-s)^{\beta-2}|p(s)|ds, \quad (21)$$

and Lyapunov-type inequalities (18) and (19) become

$$\frac{\Gamma(\beta)(2\beta-3)^{2\beta-3}}{(d-c)^{\beta-1}(\beta-1)^{\beta-1}(\beta-2)^{\beta-2}} \leq \int_c^d |p(s)|ds \quad (22)$$

and

$$\frac{\Gamma(\beta)(\beta-1)^{\beta-1}}{(d-c)^{\beta-1}(\beta-2)^{\beta-2}} \leq \int_c^d |p(s)|ds. \quad (23)$$

**Remark 1.** (17) implies that inequality (20) improves inequality (21), and Lemma 3 implies that inequality (22) improves inequality (23).

Next, we will compare these inequalities with the existed results. When  $\beta = 3$ , inequality (23) coincides with inequality (9) obtained by [9], and inequality (22) becomes

$$\int_c^d |p(s)|ds \geq \frac{27}{2(d-c)^2}$$

which is clearly better than (9). As for  $\beta \in (2, 3)$ , from the above proof of Theorem 1, if we assume that the nontrivial solution of (6) does not change signs on  $[c, d]$ , we obtain the following inequality:

$$\Gamma(\beta)(d-c)^{\beta-2} < \int_c^d (s-c)^{\beta-1}(d-s)^{\beta-2} p_+(s)ds$$

and Lyapunov-type inequality:

$$\frac{\Gamma(\beta)(2\beta-3)^{2\beta-3}}{(d-c)^{\beta-1}(\beta-1)^{\beta-1}(\beta-2)^{\beta-2}} < \int_c^d p_+(s)ds. \quad (24)$$

Finally, we show that inequality (24) is the improvement of inequality (7). In fact, we only need to prove the following inequality:

$$(2\beta-3)^{2\beta-3} > (\beta-1)^{2\beta-2}, \quad \beta \in (2, 3]. \quad (25)$$

Consider the function  $\varphi$  defined on  $[2, 3]$  by

$$\varphi(x) = (2x-3)\ln(2x-3) - (2x-2)\ln(x-1), \quad x \in [2, 3].$$

We see that  $\varphi$  is continuous on  $[2, 3]$  and differentiable on  $(2, 3)$ , and  $\varphi(2) = 0$ . Differentiating  $\varphi$  on  $(2, 3)$ , we infer that

$$\varphi'(x) = 2\ln(2x-3) - 2\ln(x-1) = 2\ln \frac{2x-3}{x-1} > 2\ln 1 = 0, \quad x \in (2, 3).$$

This implies that  $\varphi$  is increasing on  $[2, 3]$ . Thus, for  $x \in (2, 3]$ ,  $\varphi(x) > \varphi(2) = 0$  and (25) holds.

#### 4. Conclusions

In this paper, we study some Lyapunov-type inequalities for fractional differential equations. We first transform the fractional differential equations (10) into an equivalent operator equation, and then construct the properties of Green's function corresponding to appropriate boundary conditions. Finally, some Lyapunov-type inequalities for (10) and (6) are obtained. It should be noted that the proof of Lyapunov-type inequalities does not depend on the maximum of Green's function of fractional differential equations (10).

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