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Decision Making for Project Appraisal in Uncertain Environments: A Fuzzy-Possibilistic Approach of the Expanded NPV Method

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Abstract: The major drawback of the classic approaches for project appraisal is the lack of the possibility to handle change requests during the project's life cycle. This fact incorporates the concept of uncertainty in the estimation of this investment's worth. To resolve this issue, the authors use fuzzy numbers, possibilistic moments of fuzzy numbers and the hybrid (fuzzy statistic) fuzzy estimators' method in order to introduce a fuzzy possibilistic version of the expanded net present value method (FPeNPV). This approach consists of two factors: the fuzzy possibilistic NPV and the fuzzy option premium. For the estimation of the fuzzy NPV, some basic assumptions are taken into consideration: (1) the opportunity cost of capital, used as the present value interest factor calculated through the weighted average cost of capital (WACC), (2) the equity cost, determined through the possibilistic set-up of the capital asset pricing model CAPM, and (3) the inflation factor, also included in the estimation of the NPV. The fuzzy estimators' method is used for the computation of the fuzzy option premium. An algorithm of nine major steps leads to the computation of the FPeNPV. This gives the administration the opportunity to adapt to potential changes in the company's internal and external environments. In this way, the symmetry between the planning and execution phase of a project can be reinstated. The results validate the statement that fuzzy and intelligent methods remain valuable tools to express uncertainty in various scientific areas. Finally, an illustrative example aims at a thorough comprehension of this new approach of the expanded NPV method.

Keywords: project appraisal; expanded NPV method; fuzzy sets; possibilistic moments; fuzzy volatility; fuzzy binomial model

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Environments: A Fuzzy-Possibilistic Approach of the Expanded NPV

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1. Introduction

The main concern for decision-making in project appraisal is the comparison between the future income and the initial cost. The net present value method approximates this challenge by forming a project portfolio that offers the maximum possible net present value (NPV) [1]. Regarding project appraisal, the NPV and discounted cash flows (DCF) methods do not provide project managers with the possibility of decisions made during the project's life. The asymmetry (caused in uncertain environments) between the planning and executing phase in the project's life cycle can be captured by incorporating the expanded NPV (eNPV) method. The expanded NPV employs the option price for more accurate evaluation of real investments (projects).

Stochastic calculus models randomness, and the fuzzy sets theory models uncertainty. Project appraisal includes both stochastic and fuzzy parameters. A future cash flow depends on determinants that can be handled either as a fuzzy or random factor. This statement constitutes the main motivation for the inclusion of fuzzy modeling in project appraisal. The present research effort tackles the vague issue of project appraisal by employing the fuzzy version of the NPV method. Fuzzy sets constitute a reliable mathematical tool to model uncertainty and vagueness in fields such as engineering and finance [2].

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Fuzzy and intelligent methods present great implementation in investment analysis and decision-making [3–13]. Possibility theory is also a popular mathematical tool for modeling uncertainty in these scientific areas [14,15].

Carlsson and Fuller [16] used fuzzy trapezoidal numbers to represent both the cash inflows and outflows. In [17,18], Muzzioli and Reynaerts used fuzzy sets and possibility theory to model the stock volatility of an option in a binomial model. Kahraman and Kaya [19] introduced fuzzy parameters into stochastic investment analysis and applied probability theory for fuzzy concepts. Collan et al. [20] used fuzzy numbers and possibility theory to price real options and handle existing uncertainty. Tsao [21] derived the NPV in an uncertain environment through pragmatic algorithms.

Chrysafis and Papadopoulos [22] made a research effort to improve the fuzzy version of the program evaluation and review technique PERT and avoid the backward recursion problem. The tools for this operation were the possibilistic moments of fuzzy numbers. Chatterjee et al. [23] presented an alternative approach for the analytical network process (ANP) methodology concerning D numbers in order to manage three types of information: complete, uncertain and incomplete. Dahooie et al. [24] used fuzziness to reinstate symmetry between sustainable development, environmental impact and human well-being. Gebrehiwet et al. [25] assessed the lag times' risks in different life cycles of construction projects using the technique for order of preference by similarity to ideal solution TOPSIS and fuzzy comprehensive evaluation. Lesniak et al. [26] used the fuzzy analytic hierarchy process (FAHP) to increase the validity of contractors' bidding decisions. Rola and Kuchta [27] found a method to estimate the product backlog items of scrum-based information technology IT projects by using fuzzy numbers and strict rules that took into account the human factor. Bolos et al. [28] formed a fuzzy tool for management decisions concerning capital budgeting. Chrysafis et al. [29] used fuzzy tools and techniques for the selection of fuel suppliers and optimal bunkering contracts. Abidin et al. [30] proposed an alternative fuzzy clustering method via fuzzy inference systems for the assessment of stock performance for different kinds of investors. Plebankiewicz and Wieczorek [31] used Mamadani's fuzzy inference model to create a cost overrun risk prediction model. Lyczkowska-Hanćkowiak [32] used fuzzy numbers to compute major investment value determinants.

In this paper, the authors propose an alternative methodology for real investment appraisal (i.e., project valuation), taking into consideration multiple factors and increasing flexibility for new decisions in the execution phase of the project. The traditional NPV criterion is modified so that it can handle the uncertain and vague environment existing in investments analysis. In this approach, the discount rate is modeled via the possibilistic set-up of CAPM—resulting from fuzzy data—taking advantage of this method's merits. Then, based on Zadeh's [33,34] extension principle, the NPV equation takes a fuzzy form. Inflation impact is also taken into account. The NPV formula is appropriately modified so that consistency is ensured. The inflation impact is calculated for both the cash flows and for the interest rates, both in real and nominal values. The results of these modifications are different formulas for converting fuzzy nominals to fuzzy real values and vice versa. Finally, the derivation of the call option price (option to expand) leads to the fuzzy possibilistic expanded net present value (FPeNPV) calculation.

2. Methods

2.1. Fuzzy Sets Principles

Klir and Yuan [35] presented the following definitions and propositions, which are necessary for this study:

Definition 1. Let X be the universal set. If A is a function from X into the interval [0,1], then A is called a fuzzy set or a fuzzy subset of X. A(x) is interpreted as the membership degree of x in the fuzzy set A.

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Definition 2. If A is a fuzzy set, by a - cuts, $a \in [0,1]$, we mean the sets $A[a] = \{x \in X : A(x) \ge a\}$. It is known that the a - cuts determine the fuzzy set A. The a - cuts of A are closed intervals. If A[a] = B[a], $\forall a \in [0,1]$ for arbitrary fuzzy sets A and B, then it holds that A = B.

Definition 3. We say that A is a fuzzy number if the following conditions hold:

- 1. The fuzzy set A is normalized;
- 2. The fuzzy set A is convex;
- 3. The fuzzy set A is upper semi-continuous;
- 4. The support of A is compact;
- 5. The fuzzy set A is normalized if there exists $x \in X$, such that A(x) = 1;
- 6. The fuzzy set A is a convex fuzzy set if $\forall t \in [0,1]$ and $x_1, x_2 \in X$, $A(tx_1 + (1-t)x_2) \ge \min\{A(x_1), A(x_2)\}$;
- 7. The s u p p $(A) = \bigcup_{a \in (0,1]} A[a] = \{ x : A(x) > 0 \}.$

Definition 4. The membership function of a fuzzy number A can be expressed as

$$A(x) = \begin{cases} 0 & for \ x \le a_1 \\ A_L(x) & for \ a_1 < x < a_2 \\ A_R(x) & for \ a_2 \le x < a_3 \\ 0 & for \ x \ge a_3 \end{cases}$$

where $A_L: [a_1, a_2] \to [0, 1]$ and $A_R: [a_3, a_4] \to [0, 1]$ are the left and right membership functions of the fuzzy number A. Zadeh [33,34], through the extension principle, extended the classical operations between real numbers to their fuzzy counterparts.

Definition 5. In the extension principle, any given function $f: X \to Y$ induces two functions, $f: F(X) \to F(Y)$ and $f^{-1}: F(X) \to F(Y)$, which are defined by $[f(A)](y): \sup_{x/y=f(x)} A(x)$ for all $A \in F(Y)$ and $[f^{-1}(B)](x): B(f(x))$ for all $B \in F(Y)$.

Based on the extension principle, any crisp function can be fuzzified (Filev and Yager 1997).

2.2. Fuzzy Arithmetic

Klir and Yuan [35] gave the fundamentals of fuzzy arithmetic, which is based on two properties of fuzzy numbers:

Property 1. Each fuzzy set, and thus also each fuzzy number, can fully and uniquely be represented by its a-cuts.

Property 2. The a-cuts of each fuzzy number are closed intervals of real numbers for all $a \in (0,1]$.

These properties enable us to define arithmetic operations on fuzzy numbers, in terms of arithmetic operations, on their a-cuts. The latter operations are a subject of interval analysis. Let \otimes denote any of the four arithmetic operations on closed intervals: addition+, subtraction-, multiplication, and division/. In this case, it can be said that the following is a general property of all arithmetic operations on closed intervals:

$$[a,b] \otimes [d,e] = [f \otimes g | a \le f \le b, d \le g \le e]$$

The exception to this is that [a,b]/[d,e] is not defined when $0 \in [d,e]$. That is, the result of an arithmetic operation on closed intervals is again a closed interval. The four arithmetic operations on closed intervals are defined as follows:

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Note that a real number r may also be regarded as a special (degenerate) interval [r, r]. When one of the above intervals is degenerate, we obtain special operations. When both of them are degenerate, we obtain the standard arithmetic of real numbers.

2.3. Non-Asymptotic Fuzzy Estimators

Sfiris and Papadopoulos [36] proved the next two propositions:

Proposition 1. Let X_1, X_2, \ldots, X_n be a random sample, and let x_1, x_2, \ldots, x_n be the sample values assumed by the sample. Additionally, let us say that $\gamma \in (0.1)$ if the size is small. In that case, then

$$\sigma_{\gamma}^{2} = \begin{cases} \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma} F\left(\frac{(n-1)s^{2}}{x}\right), \frac{(n-1)s^{2}}{F^{-1}(\frac{2-\gamma}{2})} \leq x \leq \frac{(n-1)s^{2}}{M} \\ \frac{2}{1-\gamma} F\left(\frac{(n-1)s^{2}}{x}\right) - \frac{\gamma}{1-\gamma}, \frac{(n-1)s^{2}}{M} \leq x \leq \frac{(n-1)s^{2}}{F^{-1}(\frac{\gamma}{2})} \\ 0, \text{ otherwise} \end{cases}$$

This is the membership function of a fuzzy number, the support of which is exactly the $(1-\gamma)100\%$ confidence interval for σ^2 , and the a-cuts of this fuzzy number are the closed intervals:

$$\sigma_{\gamma}^{2} = V_{\gamma o}[a] = \left[V_{o}^{l}(a), V_{o}^{r}(a)\right] = \left[\frac{(n-1)s^{2}(\gamma)}{\chi_{n-1;h(a)}^{2}}, \frac{(n-1)s^{2}(\gamma)}{\chi_{n-1;1-h(a)}^{2}}\right], \forall a \in (0,1]$$

where x_{n-1}^2 is the value of the chi-squared distribution with k=n-1 degrees of freedom, $x_{n-1,h(a)}^2=$ $F^{-1}(1-h(a))$, $h(a)=(1/2-\gamma/2)\alpha+\gamma/2$ and F denotes the cumulative distribution function of the chi-squared distribution.

Proposition 2. Let X_1, X_2, \ldots, X_n be a random sample, and let x_1, x_2, \ldots, x_n be sample values assumed by the sample. Additionally, let us say that $\gamma \in (0.1)$ if the size is large. In that case, then

$$\sigma_{\gamma}^{2} = \begin{cases} \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma} \Phi\Big(\sqrt{(n-1)/2}(s^{2}/x - 1)\Big), \frac{s^{2}}{1+\Phi^{-1}(1-\frac{\gamma}{2})\sqrt{2/(n-1)}} \leq x \leq s^{2} \\ \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma} \Phi\Big(\sqrt{(n-1)/2}(-s^{2}/x + 1)\Big), s^{2} \leq x \leq \frac{s^{2}}{1-\Phi^{-1}(1-\frac{\gamma}{2})\sqrt{2/(n-1)}} \\ 0, \text{ otherwise} \end{cases}$$

This is the membership function of a fuzzy number, the support of which is exactly the

$$(1-\gamma)100\% \ confidence \ interval \ for \ \sigma^2, \ and \ the \ a-cuts \ of \ this \ fuzzy \ number \ are \ the \ closed \ intervals:$$

$$\sigma_{\gamma}^2[\alpha] = V_{\gamma}[a] = \left[V_{\gamma c}^l(a), V_{\gamma}^r(a)\right] = \left[\frac{s^2(\gamma)}{1+z_{h(a)}\sqrt{2/(n-1)}}, \frac{s^2(\gamma)}{1-z_{h(a)}\sqrt{2/(n-1)}}\right], \forall a \in (0,1]$$

where $Z_{h(a)}=\Phi^{-1}(1-h(a))$, $h(a)=(1/2-\gamma/2)\alpha+\gamma/2$ and Φ denotes the cumulative distribution function of the standard normal distribution.

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3. Assumprions for the Present Research Work

3.1. NPV Formulas

The net present value is the monetary project's profit or loss. It can be calculated through the discount of the cash inflows and outflows [1]:

$$NPV = \sum_{i=0}^{T} (CIF_t - TCOF_t)(1+r)^{-t}$$
 (1)

The difference of $CIF_t - COF_t$ is the net cash flow (NCF).

The cash flows are subdivided into the initial investment outlay, liquidation value (after the project's closing phase), cash inflows and outflows. Then, the above formula can be reformed as the following:

$$NPV = -I_0 + \sum_{t=0}^{T} ((p_t - cof_t)x_t - COF_t)(1+r)^{-t} + L_t(1+r)^{-t}$$
 (2)

To incorporate inflation in the NPV formulae, the following must happen:

- The real cash flows have to be converted to nominal cash flows (the use of a nominal discount rate is also necessary);
- The cash flows are estimated in real values (the use of a real discount rate is also necessary).

The relationships between the real and nominal values at time *t* are as follows:

Nominal Cash (t) =
$$(1 + Inflation Rate)^t \times Real Cash flows$$

Additionally,

Nominal Discount Rate =
$$(1 + Inflation Rate) \times (1 + Real Discount Rate) - 1$$
 (3)

Next, the basic assumptions for the conduct of this research work are presented:

- The time value of money is represented by the opportunity cost of capital, calculated through the weighted average cost of capital (WACC);
- The equity cost is determined through the possibilistic set-up of CAPM;
- The inflation factor is also included in the estimation of the NPV;
- The equation for the NPV is fuzzified by considering Zadeh's [33,34] extension principle;
- The value from the expansion of the project is calculated through the fuzzy binomial model.

The formula for the expanded NPV is the following [1]:

3.2. Possibilistic Discount Rate via Possibilistic CAPM

Chrysafis [37] derived a possibilistic CAPM beta value and a possibilistic value for the market premium. These values lead to a possibilistic set-up of CAPM and to the computation of the after tax weighted average cost of capital (WACC). The possibilistic CAPM beta is

$$beta_{poss} = \frac{\int_{0}^{1} a \left(\left(\frac{D_{t}^{r}(a) + P_{t}^{l}(a)}{P_{t-1}^{l}(a)} - 1 \right) - \left(\frac{D_{t}^{l}(a) + P_{t}^{l}(a)}{P_{t-1}^{r}(a)} - 1 \right) \right) \left(\left(\frac{d_{t}^{r}(a) + I_{t}^{r}(a)}{I_{t-1}^{l}(a)} - 1 \right) - \left(\frac{d_{t}^{l}(a) + I_{t}^{l}(a)}{I_{t-1}^{r}(a)} - 1 \right) \right) da}{\int_{0}^{1} a \left(\left(\frac{d_{t}^{r}(a) + I_{t}^{r}(a)}{I_{t-1}^{l}(a)} - 1 \right) - \left(\frac{d_{t}^{l}(a) + I_{t}^{l}(a)}{I_{t-1}^{r}(a)} - 1 \right) \right)^{2} da}$$

$$(5)$$

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The possibilistic value for the market risk premium is

$$R_M = \int_0^1 a((\frac{D_t^r(a) + I_t^r(a)}{I_{t-1}^l(a)} - 1) + (\frac{d_t^l(a) + I_t^l(a)}{I_{t-1}^r(a)} - 1))da - R_f$$
 (6)

The possibilistic value for the equity cost rposs via the possibilistic set-up of CAPM is

$$\begin{split} r_{E} &= \\ R_{f} \\ &+ \frac{\int_{0}^{1} a \; ((\frac{D_{t}^{r}(a) + P_{t}^{t}(a)}{P_{t-1}^{l}(a)} - 1) - (\frac{D_{t}^{l}(a) + P_{t}^{l}(a)}{P_{t-1}^{r}(a)} - 1)) \; ((\frac{d_{t}^{r}(a) + l_{t}^{r}(a)}{l_{t-1}^{l}(a)} - 1) - (\frac{d_{t}^{l}(a) + l_{t}^{l}(a)}{I_{t-1}^{r}(a)} - 1)) da}{\int_{0}^{1} a \; ((\frac{d_{t}^{r}(a) + l_{t}^{r}(a)}{I_{t-1}^{l}(a)} - 1) - (\frac{d_{t}^{l}(a) + l_{t}^{l}(a)}{I_{t-1}^{r}(a)} - 1)) da} \\ &\times \; \left(\int_{0}^{1} a \; ((\frac{D_{t}^{r}(a) + l_{t}^{r}(a)}{I_{t-1}^{l}(a)} - 1) + (\frac{d_{t}^{l}(a) + l_{t}^{l}(a)}{I_{t-1}^{r}(a)} - 1)) da - R_{f} \right) \end{split}$$

The appropriate cost of capital to be applied in the project results from after the tax weighted average cost of capital (WACC) is

$$WACC = rD (1 - TC) DA/V + rE EA/V = rposs$$
 (8)

where rD is the expected rate on return on debt and rE is the expected rate of return on equity; in other words, these are the debt cost and the equity cost. The weights DA/V and EA/V are the fractions of debt and equity, based on market values. V, the total market value of the firm, is the sum of DA and EA. TC is the corporate tax rate. r_D is considered to be the gross redemption yield (GRY%) [37]. See Abbreviations for the basic notation.

4. Fuzzy Possibilistic Net Present Value

Taking into account the equations for the NPV (Equations (1) and (2)), the adjustment to inflation and the assumptions presented above, the definitions for certain approaches of the FPNPV are derived:

Definition 6. A project's fuzzy possibilistic NPV, with the fuzzy numbers for CIF_t and COF_t and the possibilistic rate resulting from fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_0 + \sum_{i=1}^{T} \left[(CIF_t^I(a) - TCOF_t^r(a))(1 + r_{poss})^{-t}, (CIF_t^r(a) - TCOF_t^I(a))(1 + r_{poss})^{-t} \right]$$
 (9)

This is eexpressed in a-cuts form. See Appendix A for the operations.

Definition 7. The conversion of a project's fuzzy possibilistic NPV from real to nominal values, with the fuzzy numbers for CIF_t , COF_t and f_t and the possibilistic rate resulting from the fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_{0}$$

$$+\sum_{i=1}^{T} \left(\frac{\left(\left[CIF_{t}^{l}(a) - TCOF_{t}^{r}(a), CIF_{t}^{l}(a) - TCOF_{t}^{l}(a) \right] \right)}{\left[1 + f_{t}^{l}(a), 1 + f_{t}^{r}(a) \right]^{t} \left[1 + \frac{r_{poss} - f_{t}^{r}(a)}{1 + f_{t}^{l}(a)}, 1 + \frac{r_{poss} - f_{t}^{l}(a)}{1 + f_{t}^{l}(a)} \right]^{-t}}{\left[1 + \frac{r_{poss} - f_{t}^{l}(a)}{1 + f_{t}^{l}(a)}, 1 + \frac{r_{poss} - f_{t}^{l}(a)}{1 + f_{t}^{l}(a)} \right]^{-t}} \right)$$

$$(10)$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $CIF_t[a] - TCOF_t[a]$ (expressed in a-cuts form). See Appendix A for the operations.

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Definition 8. The conversion of a project's fuzzy possibilistic NPV from nominal to real values, with the fuzzy numbers for CIF_t , COF_t and f_t and the possibilistic rate resulting from the fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_{0}$$

$$+ \sum_{i=1}^{T} \left(\frac{[CIF_{t}^{l}(a) - TCOF_{t}^{r}(a), CIF_{t}^{l}(a) - TCOF_{t}^{l}(a)]}{[(1 + f_{t}^{l}(a))^{t}, (1 + f_{t}^{r}(a))^{t}]} \right) (11)$$

$$[(1 + r_{poss} + f_{t}^{r}(a) + r_{poss}f_{t}^{r}(a))^{-t}, (1 + r_{poss} + f_{t}^{l}(a) + r_{poss}f_{t}^{l}(a))^{-t}]$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $CIF[a] - TCOF_t[a]$ (expressed in a-cuts form). See Appendix A for the operations.

Definition 9. A project's fuzzy possibilistic NPV, with the fuzzy numbers for p_t , cof_t , x_t , COF_t , f_t and L and the possibilistic rate resulting from the fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_0$$

$$+\sum_{i=1}^{T} \left(\begin{array}{c} ([(p_t^l(a) - cof_t^r(a))x_t^l(a) - COF_t^r(a), (p_t^r(a) - cof_t^l(a))x_t^r(a) - COF_t^l(a)]) \\ (1 + r_{poss})^{-t} + [(1 + r_{poss})^{-T}L_t^l(a), (1 + r_{poss})^{-T}L_t^r(a)] \end{array} \right)$$

$$(12)$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the calculation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$ (expressed in a-cuts form). See Appendix A for the operations.

Definition 10. The conversion of a project's fuzzy possibilistic NPV from real to nominal values, with the fuzzy numbers for p_t , cof_t , x_t , COF_t , f_t and L and the possibilistic rate resulting from the fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_0$$

$$+\sum_{i=1}^{T} \begin{pmatrix} \left[(p_t^l(a) - cof_t^r(a)) x_t^l(a) - COF_t^r(a), (p_t^r(a) - cof_t^l(a)) x_t^r(a) - COF_t^l(a) \right] \\ +\sum_{i=1}^{T} \begin{pmatrix} \left[(p_t^l(a) - cof_t^r(a)) x_t^l(a) - COF_t^l(a) - COF_t^l(a) \right] \\ \left[(1 + f_t^l(a), 1 + f_t^r(a)) \right]^t \left[(1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a) \right]^{-t} \\ + \left[\left[(L_t^l(a), L_t^r(a)) \right] \left[(1 + r_{poss} + f_t^l(a) + r_{poss} f_t^r(a) + r_{poss} f_t^r(a) \right]^{-T} \end{pmatrix}$$

$$(13)$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$ (expressed in a-cuts form). See Appendix A for the operations.

Definition 11. The conversion of a project's fuzzy possibilistic NPV from nominal to real values, with fuzzy numbers for p_t , cof_t , x_t , COF_t , f_t and L and the possibilistic rate resulting from the fuzzy data for the discount rate r_{poss} , is defined as

$$FPNPV = -I_{c}$$

$$+\sum_{i=1}^{T} \left([(p_{t}^{l}(a) - cof_{t}^{r}(a))x_{t}^{l}(a) - COF_{t}^{r}(a), (p_{t}^{r}(a) - cof_{t}^{l}(a))x_{t}^{r}(a) - COF_{t}^{l}(a)] + \sum_{i=1}^{T} \left([1 + f_{t}^{l}(a), 1 + f_{t}^{r}(a)]^{t} [1 + \frac{r_{poss} - f_{t}^{r}(a)}{1 + f_{t}^{r}(a)}, 1 + \frac{r_{poss} - f_{t}^{l}(a)}{1 + f_{t}^{l}(a)}]^{-t} + [L_{t}^{l}(a), L_{t}^{r}(a)][1 + f_{t}^{l}(a), 1 + f_{t}^{r}(a)][1 + \frac{r_{poss} - f_{t}^{r}(a)}{1 + f_{t}^{r}(a)}, 1 + \frac{r_{poss} - f_{t}^{l}(a)}{1 + f_{t}^{l}(a)}]^{-T} \right)$$

$$(14)$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$ (expressed in a-cuts form). See Appendix A for the operations.

5. Fuzzy Binomial Model

5.1. The Classic Binomial Model

The methodology for the binomial model was presented by Cox et al. [38]. According to this model, the investment duration is divided into n discrete time intervals. The stock

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price may rise by a multiple of e_u (greater than 1) with a probability p or fall by a multiple of e_d (less than 1) with a probability q = 1 - p.

Next, the equations for the call and option price are presented with the assumption that there are no dividends paid during the investment's life [39]. The call option price can be expressed as

$$C_t = e^{-rf(t)} \sum_{j=0}^n {}^{n}C_j p^j (1-p)^{n-j} \max \left\{ 0, S_t e^{ju + (n-j)d} - X \right\}$$
 (15)

The put option price can be expressed as

$$P_{t} = e^{-rf(t)} \sum_{j=0}^{n} {^{n}C_{j}p^{j}(1-p)^{n-j}} \max \left\{ 0, -S_{t}e^{ju+(n-j)d} + X \right\}$$
 (16)

where C_t is call option price at time t, P_t is the put option price at time t, j is the number of upward jumps, k - j is the number of downward jumps, S_t is the cash flow at time t and X is the strike price.

5.2. Fuzzy Up and Down Probabilities

Then, the up and down probabilities are derived. The author applies the algorithm of Muzzioli and Reynaerts [17] for solving fuzzy systems to the financial application of the binomial model by using the continuously compounding discount factor $e^{rf(t)}$ where $f:(0,+\infty)\to(0,+\infty)$:

$$\left. \begin{array}{l} p_u + p_d = 1 \\ up_u + dp_d = e^{rf(t)} \end{array} \right\}$$

To find the solutions of the above system, the following linear programming problem has to be solved:

$$\max_{u,d}(resp.\min_{u,d}) \frac{e^{rf(t)} - d}{u - d} \text{ and } \max_{u,d}(resp.\min_{u,d}) \frac{u - e^{rf(t)}}{u - d}$$

where

$$e^{rf(t)} \le u^l(a)d^r \le e^{rf(t)}e^{rf(t)} \ge 10 < d \le 1u \ge 1$$

The first step is the derivation of the following partial derivatives:

1.
$$\frac{\partial p_{u}}{\partial u} = \frac{d - e^{rf(t)}}{(u - d)^{2}} < 0$$
2. $\frac{\partial p_{u}}{\partial d} = \frac{e^{rf(t)} - u}{(u - d)^{2}} < 0$
3. $\frac{\partial p_{d}}{\partial u} = \frac{e^{rf(t)} - d}{(u - d)^{2}} > 0$
4. $\frac{\partial p_{d}}{\partial d} = \frac{u - e^{rf(t)}}{(u - d)^{2}} > 0$

Thus, the maximum of P_u is obtained for $u_{\text{max}} = u^l(\alpha)$ (resp. $d_{\text{max}} = d^l(\alpha)$) and the minimum for $u_{\text{min}} = u^r(\alpha)$ (resp. $d_{\text{max}} = d^r(\alpha)$).

Additionally, the maximum of P_d is obtained for $u_{\max} = u^r(\alpha)$ (resp. $d_{\max} = d^r(\alpha)$) and the minimum for $u_{\min} = u^l(\alpha)$ (resp. $d_{\max} = d^l(\alpha)$).

As such, the solution of the system is

$$\left(\left\lceil \frac{e^{rf(t)} - d^r(a)}{u^r(a) - d^r(a)}, \frac{e^{rf(t)} - d^l(a)}{u^l(a) - d^l(a)} \right\rceil, \left\lceil \frac{u^l(a) - e^{rf(t)}}{u^l(a) - d^l(a)}, \frac{u^r(a) - e^{rf(t)}}{u^r(a) - d^r(a)} \right\rceil \right)$$

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Equivalently, it could also be

$$p_{u} = \left[\frac{e^{rf(t)} - d^{r}(a)}{u^{r}(a) - d^{r}(a)}, \frac{e^{rf(t)} - d^{l}(a)}{u^{l}(a) - d^{l}(a)} \right]$$

For p_d , the solution is

$$p_d = \left[\frac{u^l(a) - e^{rf(t)}}{u^l(a) - d^l(a)}, \frac{u^r(a) - e^{rf(t)}}{u^r(a) - d^r(a)} \right]$$

For the estimation of the up and down jump factors, the original methodology gives the following formulas:

 $u = e^{\sigma \sqrt{f(t)}}$ and $d = e^{-\sigma \sqrt{f(t)}}$

Under the extension principle, these factors can be fuzzified as the following:

$$u[a] = [u^{l}(a), u^{r}(a)] = [e^{\sigma_{YZ}^{l}(a)} \sqrt{f(t)}, e^{\sigma_{YZ}^{r}(a)} \sqrt{f(t)}]$$
(17)

This could also be expressed as

$$u^{l}(a) = e^{\sigma_{YZ}^{l}(a)} \sqrt{f(t)} \text{ and } u^{r}(a) = e^{\sigma_{YZ}^{r}(a)} \sqrt{f(t)}$$

$$d[a] = [d^{l}(a), d^{r}(a)] \stackrel{relation \ x}{=} [e^{-\sigma_{YZ}^{l}(a)} \sqrt{f(t)}, e^{-\sigma_{YZ}^{r}(a)} \sqrt{f(t)}]$$

$$(18)$$

Additionally, it can be said that

$$d^{l}(a) = e^{-\sigma_{YZ}^{l}(a)} \sqrt{f(t)}$$
 and $d^{r}(a) = e^{-\sigma_{YZ}^{r}(a)} \sqrt{f(t)}$

where $\sigma_{\Upsilon Z_{i}}$ is the fuzzy estimator for the standard deviation of the asset.

5.3. Fuzzy Volatility

Yang and Zhang [40] introduced the following estimator for the volatility of the company's stock:

$$Var = V_0 + kV_{CP} + (1 - k)V_{RS} (19)$$

where

$$V_{o} = \frac{1}{n-1} \sum_{i=1}^{n} (o_{i} - \overline{o})^{2}$$

$$V_{c} = \frac{1}{n-1} \sum_{i=1}^{n} (c_{i} - \overline{c})^{2}$$

$$V_{RS} = \frac{1}{n} \sum_{i=1}^{n} [u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i})]$$

where T is the time interval of each period, f is the fraction of the period (between [0,1]) when trading is closed, V is the unknown variance, which is the unknown volatility squared (σ^2) , CP_0 is the closing price of the previous period (at time 0), OP_1 is the opening price of the current period (at time f), HP_1 is the the current period's high during the trading interval (between [f, 1]), LP_1 is the current period's low during the trading interval (between [f, 1]), CP_1 is the closing price of the current period (at time 1), $v_0 = \ln OP_1 - \ln CP_0$, the normalized open, $v_0 = \ln HP_1 - \ln OP_1$, the normalized high, $v_0 = \ln LP_1 - \ln OP_1$, the normalized close.

In this research work, the authors employed the fuzzy estimator for V_0 and V_c and finally derived the fuzzy estimator for the Yang and Zhang variance.

Based on Proposition 2, the fuzzy estimators for the variance of each variable for a large sample are as follows.

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The support for the fuzzy membership function for the fuzzy variance of the random sample of O_1, O_2, \ldots , O_n is exactly the $(1-\gamma)100\%$ confidence interval for σ^2 , and the a-cuts of this fuzzy number are the closed intervals

$$V_o[a] = \left[V_o^I(a), V_o^r(a)\right] = \left[\frac{s^2(o)}{1 + z_{h(a)}\sqrt{2/(n-1)}}, \frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}}\right], \forall a \in [0, 1] \quad (20)$$

where $Z_{h(a)}$, $h_{(a)}$ and Φ have already been defined.

Similarly, the support for the fuzzy membership function for the fuzzy variance of the random sample of C_1, C_2, \ldots, C_n is exactly the $(1 - \gamma)100\%$ confidence interval for σ^2 , and the a-cuts of this fuzzy number are the closed intervals

$$V_c[a] = \left[V_c^l(a), V_c^r(a)\right] = \left[\frac{s^2(c)}{1 + z_{h(a)}\sqrt{2/(n-1)}}, \frac{s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}}\right], \forall a \in [0, 1] \quad (21)$$

where $Z_{h(a)}$, $h_{(a)}$ and Φ have already been defined.

Based on Proposition 1, the fuzzy estimators for the variance of each variable for a small sample are as follows.

The support for the fuzzy membership function for the fuzzy variance of the random sample of O_1, O_2, \ldots, O_n is exactly the $(1 - \gamma)100\%$ confidence interval for σ^2 , and the a-cuts of this fuzzy number are the closed intervals

$$V_o[a] = \left[V_o^l(a), V_o^r(a)\right] = \left[\frac{(n-1)s^2(o)}{\chi_{n-1;h(a)}^2}, \frac{(n-1)s^2(o)}{\chi_{n-1;1-h(a)}^2}\right], \forall a \in [0, 1]$$
(22)

where $Z_{h(a)}$, $h_{(a)}$ and Φ have already been defined.

Similarly, the support for the fuzzy membership function for the fuzzy variance of the random sample of C_1, C_2, \ldots, C_n is exactly the $(1 - \gamma)100\%$ confidence interval for σ^2 , and the a-cuts of this fuzzy number are the closed intervals

$$V_{c}[a] = \left[V_{c}^{l}(a), V_{c}^{r}(a)\right] = \left[\frac{(n-1)s^{2}(c)}{\chi_{n-1;h(a)}^{2}}, \frac{(n-1)s^{2}(c)}{\chi_{n-1;1-h(a)}^{2}}\right], \forall a \in [0, 1]$$
(23)

where $Z_{h(a)}$, $h_{(a)}$ and Φ have already been defined.

In this work, the authors propose a fuzzy version of the constant k in order to include the prices proposed from various experts, as mentioned above. Under the extension principal, the k factor for the minimization of the variance can be fuzzified. That is, it can be expressed as

$$k_0[a] = \frac{[\theta^l(a), \theta^r(a)] - 1}{[\theta^l(a), \theta^r(a)] + \frac{n+1}{n-1}} = \left[\frac{\theta^l(a) - 1}{\theta^r(a) + \frac{n+1}{n-1}}, \frac{\theta^r(a) - 1}{\theta^l(a) + \frac{n+1}{n-1}}\right]$$
(24)

Taking into consideration Equations (19)–(24), the fuzzy estimator for the Yang and Zhang variance derives as follows.

When the sample is large, then

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$$FE\sigma_{YZ} = \sigma_{YZ}[\alpha] = \sqrt{V_o[a] + k[a]V_c[a] + (1 - k[a])V_{RS}} = \sqrt{[V_o^l(a), V_o^r(a)] + [k^l(a), k^r(a)][V_c^l(a), V_c^r(a)] + [1 - k^r(a), 1 - k^l(a)]V_{RS}}$$

$$= \sqrt{\left[\frac{s^2(o)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^r(a))V_{RS}, \right]}$$

$$= \sqrt{\left[\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^l(a))V_{RS}\right]}$$

$$= \sqrt{\left[\frac{s^2(o)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^l(a))(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}$$

$$= \sqrt{\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^l(a))(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}$$

$$= \sqrt{\frac{s^2(o)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^l(a))(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}$$

$$= \sqrt{\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - k^l(a))(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}$$

$$= \sqrt{\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - \frac{\theta^l(a) - 1}{\theta^l(a) + \frac{n+1}{n-1}})(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}}$$

$$= \sqrt{\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - \frac{\theta^l(a) - 1}{\theta^l(a) + \frac{n+1}{n-1}})(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}}$$

$$= \sqrt{\frac{s^2(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{k^l(a)s^2(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + (1 - \frac{\theta^l(a) - 1}{\theta^l(a) + \frac{n+1}{n-1}})(\frac{1}{n}\sum_{i=1}^{n} (u_i(u_i - c_i) + d_i(d_i - c_i)), \right]}}$$

$$\sigma_{YZ}^{l}(a) = \sqrt{\frac{s^{2}(o)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + \frac{\left(\frac{\theta^{l}(a) - 1}{\theta^{r}(a) + \frac{n+1}{n-1}}\right)s^{2}(c)}{1 + z_{h(a)}\sqrt{2/(n-1)}} + \left(1 - \frac{\theta^{r}(a) - 1}{\theta^{l}(a) + \frac{n+1}{n-1}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i})\right)\right)}$$

and

$$\sigma_{YZ}^{r}(a) = \sqrt{\frac{s^{2}(o)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \frac{\left(\frac{\theta^{l}(a) - 1}{\theta^{r}(a) + \frac{n+1}{n-1}}\right)s^{2}(c)}{1 - z_{h(a)}\sqrt{2/(n-1)}} + \left(1 - \frac{\theta^{r}(a) - 1}{\theta^{l}(a) + \frac{n+1}{n-1}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i})\right)\right)}$$

When the sample is small, then

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$$FE\sigma_{YZ} = \sqrt{V_{o}[a] + k[a]V_{c}[a] + (1 - k[a])V_{RS}} = \sqrt{[V_{o}(a), V_{o}^{r}(a)] + [k^{l}(a), k^{r}(a)][V_{c}^{l}(a), V_{c}^{r}(a)] + [1 - k^{r}(a), 1 - k^{l}(a)]V_{RS}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{r}(a))V_{RS}, \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))V_{RS}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i})), \frac{1}{n}(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}{k^{l}(a)(n-1)s^{2}(c)} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}{\chi_{n-1,1-h(a)}^{2}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}{\chi_{n-1,1-h(a)}^{2}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}{\chi_{n-1,1-h(a)}^{2}}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}}{\chi_{n-1,1-h(a)}^{2}}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}}{\chi_{n-1,1-h(a)}^{2}}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}}{\chi_{n-1,1-h(a)}^{2}}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} + (1 - k^{l}(a))(\frac{1}{n}\sum_{i=1}^{n} (u_{i}(u_{i} - c_{i}) + d_{i}(d_{i} - c_{i}))}}{\chi_{n-1,1-h(a)}^{2}}}$$

$$= \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1,1-h(a)}^{2}} + \frac{k^{l}(a)(n-1)s^{2}(c)}{\chi_{n-1,1-h(a)}^{2}} +$$

$$\sigma_{YZ}^{l}(a) = \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1;h(a)}^{2}} + \frac{\left(\frac{\theta^{r}(a)-1}{\theta^{l}(a) + \frac{n+1}{n-1}}\right)(n-1)s^{2}(c)}{\chi_{n-1;h(a)}^{2}} + \left(1 - \frac{\theta^{l}(a)-1}{\theta^{r}(a) + \frac{n+1}{n-1}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(u_{i}(u_{i}-c_{i}) + d_{i}(d_{i}-c_{i})\right)\right)}$$

and

$$\sigma_{YZ}^{r}(a) = \sqrt{\frac{(n-1)s^{2}(o)}{\chi_{n-1;1-h(a)}^{2}} + \frac{\left(\frac{\theta^{r}(a)-1}{\theta^{l}(a)+\frac{n+1}{n-1}}\right)(n-1)s^{2}(c)}{\chi_{n-1;1-h(a)}^{2}} + \left(1 - \frac{\theta^{l}(a)-1}{\theta^{r}(a)+\frac{n+1}{n-1}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(u_{i}(u_{i}-c_{i})+d_{i}(d_{i}-c_{i})\right)\right)^{2}}$$

Next, an illustrative example is presented for thorough comprehension of the computational steps of the proposed method.

6. Example (Revisited and Substantially Extended for the Illustration of the FPeNPV Method)

Let us now consider Company A, with an equity/debt ratio of 5/3. The corporate debt has a gross redemption yield of 10%. The corporate debt is also assumed to be risk-free. The corporate tax rate is 30%. Company A is considering Projects P1, P2 and P3. In the same industry, there is Company B, which runs Project P1, and Company C, which runs Projects P2 and P3. The expert has expressed his opinion (Tables 1–4) for the quantities for the stock return's determinants and for each project data as follows [37].

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Table 1. Expert oponions in the form of the a-	-cuts of fuzzy numbers.
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	Company B	Comp	any C
Variable	Project P1	Project P2	Project P3
Dt [α]	[2.00,2.50]	[2.20,2.60]	[3.00,3.50]
Pt [α]	[12.00,14.50]	[11.00,12.00]	[14.00,15.00]
$Pt-1[\alpha]$	[11.50,11.50]	[10.50,12.00]	[11.70,12.00]
dt [α]	[3.50,4.00]	[3.50,4.00]	[3.50,4.00]
It [α]	[12.00,14.50]	[12.50,14.50]	[12.00,14.50]
$It-1[\alpha]$	[15.00,15.00]	[15.00,15.00]	[15.00,15.00]

Table 2. Data for Project P1.

Type of Variable	Variable	Left Tail of A-Cut	Right Tail of A-Cut
Crisp	I	4000.00 €	
Fuzzy	pt	15.00 €	15.50 €
Fuzzy	coft	6.00 €	6.30 €
Fuzzy	xt	1500.00 units	1800.00 units
Fuzzy	COFt	5000.00 €	5400.00 €
Fuzzy	ft	4.00%	4.20%
Crisp	economic life T	3 y	rears
Fuzzy	Lt	1000.00 €	1,250,00 €

Table 3. Data for Project P2.

Type of Variable	Variable	Left Tail of A-Cut	Right Tail of A-Cut
Crisp	I	4000.00 €	
Fuzzy	pt	16.00 €	16.70 €
Fuzzy	coft	5.00 €	6.50 €
Fuzzy	xt	2000.00 units	2100.00 units
Fuzzy	COFt	4000.00 €	4300.00 €
Fuzzy	ft	4.00%	4.20%
Crisp	economic life T	3 y	rears
Fuzzy	Lt	1100.00 €	1400.00 €

Table 4. Data for Project P3.

Type of Variable	Variable	Left Tail of A-Cut	Right Tail of A-Cut
Crisp	I	6000.00 €	
Fuzzy	pt	12.00 €	12.50 €
Fuzzy	coft	4.00 €	5.00 €
Fuzzy	xt	3000.00units	3200.00 units
Fuzzy	COFt	4000.00 €	4300.00 €
Fuzzy	ft	4.00%	4.20%
Crisp	economic life T	3 y	rears
Fuzzy	Lt	1200.00 €	1300.00 €

The equity/debt ratio of Company B is 7/3, and for Company C it is 3/1. Company A maintains the same capital structure after the potential implementation of Projects P1, P2 and P3. The variables for the projects are expressed in real values. Let us follow the following algorithm to find the FPeNPV for each project in nominal values.

Step 1.

Identify a suitable equity beta coming from a company running the project in the same industry. Use Equation (5).

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Project P1	Project P2	Project P3
equity beta	equity beta	equity beta
1.5600	1.7423	0.9870

Step 2.

Degear the equity beta to the asset beta. Strip out the gearing risk to calculate the asset beta for the project ($\beta_{asset} = E_B \times \beta_{equity}/[E_B + D_B \times (1 - T_C)]$).

Project P1	Project P2	Project P3
asset beta	asset beta	asset beta
1.2000	1.4126	0.8003

Step 3.

Regear the asset beta to the equity beta. Re-work the same formula to add back the unique gearing relating to the project $(\beta_{equity} = [E_A + D_A \times (1 - T_C)] \times \beta_{asset}/E_A)$.

Project P1	Project P2	Project P3
equity beta	equity beta	equity beta
1.7040	2.0060	1.1364

Step 4.

Calculate the equity cost via the possibilistic set-up of CAPM. Use Equation (7).

Project P1	Project P2	Project P3
CAPM equity cost	CAPM equity cost	CAPM equity cost
0.0283	0.0333	0.0189

Step 5.

Find the cost of capital (discount rate) via the WACC. Use Equation (8).

Project P1	Project P2	Project P3
WACC	WACC	WACC
4.39%	4.71%	3.80%

Step 6.

Apply the FPNPV method. To apply the analytical form, choose a definition from Definitions 9–11 (Equations (12)–(14)).

Step 7.

Apply Definition 10 for each project to turn the real values into nominal values (Equation (13). See the results in Table 5 (the aggregate results). Based on the FPNPV criterion, P3 must be chosen. Let us now see what happens if we consider an expansion of P3 for one period with the additional expense for the expansion to be $1000 \in$.

Step 8.

Combine Equations (15), (17) and (18) and Equations (24) and (26) to find the value of the option to expand (call option). We considered that we have a small sample formed by expert opinions.

Step 9.

Find the fuzzy possibilistic expanded NPV value by adding the static NPV to the call option value (use Equation (4)).

FPeNPV = [10.125,32 €, 16.001.54 €].

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Table 5. Aggregated results for	r the fuzzy possibilistic net present v	value (NPV) of each project.
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	Period	Net Cash Flow	
	renou	Right Tail of A-Cut	Left Tail of A-Cut
	1	3911.45 €	6159.80 €
Dundant D1	2	1998.97 €	3134.81 €
Project P1	3	1045.33 €	1625.05 €
	FPNPV	2955.75 €	6919.67 €
	1	7423.65 €	10,383.77 €
Dundant Do	2	3748.72 €	5241.33 €
Project P2	3	1919.12 €	2678.76 €
	FPNPV	9091.49 €	14,303.87 €
	1	8471.05 €	11,742.01 €
	2	4296.05 €	5942.02 €
Project P3	3	2208.11 €	3038.73 €
	FPNPV	8975.22 €	14,722.76 €

7. Results

The traditional NPV criterion was modified so that it could handle the uncertain and vague environment existing in financial markets regarding investments. The possibilistic set-up of CAPM was used for the discount rate. The inflation impact in future cash flows and in the discount rate was included. Different formulas for converting fuzzy nominal to fuzzy real values and vice versa were provided. The fuzzy estimator for the Yang and Zhang variance [40] was appied in the binomial option pricing model. This could be derived either from statistical data (in the case of a large sample) or by expert opinions (when the sample is small). The fuzzy estimators method was employed as a risk aversion measure in the FPeNPV, since it gave the choice of choosing the inteval range (through a-cuts). Finally, the calculation of the call option price (option to expand) led to the FPeNPV.

8. Discussion

The initial novelties of this research work are the use of the possibilistic beta and possibilistic CAPM [37] for the computation of the equity cost and the possibilistic set up of the WACC as a discount rate for the expanded FPeNPV. The benefits from these applications are the following: (1) the degearing of equity to asset beta, and then the regearing of the found asset beta to equity beta, is a necessary process to add beta to the unique gearing relating to the project. It is shown that the beta adjusted to the relating project is remarkably different than the initial one (e.g., $P_3 = 0.987$, and after the process $P_3 = 1.1364$); (2) the use of the possibilistic beta for each project does not require statistical information (historical data), which could be biased, and it is based on expert opinions; and (3) the possibilistic beta is a safe translation of fuzzy information. Expert fuzzy estimates conclude to a possibilistic crisp value for the equity cost (from CAPM), which is incorporated in the calculation of the WACC.

The second novelty of the proposed method is the derivation of fuzzy formulae for the conversion from real to nominal values and vice versa. The proposed method includes the appropriate adjustment to inflation both for the cash flows and for the rate. As one can notice, inflation has a direct impact not only on the cash flows, but also on the rate of return of each project. This adjustment to inflation does not necessarily offer accuracy in the calculation of the expanded FPeNPV, but most probably more realistic interval estimates for the project value.

Another novelty of the proposed method is the use of non-asymptotic fuzzy estimators for the volatility of the binomial model. The fuzzy estimators method for volatility uses a set of confidence intervals, producing fuzzy numbers for the estimation of probability distribution parameters [36]. The fuzzy estimators method acts as risk aversion measure in the expanded FPeNPV, since the financial analyst can decide how close to be to the point estimation. The fuzzy estimator for the variance (that was finally used in the binomial

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option pricing model) can be derived either from statistical data (considering that we have a large sample) or by expert opinions (in the case of a small sample). In this example, we considered a small sample formed by expert opinions.

Finally, the derivation of the call option price (option to expand) leads to the expanded FPeNPV calculation and to its offered merits, as discussed in previous sections. The current expectation is that the cash flows from the investment belong to the interval [8.975.22 $\mbox{\ensuremath{$\ell$}}$, 14.722.,76 $\mbox{\ensuremath{$\ell$}}$]. We find the expanded FPeNPV for the examined scenario belongs to the interval [10.125,32 $\mbox{\ensuremath{$\ell$}}$, 16.001.54 $\mbox{\ensuremath{$\ell$}}$]. Thus, the expansion can be chosen.

9. Conclusions and Further Work

Intelligent and fuzzy methods remain reliable methods in financial management and investment appraisal. Possibility theory also gives the opportunity to safely translate the fuzzy information. Hybrid methods, such as the fuzzy estimators' method, give the choice of deriving fuzzy information either from statistical data or from expert opinions. The FPeNPV method offers flexibility for future decision-making in investment appraisal.

The contribution of the proposed method, the fuzzy possibilistic expanded net present value (FPeNPV), is the combination of several methods in each step in order to provide the administration with more realistic interval results for a project's value. The main authors' effort is to provide a set of computational steps that improve all the intermediate assumptions and variable estimates until reaching the final goal, which is a realistic and flexible approximation for a project's value. The traditional NPV criterion is modified so that it can handle the uncertain and vague environment existing in financial markets regarding investments. The discount rate is modeled via the possibilistic set-up of CAPM resulting from fuzzy data—taking advantage of this method's merits. Inflation impact is also taken into account. The NPV formula is appropriately modified so that consistency is ensured. The results of these modifications are different formulas for converting fuzzy nominals to fuzzy real values and vice versa. The fuzzy estimator for the Yang and Zhang variance (that was finally used in the binomial option pricing model) can be derived either from statistical data (considering that we have a large sample) or by expert opinions (in the case of a small sample). The fuzzy estimators method acts as a risk aversion measure in the FPeNPV, since the financial analyst can decide how close to be to the point estimation. The derivation of the call option price (option to expand) leads to the FPeNPV calculation.

Due to the incorporation of (1) the possibilistic equity cost (and as a consequence, the possibilistic WACC), (2) the fuzzy NPV adjusted for inflation (both cash flows and rate) and (3) the fuzzy volatility in the call option calculation (in the fuzzy binomial model), the expanded FPeNPV offers results closer to market reality. The limitation of this research work is the assumption that the project's value volatility can be approximated by the company's (running the project) stock volatility. Thus, a direction for future research could be setting the examination of the similarity degree between these two factors. Moreover, the application of this methodology to construction, transition, reform and various types of projects would be of great interest.

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Abbreviations

Basic notation is as follows: NPV net present value
TFN triangular fuzzy number

FPNPV fuzzy possibilistic net present value

t time index R_t stock return in t

 P_t stock price at the end of t P_{t-1} stock price at the end of t-1

 D_t stock dividend in t

 R_{Mt} market portfolio return on t I_t level of the index at the end of t I_{t-1} level of the index at the end of t-1 d_t dividend paid on the index in t

 R_f risk-free rate

GRY gross redemption yield

 T_C corporate tax

 $E_{company}$ equity part of a company's equity/debt ratio $D_{company}$ debt part of a company's equity/debt ratio

 $V_{company}$ total market value of firm (the sum of $E_{Company} + D_{Company}$)

WACC weighted average cost of capital

 I_0 initial investment outlay p_t sales price in period t $TCOF_t$ total cash outflows in t

 cof_t production- and sales-dependent (variable) cash outflows per unit in period t

 x_t production and sales volume in period t

 CIF_t cash inflows in t

 COF_t production- and sales-independent (fixed) cash outflows in period t

 r_{poss} discount rate resulting from possibilistic CAPM

 L_t liquidation index

T the last year when cash flows take place

FNPV fuzzy net present value

Appendix A

Operations for Definition 6

No conversion needed

$$FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r)^{-t} \rightarrow$$

➤ The extension principle:

$$\begin{split} FPNPV &= -I_{0} \\ &+ \sum_{i=1}^{T} \left(\left[CIF_{t}^{l}(a), CIF_{t}^{r}(a) \right] - \left[TCOF_{t}^{l}(a), TCOF_{t}^{r}(a) \right] \right) (1 + r_{poss})^{-t} \rightarrow \\ FPNPV &= -I_{0} \\ &+ \sum_{i=1}^{T} \left[\left(CIF_{t}^{l}(a) - TCOF_{t}^{r}(a), CIF_{t}^{r} - TCOF_{t}^{l}(a) \right] (1 + r_{poss})^{-t} \\ FPNPV &= -I_{0} \\ &+ \sum_{i=1}^{T} \left[\left(CIF_{t}^{l}(a) - TCOF_{t}^{r}(a) \right) (1 + r_{poss})^{-t}, \left(CIF_{t}^{r}(a) - TCOF_{t}^{l}(a) \right) (1 + r_{poss})^{-t} \right] \end{split}$$

Operations for Definition 7

Real to Nominal

• The basic formula:

$$FPNPV = -I_0 + \sum_{t=1}^{T} (CIF_t - TCOF_t)(1+r)^{-t} \rightarrow$$

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Convert the real cash flows and real discount rate to nominal values:

$$FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t ((1+r)(1+f))^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^t (1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r+f+rf)^{-t} \to FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+rf+rf+rf)^{-t}$$

• The extension principle:

$$\begin{split} FPNPV &= -I_0 \\ &+ \sum\limits_{i=1}^{T} \ \left(\left[CIF_t^l(a) - CIF_t^r(a) \right] - \left[TCOF_t^l(a), TCOF_t^r(a) \right] \right) \\ &+ \sum\limits_{i=1}^{T} \ \left[1 + f_t^l(a), 1 + f_t^r(a) \right]^t \left[1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a) \right]^{-t} \\ &+ FPNPV &= -I_0 \\ &\quad \left[CIF_t^l(a) - TCOF_t^r(a), CIF_t^l(a) - TCOF_t^l(a) \right] \\ &+ \sum\limits_{i=1}^{T} \ \left[\left(1 + f_t^l(a) \right)^t, \left(1 + f_t^r(a) \right)^t \right] \\ &\quad \left[\left(1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a) \right)^{-t}, 1 + \left(r_{poss} + f_t^l(a) + r_{poss} f_t^l(a) \right)^{-t} \right] \end{split}$$

Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $CIF_t[a] - TCOF_t[a]$.

Operations for Definition 8

Nominal to Real

• The basic formula:

$$FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+r)^{-t} \to$$

• Convert the nominal cash flows and nominal discount rate to real values:

$$FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t - TCOF_t)(1+f)^{-t} ((1+r)(1+f)^{-1})^{-t} \rightarrow FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t[a]_t - TCOF_t)(1+f)^{-t} (1+\frac{r-f}{1+f})^{-t} \rightarrow FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t[a]_t - TCOF_t)(1+f)^{-t} (1+\frac{r-f}{1+f})^{-t} \rightarrow FPNPV = -I_0 + \sum_{i=1}^{T} (CIF_t[a]_t - TCOF_t)(1+f)^{-t} (1+f)^{-t} (1+f$$

• The extension principle:

• Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $CIF[a] - TCOF_t[a]$.

Operations for Definition 9

No conversion needed

$$FPNPV = -I_0 + \sum_{t=1}^{T} ((p_t - cof_t)x_t - COF_t)(1+r)^{-t} + L(1+r)^{-T} \rightarrow$$

• The extension principle:

$$\begin{split} FPNPV &= -I_0 \\ &+ \sum_{i=1}^{T} \ (([p_t^l(a), p_t^r(a)] - [cof_t^l(a), cof_t^r(a)])[x_t^l(a), x_r^r(a)] - [COF_t^l(a), COF_t^r(a)]) \\ &+ \sum_{i=1}^{T} \ (1 + r_{poss})^{-t} + [L_t^l(a), L_t^r(a)](1 + r_{poss})^{-T} \\ FPNPV &= -I_0 \\ &+ \sum_{i=1}^{T} \ [(p_t^l(a) - cof_t^r(a))x_t^l(a) - COF_t^r(a), (p_t^r(a) - cof_t^l(a))x_t^r(a) - COF_t^l(a)] \\ &+ \sum_{i=1}^{T} \ (1 + r_{poss})^{-t} + [(1 + r_{poss})^{-T}L_t^l(a), (1 + r_{poss})^{-T}L_t^r(a)] \end{split}$$

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• Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$.

Operations for Definition 10

Real to Nominal

The basic formula:

$$FPNPV = -I_0 + \sum_{i=1}^{T} ((p_t - cof_t)x_t - COF_t)(1+r)^{-t} + L(1+r)^{-T} \rightarrow$$

Convert the real cash flows and real discount rate to nominal values:

$$FPNPV = -I_0 + \sum_{i=1}^{T} \frac{((p_t - cof_t)x_t - COF_t)(1+f)^t((1+r)(1+f))^{-t}}{L(1+f)^t((1+r)(1+f))^{-T}} \rightarrow FPNPV = -I_0 + \sum_{i=1}^{T} \frac{((p_t - cof_t)x_t - COF_t)(1+f)^t(1+r+f+rf)^{-t}}{+L(1+f)^t(1+r+f+rf)^{-T}} \rightarrow FPNPV = -I_0 + \sum_{i=1}^{T} \frac{((p_t - cof_t)x_t - COF_t)(1+f)^t(1+r+f+rf)^{-t}}{+L(1+f)^t(1+r+f+rf)^{-T}}$$

The extension principle:

$$\begin{split} FPNPV &= -I_0 \\ & \quad (([p_t^l(a), p_t^r(a)] - [cof_t^l(a), cof_t^r(a)])[x_t^l(a), x_r^r(a)] - [COF_t^l(a), COF_t^r(a)]) \\ & \quad + \sum_{t=1}^{T} \quad [1 + f_t^l(a), 1 + f_t^r(a)]^t [1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a)]^{-t} \\ & \quad + [L_t^l(a), L_t^r(a)][1 + f_t^l(a), 1 + f_t^r(a)] \\ & \quad = [1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a)]^{-T} \\ & \quad FPNPV &= -I_0 \\ & \quad [(p_t^l(a) - cof_t^r(a))x_t^l(a) - COF_t^r(a), (p_t^r(a) - cof_t^l(a))x_t^r(a) - COF_t^l(a)] \\ & \quad + \sum_{t=1}^{T} \quad [1 + f_t^l(a), 1 + f_t^r(a)]^t [1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} + f_t^r(a) + r_{poss} f_t^r(a)]^{-t} \\ & \quad + [[L_t^l(a), L_t^r(a)][1 + f_t^l(a), 1 + f_t^r(a)] \\ & \quad [1 + r_{poss} + f_t^l(a) + r_{poss} f_t^l(a), 1 + r_{poss} f_t^r(a)]^{-T} \end{split}$$

• Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$.

Operations for Definition 11

Nominal to Real

• The basic formula:

$$FPNPV = -I_0 + \sum_{t=1}^{T} ((p_t - cof_t)x_t - COF_t)(1+r)^{-t} + L(1+r)^{-T} \rightarrow$$

Converting nominal cash flows and nominal discount rate to real values:

$$\begin{split} FPNPV &= -I_0 + \sum_{i=1}^{T} \frac{\left((p_t - cof_t)x_t - COF_t\right)(1+f)^{-t} \left((1+r)(1+f)\right)^{-t}}{+L(1+f)^t \left((1+r)(1+f)\right)^{-T}} \rightarrow \\ FPNPV &= -I_0 + \sum_{i=1}^{T} \left((p_t - cof_t)x_t - COF_t\right)(1+f)^{-t} \left(1 + \frac{r_{pss} - f}{1+f}\right)^{-t} + L(1+f)^t \left(\frac{r_{poss} - f}{1+f}\right)^{-T} \rightarrow \end{split}$$

• The extension principle:

$$\begin{split} FPNPV &= -I_0 \\ & \quad (([p_t^l(a), p_t^r(a)] - [cof_t^l(a), cof_t^r(a)])[x_t^l(a), x_t^r(a)] - [COF_t^l(a), COF_t^r(a)]) \\ &+ \sum_{i=1}^T \quad [1 + f_t^l(a), 1 + f_t^r(a)]^t (1 + \frac{[r_{poss} - f_t^r(a), r_{poss} - f_t^l(a)]}{[1 + f_t^l(a), 1 + f_t^r(a)]})^{-t} \\ &+ [L_t^l(a), L_t^r(a)][1 + f_t^l(a), 1 + f_t^r(a)](1 + \frac{[r_{poss} - f_t^r(a), r_{poss} - f_t^l(a)]}{[1 + f_t^l(a), 1 + f_t^r(a)]})^{-T} \end{split}$$

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Next, the operation of multiplication on intervals is applied, depending on the sign of the operation $(p_t[a] - cof_t[a])x_t[a] - COF_t[a]$:

$$\begin{split} FPNPV &= -I_0 \\ & [(p_t^l(a) - cof_t^r(a))x_t^l(a) - COF_t^r(a), (p_t^r(a) - cof_t^l(a))x_t^r(a) - COF_t^l(a)] \\ &+ \sum_{i=1}^T \left[1 + f_t^l(a), 1 + f_t^r(a)\right]^t \left[1 + \frac{r_{poss} - f_t^l(a)}{1 + f_t^l(a)}, 1 + \frac{r_{poss} - f_t^l(a)}{1 + f_t^l(a)}\right]^{-t} \\ &+ [L_t^l(a), L_t^r(a)][1 + f_t^l(a), 1 + f_t^r(a)][1 + \frac{r_{poss} - f_t^l(a)}{1 + f_t^l(a)}, 1 + \frac{r_{poss} - f_t^l(a)}{1 + f_t^l(a)}]^{-T} \end{split}$$

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