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Group Theoretical Approach to Pseudo-Hermitian Quantum Mechanics with Lorentz Covariance and $c \to \infty$ Limit

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Abstract: We present the formulation of a version of Lorentz covariant quantum mechanics based on a group theoretical construction from a Heisenberg–Weyl symmetry with position and momentum operators transforming as Minkowski four-vectors. The basic representation is identified as a coherent state representation, essentially an irreducible component of the regular representation, with the matching representation of an extension of the group C^*-algebra giving the algebra of observables. The key feature is that it is not unitary but pseudo-unitary, exactly in the same sense as the Minkowski spacetime representation. The language of pseudo-Hermitian quantum mechanics is adopted for a clear illustration of the aspect, with a metric operator obtained as really the manifestation of the Minkowski metric on the space of the state vectors. Explicit wavefunction description is given without any restriction of the variable domains, yet with a finite integral inner product. The associated covariant harmonic oscillator Fock state basis has all the standard properties in exact analog to those of a harmonic oscillator with Euclidean position and momentum operators. Galilean limit and the classical limit are retrieved rigorously through appropriate symmetry contractions of the algebra and its representation, including the dynamics described through the symmetry of the phase space.

Keywords: Lorentz covariant quantum mechanics; Minkowski Metric Operator; pseudo-unitary representation; pseudo-hermitian quantum mechanics; symmetry contraction limits; quantum nonrelativistic and classical limits; quantum relativity; WWGM formulation; coherent state representation; noncommutative spacetime

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1. Introduction

Our group had implemented a, quantum relativity symmetry, group theoretical formulation of the full dynamical theory of the familiar quantum mechanics with rigorous classical limit given as the Newtonian theory, obtained through a contraction of the relativity symmetry applied to the specific representation [1]. The latter is taken as essentially an irreducible component of the regular representation of $H(3)$, the Heisenberg–Weyl group. The full quantum relativity symmetry, denoted $\tilde{G}(3)$, can naturally be seen as a $U(1)$ central extension [2] of the Galilean symmetry. $H_R(3)$ is (or is isomorphic to) its subgroup, left after the 'time-translation' is taken out. A $H(3)$ representation is a spin zero, time independent representation of $\tilde{G}(3)$. The representation is really the one of the canonical coherent states [3–5]. The matching representation of the group C^*-algebra [6,7], further extended to a proper class of distributions, gives the observable algebra as functions, and distributions, of position and momentum operators, $\hat{X}_i = x_i\star$ and $\hat{P}_i = p_i\star$, as given by the Weyl–Wigner–Groenewold–Moyal(WWGM) formulation [8–11]. The operators $\alpha(p_i\star, x_i\star) = \alpha(p_i, x_i)\star$ act as differential operators on the wavefunctions on coherent state basis $\phi(p^i, x^i)$ by the Moyal star-product $\alpha \star \phi$; $\alpha \star \beta\star = (\alpha \star \beta)\star$. $\hat{X}_i$ and $\hat{P}_i$ can be seen as operator coordinates of the quantum phase space [12,13], which has been argued to serve as a proper quantum model for the physical space [1,14]. We naturally seek a Lorentz covariant version of that with a $c \to \infty$ contraction of the symmetry taking the Lorentz boosts to that of the Galilean ones [15]. Such a contraction is the mathematically rigorous way to look at the full approximation of a theory under a certain limit, from the symmetry theoretical perspective.

The relativity symmetry for the quantum theory is one of $H_R(1,3)$, which fits well into the contraction chain, at least at the symmetry and coset space level [16,17]. It has been well known that from a group theoretical perspective, a general overcomplete coherent state basis can naturally be identified with points of the appropriate coset. The latter in our cases corresponds to something like the classical phase space. The formulation of a fully Lorentz covariant version of quantum mechanics, with position and momentum operators $\hat{X}_\mu$ and $\hat{P}_\mu$ transforming as Minkowski four-vectors, has been around since the early days of quantum mechanics. A naive thinking would be to represent those operators as x_μ and $-i\hbar\partial_{x^\mu}$, respectively, acting on the wavefunctions $\psi(x^\mu)$ with the simple inner product giving the squared integral norm, and to take a unitary Schrödinger evolution under the Einstein proper time τ, which gives the Klein–Gordon equation as the τ-independent equation of motion. Explicit group theoretical picture of that has been available since the sixties [18,19]. The truth is, in any theory of quantum mechanics with wavefunctions on Minkowski four-vector variable(s), the real symmetry behind the system is the $H_R(1,3)$ group instead of only its Poincaré subgroup. There are, however, difficulties with the unitary theory, especially well illustrated in the covariant harmonic oscillator problem [18,20], which we show explicitly in the Appendix A.

Other than being of interest on its own, the harmonic oscillator problem is of great theoretical importance. For our usual quantum mechanics, we have the well appreciated close connection between the Fock states, as the eigenstates of the harmonic oscillator Hamiltonian, and the canonical coherent states. The set of Fock states is one of the most useful orthonormal basis for the Hilbert space and the latter, as the space of rapidly decreasing functions spanned by their wavefunctions, giving the states on which the position and momentum operators are truly Hermitian in a completely consistent formulation [8]. Upon a more careful inspection, the Fock states are simultaneous eigenstates of the number operators $\hat{N}_i$, or equivalently of $\hat{X}_i^2 + \hat{P}_i^2$. The subspace spanned by the Fock states of a fixed eigenvalue n of the total number operator $\sum \hat{N}_i$ corresponds exactly to the space of symmetric n-tensors of the three-dimensional Euclidean space. In particular, the three $n = 1$ states transform exactly as components of a three-vector in a complexified Newtonian space. A perfectly nice embedding of all that into the space spanned by Fock states of the Lorentz covariant harmonic oscillator problem should be expected to have a fully parallel structure of symmetric n-tensors in the $(1+3)$-dimensional Minkowski spacetime [20]. Unlike for the $SO(3)$ symmetry, however, the noncompact nature of Lorentz $SO(1,3)$ symmetry means that the corresponding spaces for the symmetric n-tensors, as its irreducible representations, cannot be unitary.

Replacing the full unitarity of the irreducible representation of $H_R(1,3)$ by a pseudo-unitarity exactly in line with the Minkowski spacetime may be a good direction to formulate a theory of covariant quantum mechanics [21]. The representation as one for the $SO(1,3)$ subgroup would reduce to a sum of finite dimensional irreducible components each labeled by two integers, the n and a nonzero positive integer characterizing the spin independent Casimir invariant. The latter corresponds to one plus the rank of the symmetric n-tensors [22].

We have presented in Ref. [22] the complete set of Fock states wavefunctions of such a pseudo-unitary representation of $H_R(1,3)$ symmetry on a space of rapidly decreasing functions, hence completely free from divergence in themselves as well as in the Lorentz invariant indefinite inner product. Formulation given there corresponds to writing the time coordinate as ict. Here, the representation is rather given in the form of the $|p^\mu, x^\mu\rangle$ coherent states, with p^μ and x^μ being real Minkowski four-vectors, from our group theoretical grand framework [1,16,17].

We want to emphasize that quantum dynamics is a symplectic dynamics and the physical Hamiltonian is just one among the many general Hamiltonians with the generated Hamiltonian flows as symmetries of the phase space. It is the symplectic structure of the latter as fixed by the inner product, or the metric for the vector space or its projective space, that is really the key. The actual symmetries of a physical system of course correspond to Hamiltonian flows the generators of which commute with the physical Hamiltonian, with the generators giving the conserved physical quantities.

As a preparation, we first sketch the notion of pseudo-Hermiticity and pseudo-unitarity clearly in the next section. In Sections 3 and 4 below, we start with an explicit presentation of the regular representation, and its irreducible components, of the $H(1,3)$ group. A major part of that is also needed to formulate the $c \to \infty$ contraction. Each such component is shown to give essentially the same physical theory of covariant quantum mechanics we present in detail on the coherent state basis, in the abstract form and in wavefunctions, with the Lorentz invariant indefinite inner product. The part that involves the inner product and the pseudo-Hermitian/pseudo-unitary nature of the theory is put in Section 4 after the presentation of the space of state vectors as the Fock space for covariant harmonic oscillator. Section 5 deals with the Lorentz to Galilean, $c \to \infty$, contraction of the representation, i.e., the retrieval of the 'nonrelativistic' limit, the part for the dynamics of which is left to the last subsection of the Section 6. The latter is first devoted to the WWGM framework or the observable algebra, focusing on the symmetry transformations and the dynamics as a specific case of such a symmetry flow, with the real parameter characterizing transformation corresponding to an evolution parameter, which is taken as the proper time in the case. In Section 7, we give a brief description of contraction to the classical theory and conclude in the last section.

2. Pseudo-Hermiticity and Pseudo-Unitarity

Pseudo-unitarity is about an inner product that is not positive definite, like the Minkowski metric. Lorentz covariant quantum theory with an indefinite inner product vector space of states was first introduced by Dirac and Pauli [23,24]. However, an explicit detailed formulation of quantum mechanics with a careful attention paid to the covariant and contravariant indices seems not to be available. More interest has been focused on quantum field theories, such as quantum electrodynamics (see Ref. [25] for a review). It has been a common strategy, especially in gauge theories since Gupta–Bleuler [26,27], to formulate a theory on such a Krein space [28] and then project it onto the 'physical' Hilbert space as the positive normed subspace (see also Ref. [20] for the harmonic oscillator case), retrieving a standard probability interpretation. Interest in the related subject matter for quantum mechanics has been brought back to popularity from works on the so-called pseudo-Hermitian quantum mechanics [29–31], which we, in a way, rediscovered in our work of Ref. [22].

Let us sketch pseudo-Hermitian quantum mechanics here. A naive direct picture starts with a Hamiltonian operator $\hat{A}_H$ that is not Hermitian with respect to the given inner product of the Hilbert space. If a Hermitian operator $\hat{\eta}$ can be found such that

$$\hat{A}_H^\dagger = \hat{\eta}\hat{A}_H\hat{\eta}^{-1}\,, \tag{1}$$

the operator $\hat{A}_H$ is called pseudo-Hermitian and $\hat{\eta}$ the (pseudo-)metric operator [24,31]. We think metric operator is the more appropriate name than pseudo-metric operator, especially because in our case it is essentially the exact manifestation of the Minkowski metric $\eta_{\mu\nu}$. The interesting thing is that a new inner product ${}_{\eta}\langle\cdot|\cdot\rangle$ can be introduced with respect to which the operator $\hat{A}_H$ is really Hermitian, namely $\hat{A}_H^{\dagger^\eta} = \hat{A}_H$ for the Hermitian conjugation satisfying

$${}_{\eta}\!\left\langle \cdot|\hat{A}^{\dagger^\eta}\cdot\right\rangle = {}_{\eta}\!\left\langle \hat{A}\cdot|\cdot\right\rangle . \tag{2}$$

(We introduce the somewhat unusual notation for a reason. Since we are talking about a second inner product on the same vector space, we want the vectors, kets, to be independent of the inner products, while the sets of bras as functionals can be defined differently [22], giving the different Dirac brackets as the different inner products). To be more specific, one can call it a η-Hermiticity.

The new inner product is not required to be positive definite. More importantly, the evolution generated by $\hat{A}_H$ is 'unitary' [31] in the sense that it preserves the inner product between any two states. In the case of an indefinite inner product, the transformations preserving it are truly represented by the pseudo-unitary, rather than unitary, matrices. Adopting from the terminology of special relativity, we have states with norms that can be spacelike (+ve), timelike ($-$ve), or lightlike (0). For the nondegenerate case, explicitly, one can find a countable orthonormal basis, like the Fock basis in our case, with L vectors of the norm -1, M vectors of $+1$ and none of the vanishing norm, in which the 'unitary' transformations generated by any pseudo-Hermitian operator satisfying Equation (2) are represented by $SU(L, M)$ matrices, including the case of $L, M \to \infty$.

Ref. [31] restricted the term 'inner product' to positive-definite products, which is not within its mathematical definition. That is the source of many 'pseudo-' terminology as in 'pseudo-inner product' and 'pseudo-metric', which we see as unnecessary. Defining an absolute pseudo-Hermiticity for the otherwise Hermitian operators which generate pseudo-unitary transformations preserving the indefinite inner product could be quite sensible though. Actually, the theory of quantum mechanics we are interested in here is a pseudo-unitary representation of the background (relativity) symmetry group. The generators of the symmetry are all pseudo-Hermitian operators. These are 'Hamiltonian operators' in the sense of a symplectic/geometric picture of the theory. An acceptable physical Hamiltonian operator in the theory, of course, has to satisfy the same pseudo-Hermiticity, namely the η-Hermiticity.

Note that the notion of pseudo-Hermiticity is a relative one. $\hat{A}_H$ *is not* Hermitian and is pseudo-Hermitian only with respect to the original inner product $\langle\cdot|\cdot\rangle$, for which the Hermitian conjugate $\hat{A}_H^\dagger$ is defined as the operator satisfying

$$\left\langle \cdot|\hat{A}^\dagger \cdot \right\rangle = \left\langle \hat{A} \cdot |\cdot \right\rangle . \tag{3}$$

Looking at the theory as a dynamical one with the physical Hamiltonian operator, the ${}_\eta\langle\cdot|\cdot\rangle$ inner product is the only one relevant. The inner product certainly gives a metric to the vector space and its projective space, which also fixes the symplectic structure. That is the meaning of the choice of the (nontrivial) metric operator. The bottom line is, two different inner products on the same vector space really make two different inner product spaces and we generally do not have any necessity to consider two different inner products for a single theory of quantum dynamics. Often time, as in Ref. [22], it is just that the simplest or the most familiar kind of inner product is the 'wrong' one, based on which one can construct the 'right' one more easily. In this case, ${}_\eta\langle\cdot|\cdot\rangle = \langle\cdot|\hat{\eta}|\cdot\rangle$, or equivalently ${}_\eta\langle\cdot| = \langle\cdot|\hat{\eta}$. Obviously, that is the same as $\langle\cdot| = {}_\eta\langle\cdot|\hat{\eta}^{-1}$, so the two sets of bras are really on the equal footing. The naive perspective that the inner product $\langle\cdot|\cdot\rangle$ is more basic is only a consequence of the presentation. Furthermore, the reality of an eigenvalue for an η-Hermitian operator follows in the same way as for a usual Hermitian one so long as the norm, i.e., η-norm here, of a corresponding eigenstate is nonzero. The latter, of course, always holds on a Hilbert space or, equivalently, for a positive definite inner product.

3. The Irreducible Representations of $H_R(1,3)$

We give the Lie algebra for $H_R(1,3)$ as

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = 2i\big(\eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}\big)\ ,\\
&[J_{\mu\nu}, Y_\rho] = 2i\big(\eta_{\mu\rho}Y_\nu - \eta_{\nu\rho}Y_\mu\big)\ ,\\
&[J_{\mu\nu}, E_\rho] = 2i\big(\eta_{\mu\rho}E_\nu - \eta_{\nu\rho}E_\mu\big)\ ,\\
&[Y_\mu, E_\nu] = 2i\eta_{\mu\nu} I\ ,
\end{aligned} \tag{4}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}$. The choice of notation with Y_μ corresponding essentially to spacetime position observables and E_μ to energy–momentum observables is somewhat unusual. The reason for it should be clear from the analysis below. Notice that the generators are all taken to have no physical dimension, and the factor 2 corresponds to $\hbar$ in the chosen units, which is at least convenient for the coherent state formulation [1]. In terms of the group element $g(p^\mu, x^\mu, \theta, \Lambda^\mu_\nu)$, we have the group product (with the indices suppressed)

$$g(p', x', \theta', \Lambda')\, g(p, x, \theta, \Lambda) = g\big(p' + \Lambda' p, x' + \Lambda' x, \theta' + \theta - x'\Lambda' p + p'\Lambda' x, \Lambda'\Lambda\big). \tag{5}$$

The story is an extension of what has been done in Refs. [1,14] for $H_R(3) = H(3) \rtimes SO(3)$ to the framework of

$$H_R(1,3) = H(1,3) \rtimes SO(1,3)\ , \tag{6}$$

the focus of which, for the spin zero case here, is only on the irreducible representation of the Heisenberg–Weyl symmetry $H(1,3)$ and $H(3)$. A key point of difference between the two cases is that $SO(1,3)$ is noncompact, the finite dimensional representations of which, as a direct extension of those compact ones of $SO(3)$, are pseudo-unitary instead of unitary. The basis of that pseudo-unitarity is the indefinite Minkowski norm associated with the metric $\eta_{\mu\nu}$ extending the Euclidean δ_{ij} [21,22]. In the case of $H_R(3)$, the representation is naturally an irreducible component of the regular representation of $H(3)$, which all can be seen actually as physically equivalent. It comes naturally as wavefunctions in the coherent state basis, on which the observables are represented as differential operators, essentially those obtained from the WWGM framework. Details of all that for the case of $H(3)$ group has been presented in Ref. [1].

We present first the results from a harmonic analysis of Heisenberg–Weyl groups adapted to our case of $H(1,3)$ [32]. We write the left regular representation in $\hbar = 2$ units as $V(p^\mu, x^\mu, \theta) = e^{i(p^\mu Y^L_\mu - x^\mu E^L_\mu + \theta I^L)}$, where

$$\begin{aligned}
Y^L_\mu &= i x_\mu \partial_\theta + i\partial_{p^\mu}\ ,\\
E^L_\mu &= i p_\mu \partial_\theta - i\partial_{x^\mu}\ ,\\
I^L &= i\partial_\theta
\end{aligned} \tag{7}$$

are the left-invariant vector fields. In an irreducible representation, the central generator I has to be represented by a multiple of identity. We write the one parameter series V_ς ($\varsigma \neq 0$) of representations for the generators as operators $\{\hat{Y}^L_\varsigma, \hat{E}^L_\varsigma, \varsigma\hat{I}\}$, where $\hat{I}$ is the identity operator and $[\hat{Y}^L_{\varsigma\mu}, \hat{E}^L_{\varsigma\nu}] = 2i\varsigma\eta_{\mu\nu}\hat{I}$. The V^L_ς set can be considered the set of equivalence classes of irreducible representations with nonzero Plancherel measure. The limit of V^L_ς as $\varsigma \to 0$ gives the whole set of irreducible one-dimensional representations. The latter set has zero Plancherel measure and together with the V^L_ς exhausts all equivalence classes of irreducible representations. Based on the measure, one should consider the expansion

$$\alpha(p^\mu, x^\mu, \theta) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int d\varsigma\, \alpha_\varsigma(p^\mu, x^\mu)\, e^{-i\varsigma\theta} |\varsigma|^n\ , \tag{8}$$

$n = 1+3$ here, given as the inverse Fourier–Plancherel transform. The actions of the left-invariant vector fields on $\alpha(p,x,\theta)$ in the form of Equation (8) are given by their actions on $\alpha_\varsigma(p,x)e^{-i\varsigma\theta}$ parts as $\varsigma x + i\partial_p$, $\varsigma p - i\partial_x$, and ς, respectively. Here, and below, we suppress the indices wherever it is unambiguous. We can see that the action at each $\varsigma \neq 0$ corresponds exactly to the V^L_ς representation with the generators represented by $\{\hat{Y}^L_\varsigma, \hat{E}^L_\varsigma, \varsigma\hat{I}\}$. That is the reduction of the regular representation into irreducible components. For positive values of ς, one can introduce the ς-independent operators

$$\hat{X}^L_{(\varsigma)\mu} \equiv \frac{1}{\sqrt{\varsigma}}\hat{Y}^L_{\varsigma\mu} = x_{(\varsigma)\mu} + i\partial_{p^\mu_{(\varsigma)}} \;,$$
$$\hat{P}^L_{(\varsigma)\mu} \equiv \frac{1}{\sqrt{\varsigma}}\hat{E}^L_{\varsigma\mu} = p_{(\varsigma)\mu} - i\partial_{x^\mu_{(\varsigma)}} \;, \tag{9}$$

where we have $x_{(\varsigma)} = \sqrt{\varsigma}x$ and $p_{(\varsigma)} = \sqrt{\varsigma}p$. $V^L_\varsigma(p_{(\varsigma)}, x_{(\varsigma)}, \theta_{(\varsigma)})$ is then given by $e^{i(p_{(\varsigma)}\hat{X}^L_{(\varsigma)} - x_{(\varsigma)}\hat{P}^L_{(\varsigma)} + \theta_{(\varsigma)}\hat{I})}$, with $\theta_{(\varsigma)} = \varsigma\theta$, hence, in a form formally independent of ς. $\hat{X}^L_{(\varsigma)}$ and $\hat{P}^L_{(\varsigma)}$ are still $SO(1,3)$ vectors, and so are $p_{(\varsigma)}$ and $x_{(\varsigma)}$. The (ς) index becomes completely dummy and analysis based on the new operators and parameters is independent of ς so long as we are looking only at a particular irreducible representation. One can even simply drop it. From a physics perspective, we have absorbed the value of ς by a choice of physical unit for measuring the observables corresponding to Y and E, here all in the unit of $\sqrt{\varsigma}$. For ς being negative, we should switch $\hat{Y}^L_\varsigma$ with $\hat{E}^L_\varsigma$ first; i.e., we take

$$\hat{X}^L_{(\varsigma)} \equiv \frac{1}{\sqrt{|\varsigma|}}\hat{E}^L_\varsigma = x_{(\varsigma)} + i\partial_{p_{(\varsigma)}} \;,$$
$$\hat{P}^L_{(\varsigma)} \equiv \frac{1}{\sqrt{|\varsigma|}}\hat{Y}^L_\varsigma = p_{(\varsigma)} - i\partial_{x_{(\varsigma)}} \;,$$

achieved by taking $x_{(\varsigma)} = -\sqrt{|\varsigma|}p$ and $p_{(\varsigma)} = -\sqrt{|\varsigma|}x$. The result for $V^L_\varsigma(p_{(\varsigma)}, x_{(\varsigma)}, \theta_{(\varsigma)})$ still maintains. (From the physical point of view, the representations corresponding to different values of ς can be seen as describing the same physics. The parameter ς may then be taken as the physical constant $\frac{\hbar c^2}{2}$. For that matter, ς cannot be negative. Physicists identify the symmetry algebra from a relevant representation with $\hat{X}^L_{(\varsigma)}$ and $\hat{P}^L_{(\varsigma)}$ as the position and momentum observables satisfying $[\hat{X}^L_{(\varsigma)\mu}, \hat{P}^L_{(\varsigma)\nu}] = 2i\eta_{\mu\nu}$, in the $\hbar = 2$ units. However, the mathematical case of a product of two representations with different ς values may have interesting physics implications if a composite physical system corresponding to that exists in nature.) ς can actually be seen as the eigenvalue of I, essentially the Casimir operator. The semi-direct product structure $H_R(1,3) = H(1,3) \rtimes SO(1,3)$ says that with each irreducible representation of the subgroup $H(1,3) \rtimes S_{\hat{O}}$, where $S_{\hat{O}} \subseteq SO(1,3)$ is the stability subgroup for an orbit $\hat{O}$ of $SO(1,3)$ in the space of equivalent classes of irreducible representations of $H(1,3)$, one can associate an induced representation which is irreducible [33]. We have seen that, apart from the set of measure zero, each of which only gives one-dimensional representations, the irreducible representations are characterized by the nonzero value of ς and the representations (though mathematically nonequivalent) can be cast in the same form as $V^L_\varsigma(p_{(\varsigma)}, x_{(\varsigma)}, \theta_{(\varsigma)})$. It is obvious that the representation is invariant under the $SO(1,3)$ transformations, hence each is an independent orbit. That is to say $S_{\hat{O}} = SO(1,3)$. The fact is of paramount importance for unambiguously identifying the nature of the coherent states below. In view of the discussion above, we can see that for any of the $V^L_\varsigma(p_{(\varsigma)}, x_{(\varsigma)}, \theta_{(\varsigma)})$ representation, we can simply write it in the simple notation $V^L(p,x,\theta)$, like taking the $\varsigma = 1$ case as a representative. That is essentially what has been done in Ref. [1] for the $H(3)$ or $H_R(3)$ case. However, for the reason to be clear below, we keep the explicit ς-notation for the most part of the manuscript.

The standard approach is to introduce the abstract canonical coherent states as

$$\left| p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)} \right\rangle \equiv V_\varsigma(p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)})|0,0\rangle \equiv e^{-i\theta_{\scriptscriptstyle(\varsigma)}} V_\varsigma(p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)}, \theta_{\scriptscriptstyle(\varsigma)})|0,0\rangle \;, \tag{10}$$

for

$$V_\varsigma(p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)}, \theta_{\scriptscriptstyle(\varsigma)}) \equiv e^{i(p^\mu_{\scriptscriptstyle(\varsigma)} \hat{X}_{\scriptscriptstyle(\varsigma)\mu} - x^\mu_{\scriptscriptstyle(\varsigma)} \hat{P}_{\scriptscriptstyle(\varsigma)\mu} + \theta_{\scriptscriptstyle(\varsigma)} \hat{I})} \;, \tag{11}$$

representing the $H(1,3)$ group element $W(p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)}, \theta_{\scriptscriptstyle(\varsigma)})$ satisfying the group product

$$W(p'^\mu_{\scriptscriptstyle(\varsigma)}, x'^\mu_{\scriptscriptstyle(\varsigma)}, \theta'_{\scriptscriptstyle(\varsigma)}) W(p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)}, \theta_{\scriptscriptstyle(\varsigma)}) = W\Big(p'^\mu_{\scriptscriptstyle(\varsigma)} + p^\mu_{\scriptscriptstyle(\varsigma)}, x'^\mu_{\scriptscriptstyle(\varsigma)} + x^\mu_{\scriptscriptstyle(\varsigma)}, \theta'_{\scriptscriptstyle(\varsigma)} + \theta_{\scriptscriptstyle(\varsigma)} - (x'_{\scriptscriptstyle(\varsigma)\mu} p^\mu_{\scriptscriptstyle(\varsigma)} - p'_{\scriptscriptstyle(\varsigma)\mu} x^\mu_{\scriptscriptstyle(\varsigma)})\Big). \tag{12}$$

Each group element can be identified with a point in the $H_R(1,3)/SO(1,3)$ coset space [14,17]. $\hat{X}_{\scriptscriptstyle(\varsigma)}$ and $\hat{P}_{\scriptscriptstyle(\varsigma)}$ are operators on the abstract representation space $\mathcal{H}_\varsigma$ spanned by the $\left| p^\mu_{\scriptscriptstyle(\varsigma)}, x^\mu_{\scriptscriptstyle(\varsigma)} \right\rangle$ vectors, and $|0,0\rangle = |0_{\scriptscriptstyle(\varsigma)}\rangle$ is a fiducial normalized cyclic vector corresponding to the points $(0,0,\theta_{\scriptscriptstyle(\varsigma)})$ in the coset space, each of which is fixed under $SO(1,3)$ transformations.

4. Pseudo-Hermitian Nature of the Representation of Symmetry Generators from the Fock States

The kind of operator representation of the four-vector observables given in Equation (9) would be naively seen as Hermitian, hence the full representation of the $H_R(1,3)$ group as unitary. To be more careful, the unitarity of a representation is really to be defined with respect to the inner product assumed for the representation space. The operator representation sure is Hermitian with respect to the usual squared-integral inner product, (with bar denoting the complex conjugation),

$$\langle \phi | \phi' \rangle = \frac{1}{\pi^4} \int d^4p\, d^4x\, \bar{\phi}(p^\mu, x^\mu)\, \phi'(p^\mu, x^\mu) \;, \tag{13}$$

for the wavefunctions that vanish at infinity. In the equivalent formulation in terms of standard Schrödinger wavefunctions $\psi(x^\mu)$, that is exactly the unitary representation given explicitly first in 1966 [18,19]. The short-comings of the formulation are best seen in the covariant harmonic oscillator problem [18,20]. We illustrate them explicitly in the Appendix A. To illuminate the pseudo-Hermitian nature of our representation, we present in the following the pseudo-unitary Fock space and complete the coherent states representation, together with the appropriate Lorentz invariant integral inner product. For convenience, in this section we drop the $_{\scriptscriptstyle(\varsigma)}$ and $_\varsigma$ subscripts.

We start with summarizing the more transparent abstract algebraic results for the Fock space [20,22] in a better logic and notation. For the Hamiltonian $\hat{X}_\mu \hat{X}^\mu + \hat{P}_\mu \hat{P}^\mu$, we consider $\hat{a}^\mu = \eta^{\mu\nu}(\hat{X}_\nu + i\hat{P}_\nu)$, $\hat{a}^{\dagger\eta}_\mu = \hat{X}_\mu - i\hat{P}_\mu$ and $\hat{N}_{(\mu)} = \frac{1}{4}\hat{a}^{\dagger\eta}_\mu \hat{a}^\mu$ without summation (the index (μ) is not a vector one) with

$$[\hat{N}_{(\mu)}, \hat{a}^\nu] = -\eta^\nu_\mu \hat{a}^\mu \;, \qquad [\hat{N}_{(\mu)}, \hat{a}^{\dagger\eta}_\nu] = \eta^\mu_\nu \hat{a}^{\dagger\eta}_\mu \;; \qquad \left[\hat{a}^\mu, \hat{a}^{\dagger\eta}_\nu\right] = 4\eta^\mu_\nu \;. \tag{14}$$

Here, we introduce the $^{\dagger\eta}$ notation for a yet unspecified $\hat{\eta}$, not excluding that being trivial, requiring however the η-Hermiticity of $\hat{X}_\mu$ and $\hat{P}_\mu$. Note that the feasible inner product is still to be determined. The Fock states are simultaneous eigenstates of the $\hat{N}_{(\mu)}$, and hence also the $\hat{N} = \sum \hat{N}_{(\mu)}$, operators. The last one is of course Lorentz invariant.

$$|n\rangle \equiv |n_0; n_1, n_2, n_3\rangle = \frac{1}{2^n \sqrt{n_0!\, n_1!\, n_2!\, n_3!}} \left(\hat{a}^{\dagger\eta}_0\right)^{n_0} \left(\hat{a}^{\dagger\eta}_1\right)^{n_1} \left(\hat{a}^{\dagger\eta}_2\right)^{n_2} \left(\hat{a}^{\dagger\eta}_3\right)^{n_3} |0\rangle \;, \tag{15}$$

with

$$\hat{a}^0|n_0;n_1,n_2,n_3\rangle = 2\sqrt{n_0}|n_0-1;n_1,n_2,n_3\rangle\ ,$$
$$\hat{a}_0^{\dagger^\eta}|n_0;n_1,n_2,n_3\rangle = 2\sqrt{(n_0+1)}|n_0+1;n_1,n_2,n_3\rangle\ , \tag{16}$$

and the exact corresponding results for $\hat{a}^i|n\rangle$ and $\hat{a}_i^{\dagger^\eta}|n\rangle$. The $H_R(1,3)$ canonical coherent states, satisfying $\hat{a}^\nu|p^\mu,x^\mu\rangle = 2(x^\nu + ip^\nu)|p^\mu,x^\mu\rangle$, can be expanded as

$$\begin{aligned}|p^\mu,x^\mu\rangle \quad &= \quad e^{-\frac{x_\mu x^\mu + p_\mu p^\mu}{2}}\sum\frac{1}{\sqrt{n_0!\,n_1!\,n_2!\,n_3!}}(x^0+ip^0)^{n_0}(x^1+ip^1)^{n_1}\\ &\qquad\times(x^2+ip^2)^{n_2}(x^3+ip^3)^{n_3}\,|n_0;n_1,n_2,n_3\rangle\ ,\end{aligned} \tag{17}$$

with $|0,0\rangle = |0\rangle$. Moreover, those are exactly the states defined earlier, obtained by an action of $V(p^\mu,x^\mu) = e^{i(p^\mu\hat{X}_\mu - x^\mu\hat{P}_\mu)}$ on $|0\rangle$ state, or equivalently

$$V(p^\mu,x^\mu)|0\rangle = e^{-\frac{x_\mu x^\mu + p_\mu p^\mu}{2}}e^{\frac{(x^\nu+ip^\nu)\hat{a}_\nu^{\dagger^\eta}}{2}}|0\rangle\ .$$

The right inner product to complete the familiar algebra of the problem is, however, nontrivial. While the operators $\hat{a}^\mu$, $\hat{a}_\mu^{\dagger^\eta}$ and $\hat{N}_{(\mu)}$ with the commutation relation of Equation (14) give a convenient generalization of the $\hat{a}_i \equiv \hat{a}^i$, $\hat{a}_i^\dagger$ and $\hat{N}_{(i)}$ system, insensitive to the metric signature yet having the right Lorentz transformation properties of the Fock state solutions built in, $\hat{a}_0^{\dagger^\eta}$ is not a naive Hermitian conjugate of $\hat{a}_0$. In fact, we can obtain from Equation (16) that $\langle\hat{a}^\mu\cdot|\cdot\rangle = \left\langle\cdot|\hat{a}_\mu^{\dagger^\eta}\cdot\right\rangle$ when the usual orthonormality $\langle m|n\rangle = \delta_{mn}$ is assumed. We need a new inner product defined as

$$_\eta\langle m|n\rangle = (-1)^{n_0}\,\delta_{mn}\ , \tag{18}$$

i.e., $\hat{\eta} = \sum(-1)^{n_0}|n\rangle\langle n|$, with the corresponding η-Hermitian conjugation, $\hat{a}_\mu^{\dagger^\eta} = \hat{\eta}^{-1}\hat{a}_\mu^\dagger\hat{\eta}$, giving $_\eta\langle\hat{a}_\mu\cdot|\cdot\rangle = {}_\eta\left\langle\cdot|\hat{a}_\mu^{\dagger^\eta}\cdot\right\rangle$. Note that specifying the inner product for a complete basis uniquely defines the inner product over the whole space. One can easily see that $(\hat{a}^\mu)^\dagger = \hat{a}_\mu^{\dagger^\eta}$ implies Hermiticity of $\hat{X}_i$ and $\hat{P}_i$ operators, while from equating $(\hat{a}^0)^\dagger = \left(\hat{X}^0 + i\hat{P}^0\right)^\dagger$ with $\hat{a}_0^{\dagger^\eta}$, we see that $\hat{X}_0$ and $\hat{P}_0$ are anti-Hermitian with respect to the usual inner product, i.e., the one with $\hat{\eta}$ being the identity. However, all of the $H_R(1,3)$ generators are represented by pseudo-Hermitian, or η-Hermitian, operators. The operators all have real spectra, as we show explicitly below. Explicitly, $\hat{X}_\mu$, $\hat{P}_\mu$ and $\hat{J}_{\mu\nu} = \hat{X}_\mu\hat{P}_\nu - \hat{X}_\nu\hat{P}_\mu$ (and $\hat{I}$) all satisfy Equation (1) in the place of $\hat{A}_H$, and we have a pseudo-unitary representation with invariant inner product $_\eta\langle\cdot|\cdot\rangle$. In particular, the coherent states are normalized as $_\eta\langle 0|0\rangle = {}_\eta\langle p^\mu,x^\mu|p^\mu,x^\mu\rangle = 1$, hiding the inner product indefiniteness. We can see the latter either through the explicit use of Equation (17), or directly from the fact that $V(p^\mu,x^\mu)$ in Equation (10) is an η-unitary operator. From the definition of $\hat{\eta}$ in Fock basis and Equation (17), we obtain

$$_\eta\langle p^\mu,x^\mu| = \langle p^\mu,x^\mu|\hat{\eta} = \langle p_\mu,x_\mu|\ , \tag{19}$$

showing explicitly the metric operator $\hat{\eta}$ is a direct manifestation of the Minkowski metric in the Krein space of our quantum theory, exactly as we are looking for [21].

As the state $|0,0\rangle = |0\rangle$ has zero expectation values for the $\hat{X}_\mu$ and $\hat{P}_\mu$ operators, we get

$$_\eta\left\langle p^\mu,x^\mu|\hat{X}_\nu|p^\mu,x^\mu\right\rangle = 2x_\nu\ ,$$
$$_\eta\left\langle p^\mu,x^\mu|\hat{P}_\nu|p^\mu,x^\mu\right\rangle = 2p_\nu\ . \tag{20}$$

The generic wavefunctions can be introduced as $\phi(p^\mu, x^\mu) \equiv {}_\eta\langle p^\mu, x^\mu|\phi\rangle$, satisfying

$$\begin{aligned} {}_\eta\langle p^\mu, x^\mu|\hat{X}_\nu|\phi\rangle &= \hat{X}^L_\nu \phi(p^\mu, x^\mu) , \\ {}_\eta\langle p^\mu, x^\mu|\hat{P}_\nu|\phi\rangle &= \hat{P}^L_\nu \phi(p^\mu, x^\mu) , \end{aligned} \tag{21}$$

with

$$\begin{aligned} \hat{X}^L_\mu &= x_\mu + i\partial_{p^\mu} , \\ \hat{P}^L_\mu &= p_\mu - i\partial_{x^\mu} , \end{aligned} \tag{22}$$

exactly in the form of Equation (9) and

$$\begin{aligned} &V^L(p^\mu, x^\mu)\phi(p'^\mu, x'^\mu) \equiv {}_\eta\langle p'^\mu, x'^\mu|V(p^\mu, x^\mu)|\phi\rangle \\ &\quad = \phi(p'^\mu - p^\mu, x'^\mu - x^\mu) e^{i(x'_\mu p^\mu - p'_\mu x^\mu)} . \end{aligned} \tag{23}$$

We see that the abstract formulation from the set of canonical coherent states based on the $H(1,3)$ manifold and the one from the irreducible component of the regular representation are really the same one. Wavefunction of a coherent state labeled by A is given by

$$\phi_A(p^\mu, x^\mu) \equiv {}_\eta\left\langle p^\mu, x^\mu|p^\mu_A, x^\mu_A\right\rangle = e^{i\left(x_\mu p^\mu_A - p_\mu x^\mu_A\right)} e^{-\frac{1}{2}\left[(x-x_A)^2 + (p-p_A)^2\right]} , \tag{24}$$

where $(x - x_A)^2$ and $(p - p_A)^2$ are the Minkowski vector magnitude squares, and can be seen as a special case of Equation (23), namely

$$\phi_A(p^\mu, x^\mu) = {}_\eta\left\langle p^\mu, x^\mu\middle|V(p^\mu_A, x^\mu_A)\middle|0, 0\right\rangle = V^L(p^\mu_A, x^\mu_A)\phi_o(p^\mu, x^\mu) . \tag{25}$$

In particular, we have

$$\phi_o(p^\mu, x^\mu) = {}_\eta\langle p, x|0\rangle = e^{-\frac{p_\mu p^\mu + x_\mu x^\mu}{2}} ,$$

which is the Lorentz invariant symmetric Gaussian.

To obtain the inner product on the space of wavefunctions, one simply has to look for the proper resolution of the identity operator on the Krein space. We have

$$\hat{I} \;=\; \sum |n\rangle_\eta\langle n|\hat{\eta} = \int d^3p d^3x dp^0 dx^0 \frac{e^{-2\left(x^0\right)^2 - 2\left(p^0\right)^2}}{\pi^4} |p^\mu, x^\mu\rangle_\eta\langle p^\mu, x^\mu|\hat{\eta} . \tag{26}$$

Therefore, the functional ${}_\eta\langle\psi|$ is represented on the space of $\phi(p^\mu, x^\mu)$ as

$$\int d^3p d^3x dp^0 dx^0 \frac{e^{-2\left(x^0\right)^2 - 2\left(p^0\right)^2}}{\pi^4} \bar{\psi}(p^i, x^i, -p^0, -x^0) \left(\cdot \right) ,$$

with the very nontrivial integration measure. The inner product ${}_\eta\langle\psi|\phi\rangle$ is then given by

$${}_\eta\langle\psi|\phi\rangle = \frac{1}{\pi^4} \int d^3p d^3x dp^0 dx^0 \frac{\bar{\psi}(p^i, x^i, -p^0, -x^0)}{e^{(x^0)^2 + (p^0)^2}} \frac{\phi(p^\mu, x^\mu)}{e^{(x^0)^2 + (p^0)^2}} . \tag{27}$$

Each of the basis functions $\phi_n(p^\mu, x^\mu)$, and hence any general $\phi(p^\mu, x^\mu)$ in the spanned space, is formally divergent at timelike infinity of the four-vector variables. On the other hand, all $\frac{\phi_n(p^\mu, x^\mu)}{e^{(x^0)^2 + (p^0)^2}}$, and hence all $\frac{\phi(p^\mu, x^\mu)}{e^{(x^0)^2 + (p^0)^2}}$, are rapidly decreasing functions. The factor $e^{-\left(x^0\right)^2 - \left(p^0\right)^2}$ takes the $e^{\frac{\left(x^0\right)^2 + \left(p^0\right)^2}{2}}$ factor in all $\phi_n(p^\mu, x^\mu)$ back to $e^{-\frac{\left(x^0\right)^2 + \left(p^0\right)^2}{2}}$, which characterizes the class of functions. The integral is finite for all wavefunctions as finite linear combinations of the Fock state basis ϕ_n. Using $\frac{\phi(p^\mu,x^\mu)}{e^{(x^0)^2+(p^0)^2}}$ as the wavefunctions cannot be correct, though. That would, for example, make the wavefunction for $|0\rangle$ not Lorentz invariant and mess up the right transformation properties of all those for the Fock states, described in Ref. [22]. Thinking further about $\psi^*(p^i, x^i, -p^0, -x^0)$ as $\psi^*(p_\mu, x_\mu)$, one can see in hindsight that the inner product expression is indeed exactly what it should be. Of course we have that here rigorously established.

One can now easily show that the Fock states wavefunctions $\phi_n(p^\mu, x^\mu)$ have the proper norm ± 1, and therefore are non-divergent without restricting the domain. The analytical feature is much better than that of the unitary representation (see the Appendix A). Note that, other than having a different inner product with a nontrivial integration measure, our formulation in terms of the wavefunctions and differential operator representation of the $\hat{X}_\mu$ and $\hat{P}_\mu$ really are the same as the usual unitary one. That illustrates clearly that the basic observables $\hat{X}_\mu$ and $\hat{P}_\mu$, as well as other observables in the form of their real functions, all have the same eigenvalues and eigenfunctions. In particular, the spectra are real.

5. Lorentz to Galilean Contraction

A contraction [34,35] of the Lorentz symmetry $SO(1,3)$, sitting inside the $H_R(1,3)$, to the Galilean $ISO(3)$ has been discussed in Ref. [17], together with the corresponding coset spaces of interest. The full (quantum) relativity symmetry group obtained by contraction is named $H_{GH}(3)$, with commutators among generators essentially given by

$$\begin{aligned}
&[J_{ij}, J_{hk}] = 2i(\delta_{jk}J_{ih} + \delta_{ih}J_{jk} - \delta_{ik}J_{jh} - \delta_{jh}J_{ik}) \,, \\
&[J_{ij}, X_k] = -2i(\delta_{jk}X_i - \delta_{ik}X_j) \,, \qquad [J_{ij}, P_k] = -2i(\delta_{jk}P_i - \delta_{ik}P_j) \,, \\
&[J_{ij}, K_k] = -2i(\delta_{jk}K_i - \delta_{ik}K_j) \,, \qquad [K_i, K_j] = 0 \,, \\
&[K_i, H] = -2iP_i \,, \qquad [K_i, P_j] = 0 \,, \qquad [X_i, P_j] = 2i\delta_{ij}I' \,, \\
&[T, H] = -2iI' \,, \qquad [K_i, T] = 0 \,, \qquad [K_i, X_j] = -2i\delta_{ij}T \,.
\end{aligned} \tag{28}$$

Note that the full result for the other commutators beyond the J_{ij} and K_i set, originated from $SO(1,3)$, is essentially fixed by the requirement of having the Galilean K_i-H and the Heisenberg X-P commutators. However, for the purpose here, the explicit contraction is to be implemented a bit differently. It is taken as the $c \to \infty$ limit of $K_i = \frac{1}{c}J_{i0}$, $P_i = \frac{1}{c}E_i$, $X_i = \frac{1}{c}Y_i$,$T = \frac{-1}{c^2}Y_0$, $I' = \frac{1}{c^2}I$, with the renaming $H \equiv -E_0$. In the contraction, K_i, as generators for the Galilean boosts are the basic starting point, and we would like to be able to trace physics, including the relative physical dimensions of quantities, by considering the speed of light c as having a physical dimension. Introducing $X_i = \frac{1}{c}Y_i$ is to keep the same physical dimensions for X_i and P_i. However, the essence of the contraction scheme as a formulation to retrieve an approximate physical theory from a more exact one is really to implement the contraction at a representation level.

To implement the contraction on V_ς^L, or the matching V_ς as a representation of the original $H(1,3)$, it is important to note that the original central charge generator I represented by $\varsigma\hat{I}$ in V_ς would give the representation of the contracted I', which remains central, as $\frac{\varsigma}{c^2}\hat{I}$. For a sensible result, one needs to consider $\varsigma = c^2\chi$ with χ staying finite at the contraction limit, hence I' represented by $\chi\hat{I}$ (recall: $\hat{I}$ is the identity operator). Therefore, V_ς contracts into V_χ. In other words, the V_ς representation of the original $H(1,3)$, and the full $H_R(1,3)$, survives as the V_χ ($\chi = \frac{\varsigma}{c^2} > 0$) representation of the $H(3)$ in the contracted $H_{GH}(3)$, as well as of the full group.

For the $c \to \infty$ limit of $V_\chi^L(p^\mu_{(\varsigma)}, x^\mu_{(\varsigma)})$, we have to consider first

$$\hat{P}^L_{\chi i} = \frac{1}{c}\hat{E}^L_{\varsigma i} \,, \quad \hat{X}^L_{\chi i} = \frac{1}{c}\hat{Y}^L_{\varsigma i} \,, \quad \hat{H}^L_\chi = -\hat{E}^L_{\varsigma 0} \,, \quad \hat{T}^L_\chi = -\frac{1}{c^2}\hat{Y}^L_{\varsigma 0} \,,$$

and take that to obtain

$$\hat{X}^L_{(\chi)i} = \frac{1}{\sqrt{\chi}} \hat{X}^L_{\chi i} = \hat{X}^L_{(\varsigma)i} \,, \qquad \hat{P}^L_{(\chi)i} = \frac{1}{\sqrt{\chi}} \hat{P}^L_{\chi i} = \hat{P}^L_{(\varsigma)i} \,,$$
$$\hat{T}^L_{(\chi)} = \frac{1}{\sqrt{\chi}} \hat{T}^L_{\chi} = -\frac{1}{c} \hat{X}^L_{(\varsigma)0} \,, \qquad \hat{H}^L_{(\chi)} = \frac{1}{\sqrt{\chi}} \hat{H}^L_{\chi} = -c \hat{P}^L_{(\varsigma)0} \,, \tag{29}$$

(with $\varsigma = c^2\chi$). The above are the basic set of operators acting on the functional space of $\phi(p_{(\varsigma)}, x_{(\varsigma)})$, with the variables properly rescaled to a new set of variables to match with the operators. There is also the exactly corresponding set of operators, $\hat{X}_{(\chi)i}$, $\hat{P}_{(\chi)i}$, $\hat{T}_{(\chi)}$, and $\hat{H}_{(\chi)}$, and V_χ on the abstract Hilbert space which are helpful for tracing the proper description. The proper labels for the states $\left|p^\mu_{(\varsigma)}, x^\mu_{(\varsigma)}\right\rangle$ at the contraction limit should be $\left|p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)}\right\rangle$, satisfying

$$2x_{(\chi)i} = {}_\eta\!\left\langle p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \middle| \hat{X}_{(\chi)i} \middle| p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \right\rangle \,,$$
$$2p_{(\chi)i} = {}_\eta\!\left\langle p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \middle| \hat{P}_{(\chi)i} \middle| p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \right\rangle \,,$$
$$2t_{(\chi)} = {}_\eta\!\left\langle p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \middle| \hat{T}_{(\chi)} \middle| p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \right\rangle \,,$$
$$2e_{(\chi)} = {}_\eta\!\left\langle p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \middle| \hat{H}_{(\chi)} \middle| p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)} \right\rangle \,, \tag{30}$$

and hence giving naively

$$\phi(p^\mu_{(\varsigma)}, x^\mu_{(\varsigma)}) \quad \longrightarrow \quad \phi(p^i_{(\chi)}, e_{(\chi)}, x^i_{(\chi)}, t_{(\chi)})$$

with

$$x_{(\chi)i} = x^i_{(\chi)} = x^i_{(\varsigma)} \,, \qquad p_{(\chi)i} = p^i_{(\chi)} = p^i_{(\varsigma)} \,,$$
$$t_{(\chi)} = \frac{1}{c}\, x^0_{(\varsigma)} \,, \qquad e_{(\chi)} = c\, p^0_{(\varsigma)} \,. \tag{31}$$

We have then, at least formally,

$$\hat{X}^L_{(\chi)i} = x_{(\chi)i} + i\partial_{p^i_{(\chi)}} \,, \qquad \hat{P}^L_{(\chi)i} = p_{(\chi)i} - i\partial_{x^i_{(\chi)}} \,,$$
$$\hat{T}^L_{(\chi)} = t_{(\chi)} - i\partial_{e_{(\chi)}} \,, \qquad \hat{H}^L_{(\chi)} = e_{(\chi)} + i\partial_{t_{(\chi)}} \,. \tag{32}$$

The crucial quantities controlling the nature of the representation are the overlaps

$${}_\eta\!\left\langle p^i_{(\chi)B}, e_{(\chi)B}, x^i_{(\chi)B}, t_{(\chi)B} \middle| p^i_{(\chi)A}, e_{(\chi)A}, x^i_{(\chi)A}, t_{(\chi)A} \right\rangle .$$

From the original ${}_\eta\!\left\langle p^\mu_{(\chi)B}, x^\mu_{(\chi)B} \middle| p^\mu_{(\chi)A}, x^\mu_{(\chi)A} \right\rangle$, given in Equation (24), we have it as

$$e^{i\left(e_{(\chi)B}t_{(\chi)A} - t_{(\chi)B}e_{(\chi)A} + \delta_{ij}x^i_{(\chi)B}p^j_{(\chi)A} - \delta_{ij}p^i_{(\chi)B}x^j_{(\chi)A}\right)} e^{-\frac{1}{2}\left[\left(x^i_{(\chi)B} - x^i_{(\chi)A}\right)^2 - c^2\left(t_{(\chi)B} - t_{(\chi)A}\right)^2 + \left(p^i_{(\chi)B} - p^i_{(\chi)A}\right)^2 - \frac{1}{c^2}\left(e_{(\chi)B} - e_{(\chi)A}\right)^2\right]}$$

to be taken at the $c \to \infty$ limit. It holds $e^{\frac{1}{2c^2}\left(e_{(\chi)B} - e_{(\chi)A}\right)^2} \to 1$, but the $e^{\frac{c^2}{2}\left(t_{(\chi)B} - t_{(\chi)A}\right)^2}$ factor diverges in the limit, except for $t_{(\chi)B} = t_{(\chi)A}$, which indicates that we should consider only the latter case. The magnitude of the overlap being independent of $e_{(\chi)B}$ and $e_{(\chi)A}$ is still puzzling. The answer to that comes from a more careful thinking about the nature of the variables $e_{(\chi)}$. Unlike $t_{(\chi)} = \frac{x^0_{(\varsigma)}}{c}$, which is to be taken to be finite as in the general spirit of symmetry contraction, $e_{(\chi)} = c\, p^0$ is of quite different nature. The Lie algebra contraction to begin with only has a relabeling $H = -E_0$ involving no c. One may wonder if the c in

$\hat{H}^L_{(x)} = -c\hat{P}^L_{(c)0}$ should be taken as giving a diverging energy observable $\hat{H}^L_{(x)}$ for any finite $\hat{P}^L_{(c)0}$. Furthermore, for an Einstein particle of the rest mass m, i.e., the particle in Einstein's theory of special relativity,

$$e = mc^2 + \frac{p_i p^i}{2m} + \cdots$$

where the neglected terms involve negative powers of c^2. At the $c \to \infty$ limit, it is indeed diverging. Even p^0 is diverging. That is the result of the rest mass as an energy. Hence, it suggests that we should take our variable $e_{(x)}$ as infinite, and the 'non-relativistic' energy we are interested in is the kinetic energy $\frac{p_i p^i}{2m}$ given by the limit of $e - mc^2$. Taking that feature into our consideration, the Hilbert space of interest under the contraction is really only the space spanned by the $H(3)$ coherent states $\left| p^i_{(x)}, x^i_{(x)} \right\rangle$ for a fixed time $t_{(x)}$ and a formally infinite $e_{(x)}$. To be exact, we should be implementing that logic from an Einstein particle to our quantum observables $\hat{H}_{(x)}$, $\hat{P}^0_{(x)}$, and $\hat{P}_{(x)i}$ or their expectation values, but the conclusion is the same. The coherent state wavefunction $\phi_A(p^\mu_{(c)}, x^\mu_{(c)})$ is equal to ${}_\eta\left\langle p^\mu_{(c)}, x^\mu_{(c)} | p^\mu_{(c)A}, x^\mu_{(c)A} \right\rangle$, hence at the contraction limit there is no more dependence on $t_{(x)}$ and $e_{(x)}$, reducing it essentially to just $\phi_A(p^i_{(x)}, x^i_{(x)})$. The operator $\hat{T}^L_{(x)}$ acts on the space of wavefunctions only as a multiplication by $t_{(x)}$ and is just like classical, while $\hat{H}^L_{(x)}$ is not physically relevant. Note that the full contracted representation is then simply unitary. The part of the inner product ${}_\eta\langle \cdot | \cdot \rangle$ independent of $p^0_{(c)}$ and $x^0_{(c)}$, hence $t_{(x)}$ and $e_{(x)}$, is exactly the usual one, i.e., $\hat{\eta}$ essentially reduces to identity under the contraction. The space of wavefunctions spanned by $\phi_A(p^i_{(x)}, x^i_{(x)})$ is a Hilbert space.

6. Group Theoretically-Based WWGM Framework with Wavefunctions in Coherent State Basis

The above analysis gives a successful picture of the phase space of the $H_R(1,3)$ theory, giving in the Galilean limit the phase space of the $H_R(3)$ theory at each fixed 'time' value. The infinite dimensional manifolds give, at the proper relativity symmetry contraction limit, the familiar finite dimensional classical models as approximation. The explicit results of the classical limit for the present case are presented in the section below. The merit of our group theoretical approach is that it gives a full dynamical theory associated with the corresponding spacetime/phase space model for each relativity symmetry, mutually connected through the contraction/deformation pattern. The dynamical theory is naturally a Hamiltonian theory from the symmetry of the phase space as symplectic geometry. The dynamics is better described on the algebra of observables as essentially the matching representation of the group C*-algebra [1,14,21]. Moreover, all those fit in well with the idea of the position and momentum operators as noncommutative coordinates of the phase space [12,13,21].

6.1. The Algebra of Observables, Symmetries and Dynamics

The algebra of observables is depicted essentially as the one from a WWGM formalism, as functions and distributions of the position and momentum operators $\hat{X}_\mu$ and $\hat{P}_\mu$. The basic dynamical variables of our representation on the space of wavefunctions $\phi(p^\mu_{(c)}, x^\mu_{(c)})$ are $\hat{X}^L = x + i\partial_p = x\star$ and $\hat{P}^L = p - i\partial_x = p\star$, where we have dropped the μ indices and the subscript ${}_{(c)}$. We may also write a general function of $(p^\mu_{(c)}, x^\mu_{(c)})$ as simply $\alpha(p,x)$, and the $\star$ is as in the Moyal star product

$$\alpha \star \beta(p,x) = \alpha(p,x) e^{-i(\overleftarrow{\partial}_p \overrightarrow{\partial}_x - \overleftarrow{\partial}_x \overrightarrow{\partial}_p)} \beta(p,x), \tag{33}$$

with $\alpha(p,x)\star = \alpha(p\star, x\star)$. Under such notation, the story looks quite the same as the case for $H_R(3)$ with only $\hat{X}^L_i$, and $\hat{P}^L_i$ as $x_i\star$ and $p_i\star$, given in details in Ref. [1]. Hence, we present here only a summary of the results, leaving the readers to consult the latter paper and references therein.

Let us take a little detour first to clarify our theoretical perspective. What we have is rather like the WWGM put up-side-down [1]. We start with the quantum theory as an irreducible representation of a (quantum) relativity symmetry, including the Heisenberg–Weyl symmetry. With the wavefunction in the coherent state basis as the natural reduction of the representation of the group algebra, the corresponding representation of the latter properly extended serves as the algebra of observables. The latter can be seen as a collection of functions and tempered distribution of the position and momentum operators represented as differential operators by $x\star$ and $p\star$. The real variables x and p are not quite the coordinates of the classical phase space. Only their rescaled counterparts under the contraction of the symmetry to the classical relativity symmetry are. Contrary to a deformation quantization, a contraction is a de-quantization procedure. From the algebraic point of view, the deformation of an observable algebra as in WWGM is really a result of a deformation of the classical relativity symmetry to the quantum one, pushed onto the group C*-algebra of the symmetry. The contraction is exactly the inverse of the deformation [36], at a Lie algebra level and beyond.

In the usual unitary quantum mechanics, on the Hilbert space $\mathcal{K}$ of wavefunctions $\phi(p,x)$, symmetries are represented in a form of unitary and antiunitary operators, factored by its closed center of phase transformations. On the set $\mathcal{P}$ of pure state density operators $\rho_\phi(p,x)\star$, corresponding to the abstract projection operator $\hat{\rho}_\phi = |\phi\rangle\langle\phi|$ for normalized $|\phi\rangle$, the automorphism group $Aut(\mathcal{P})$ is characterized by the subgroup of the group of real unitary transformations $\mathcal{O}(\tilde{\mathcal{K}}_R)$ compatible with the star product, $\tilde{\mathcal{K}}_R$ being the real span of all $\rho_\phi(p,x)\star$, the complex extension of which is the Hilbert space of Hilbert-Schmidt operators, as in the Tomita representation. We write the unitary transformations in the form

$$\tilde{U}_\star \alpha\star = \mu(\alpha)\star = U_\star \star \alpha \star \bar{U}_\star\star \,,$$

with $\mu \in Aut(\mathcal{P})$, where $U_\star\star \equiv U_\star(p,x)\star$ is a unitary operator on $\mathcal{K}$, generated by the Hermitian operator in the form of a real function $G_s(p\star, x\star)$, and $\bar{U}_\star\star$ is its inverse obtained by the complex conjugation and $\tilde{U}_\star \in \mathcal{O}(\tilde{\mathcal{K}}_R)$. We refer to the $U_\star\star$ as star-unitary, in particular whenever necessary to highlight it being a function of the $p\star$ and $x\star$ operators.

The above, illustrated for the case of $H_R(3)$ formulation of standard quantum mechanics in Ref. [1], can be applied to our $H_R(1,3)$ case with a slight modification. We need to use the invariant inner product with $\hat{\rho}_\phi = |\phi\rangle_\eta\langle\phi|$ for normalized $|\phi\rangle$, and replace the Hermitian and unitary requirements by η-Hermitian and η-unitary ones. Our relevant symmetry transformations are to be given by η-unitary operator $V_{\star(s)}\star$ generated by η-Hermitian $G_s(p\star, x\star)$, which are real functions of the basic η-Hermitian operators $(p\star, x\star)$, i.e., $G_s(\hat{P}^L_\mu, \hat{X}^L_\mu) = \overline{G}_s(\hat{P}^L_\mu, \hat{X}^L_\mu)$, and we use the $\bar{\alpha}$ to denote the 'complex conjugate' of α as a function which correspond to $\bar{\alpha}\star$ as the η-Hermitian conjugate of $\alpha\star$ as an operators as an element of the observable algebra. The conjugation is the involution of the latter as a *-algebra. $\overline{V}_{\star(s)}\star$ of a η-unitary $V_{\star(s)}\star$ is to be interpreted in the same manner. The feature of $\overline{V}_{\star(s)}\star$ to be the inverse of $V_{\star(s)}\star$ is exactly η-unitarity. Again, η-Hermiticity is the Hermiticity so long as the algebraic analysis is concerned. Though η-unitarity here is really pseudo-unitarity, only the inner product preserving nature of it is relevant here and it is as good as unitarity. Of course, Krein spaces are to be allowed in the place of the Hilbert spaces.

Generators of our relativity symmetry $H_R(1,3)$ are to be represented as a subgroup of $Aut(\mathcal{P})$ of the observable algebra. All $H_R(1,3)$ generators are η-Hermitian, hence each is given by a real G_s, generating (star-)η-unitary $V_{\star(s)}\star = e^{\frac{-is}{2}G_s\star}$ as one-parameter groups of symmetry transformations. Note that the factor 2 is really $\hbar$. We have $\tilde{V}_{\star(s)} = e^{\frac{-is}{2}\tilde{G}_s}$,

$$\tilde{V}_\star \alpha\star = \mu(\alpha)\star = V_\star \star \alpha \star \overline{V}_\star\star \tag{34}$$

with

$$\tilde{G}_s\rho = G_s\star\rho - \rho\star G_s = 2i\{G_s, \rho\}_\star \,, \tag{35}$$

where $\rho(p,x) \in \tilde{\mathcal{K}}$ and $\{\cdot,\cdot\}_\star$ is the Moyal bracket. Hence, with $\rho(s) = \tilde{V}_{\star(s)}\rho(s=0)$,

$$\frac{d}{ds}\rho(s) = \{G_s, \rho(s)\}_\star \,. \tag{36}$$

The equation is the Liouville equation of motion for a mixed state ρ in $\tilde{\mathcal{D}}$, the self-dual cone of $\tilde{\mathcal{K}}$. The class of operators on $\tilde{\mathcal{K}}$ representing symmetry generators are important, especially for tracing the symmetries to the classical limit where all $G_s\star$ reduce essentially to the commutative G_s, as multiplicative operators on the functional space of classical observables. We can write $\tilde{G}_s = \hat{G}_s^L - \hat{G}_s^R$, where $\hat{G}_s^L \equiv G_s(p,x)\star = G_s(\hat{P}^L, \hat{X}^L)$ is a left action and $\hat{G}_s^R$ is the corresponding right action defined by $\hat{G}_s^R\alpha \equiv \alpha \star G_s(p,x) = G_s(\hat{P}^R, \hat{X}^R)\alpha$. Analogously to $\hat{X}^L$ and $\hat{P}^L$ coming from the left-invariant vector fields of the Heisenberg–Weyl group, there are those from the right-invariant ones given by

$$\hat{X}^R = x - i\partial_p \,, \qquad \hat{P}^R = p + i\partial_x \,. \tag{37}$$

From Equation (23), we see that

$$\begin{aligned}
V_{\star(-x'^\mu)}\star\phi(p^\mu, x^\mu) &= e^{\frac{-ix'^\mu}{2}(-p_\mu\star)}\phi(p^\mu, x^\mu) = \phi\left(p^\mu, x^\mu + \frac{x'^\mu}{2}\right)e^{\frac{ix'_\mu p^\mu}{2}} \,, \\
V_{\star(p'^\mu)}\star\phi(p^\mu, x^\mu) &= e^{\frac{-ip'^\mu}{2}(x_\mu\star)}\phi(p^\mu, x^\mu) = \phi\left(p^\mu + \frac{p'^\mu}{2}, x^\mu\right)e^{\frac{-ip'_\mu x^\mu}{2}} \,.
\end{aligned} \tag{38}$$

In the above, for the wavefunctions, we show only the involved pair of variables in each case, and there is always no summation over indices. The other variables are simply not affected by the transformations. In terms of the parameters x^μ and p^μ, we have

$$\begin{aligned}
G_{-x^\mu}\star &= p_\mu\star \,, & \tilde{G}_{-x^\mu} &= -2i\partial_{x^\mu} \,, \\
G_{p^\mu}\star &= x_\mu\star \,, & \tilde{G}_{p^\mu} &= 2i\partial_{p^\mu} \,,
\end{aligned} \tag{39}$$

all in the same form as in the $H_R(3)$ case. The factors of 2 in the translations $V_{\star(x)}\star$ and $V_{\star(p)}\star$, though somewhat suspicious at the first sight, are related to the fact that the arguments of the wavefunction correspond to half of the expectation values, due to our coherent state labeling. Thus, $x_\mu\star$ and $p_\mu\star$ generate translations of the expectation values, which is certainly the right feature to have. For the Lorentz transformations, we have $G_{\omega^{\mu\nu}} = (x_\mu p_\nu - x_\nu p_\mu)$,

$$\begin{aligned}
G_{\omega^{\mu\nu}}\star &= (x_\mu p_\nu - ix_\mu\partial_{x^\nu} + ip_\nu\partial_{p^\mu} + \partial_{x^\nu}\partial_{p^\mu}) - (\mu \leftrightarrow \nu) \,, \\
\tilde{G}_{\omega^{\mu\nu}} &= -2i(x_\mu\partial_{x^\nu} - p_\nu\partial_{p^\mu}) - (\mu \leftrightarrow \nu) \,.
\end{aligned} \tag{40}$$

with the explicit action (no summation over the indices)

$$V_{\star(\omega^{\mu\nu})}\star\phi(p,x) = e^{\frac{-i\omega^{\mu\nu}}{2}(G_{\omega^{\mu\nu}}\star)}\phi(p,x) = \phi\left(e^{\frac{i\omega^{\mu\nu}}{2}\widehat{G}_{\omega^{\mu\nu}}}[p,x]\right), \tag{41}$$

where $\widehat{G}_{\omega^{\mu\nu}}$ are the infinitesimal $SO(1,3)$ transformation operators corresponding to the coset space action to be obtained from Equation (5).

All the G_{-x^μ}, G_{p^μ} and $G_{\omega^{\mu\nu}}$ (and $G_\theta = 1$) make the full set of operators for the generators $\hat{G}_s^L = G_s\star$ of the $H_R(1,3)$ group representing the symmetry on $\mathcal{K}$, and constitute a Lie algebra within the algebra of physical observables. $\hat{G}_s^R$ set does the same as a right action, and $\hat{G}_s^L$ always commute with $\hat{G}_{s'}^R$ since, in general, $[\hat{\alpha}^L, \hat{\gamma}^R] = 0$. These fourteen G_s as multiplicative operators, of course, all commute among themselves. The commutators for $\tilde{G}_s$ are same as those for $\hat{G}_s^L$, with however the vanishing $\tilde{G}_\theta$ giving a vanishing $[\tilde{G}_{p^\mu}, \tilde{G}_{-x^\nu}]$. For any function $\alpha(p^\mu, x^\mu)$, there are four associated operators on $\tilde{\mathcal{K}}$. Those are $\alpha, \hat{\alpha}^L, \hat{\alpha}^R$ and $\tilde{\alpha}$, but only two of them are linearly independent. For our relativity symmetry operators, the independent set $\{G_{-x^\mu}, G_{p^\mu}, G_{\omega^{\mu\nu}}, \tilde{G}_{-x^\mu}, \tilde{G}_{p^\mu}, \tilde{G}_{\omega^{\mu\nu}}\}$ has the only non-vanishing commutators among them given by (we also have $G_\theta = 1$, the identity, and $\tilde{G}_\theta = 0$)

$$\begin{aligned}
&[G_{\omega^{\mu\nu}}, \tilde{G}_{\omega^{\alpha\beta}}] = 2i(\eta_{\nu\beta}G_{\omega^{\mu\alpha}} - \eta_{\nu\alpha}G_{\omega^{\mu\beta}} + \eta_{\mu\alpha}G_{\omega^{\nu\beta}} - \eta_{\mu\beta}G_{\omega^{\nu\alpha}})\ ,\\
&[G_{\omega^{\mu\nu}}, \tilde{G}_{-x^\alpha}] = -2i(\eta_{\nu\alpha}G_{-x^\mu} - \eta_{\mu\alpha}G_{-x^\nu})\ ,\\
&[G_{\omega^{\mu\nu}}, \tilde{G}_{p^\alpha}] = -2i(\eta_{\nu\alpha}G_{p^\mu} - \eta_{\mu\alpha}G_{p^\nu})\ ,\\
&[\tilde{G}_{\omega^{\mu\nu}}, G_{-x^\alpha}] = -2i(\eta_{\nu\alpha}G_{-x^\mu} - \eta_{\mu\alpha}G_{-x^\nu})\ ,\\
&[\tilde{G}_{\omega^{\mu\nu}}, G_{p^\alpha}] = -2i(\eta_{\nu\alpha}G_{p^\mu} - \eta_{\mu\alpha}G_{p^\nu})\ ,\\
&[G_{p^\mu}, \tilde{G}_{-x^\nu}] = -[G_{-x^\mu}, \tilde{G}_{p^\nu}] = 2i\eta_{\mu\nu}\ ,\\
&[G_{p^\mu}, \tilde{G}_{p^\nu}] = [G_{-x^\mu}, \tilde{G}_{-x^\nu}] = 0\ .
\end{aligned} \tag{42}$$

Quantum dynamics is completely symplectic, whether described in the Schrödinger picture in terms of real/complex coordinates of the (projective) Hilbert space or the Heisenberg picture as a description in terms of the noncommutative coordinates [12]. The explicit dynamical equation of motion is to be seen as the transformations generated by a physical Hamiltonian characterized by an evolution parameter. In the $H_R(3)$ case of the usual ('non-relativistic') quantum mechanics, it is $G_t = \frac{p_i p^i}{2m}$. For our $H_R(1,3)$ case, we consider $G_\tau = \frac{p_\mu p^\mu}{2m}$ with the parameter τ being the Einstein proper time, which is expected to give the standard covariant description of Einstein particle dynamics, as we see explicitly below.

For some s-dependent operator $\alpha(p^\mu(s), x^\mu(s))\star$ and a general Hamiltonian G_s, the Heisenberg equation of motion is given by

$$\frac{d}{ds}\alpha\star = \frac{1}{2i}[\alpha\star, G_s\star]\ . \tag{43}$$

The right-hand side of the equation is simply the Poisson bracket of $\alpha(p\star, x\star)$ and $G_s(p\star, x\star)$, functions of the noncommutative canonical variables $p^\mu\star$ and $x^\mu\star$. The equation can simply be written as

$$\frac{d}{ds}\alpha = \{\alpha, G_s\}_\star = \frac{-1}{2i}\tilde{G}_s\alpha\ , \tag{44}$$

and is exactly the differential version of the automorphism flow given in Equation (34), here with our η-unitary symmetry flows $V_{\star(s)}\star = e^{\frac{-is}{2}G_s\star}$ generated by a η-Hermitian $G_s\star$. $\frac{-1}{2i}\tilde{G}_s$ is really a Hamiltonian vector field for a Hamiltonian function $G_s(p\star, x\star)$ [12].

Our physical Hamiltonian operator $G_\tau(p\star)$ is such a η-Hermitian $G_s\star$. The corresponding Heisenberg equation gives, in particular,

$$\begin{aligned}
\frac{d}{d\tau}x_\mu\star &= \frac{1}{2i}\frac{1}{2m}[x_\mu\star, p_\nu\star p^\nu\star] = \frac{p_\mu\star}{m} = \frac{\partial G_\tau(p\star)}{\partial(p^\mu\star)}\ ,\\
\frac{d}{d\tau}p_\mu\star &= \frac{1}{2i}\frac{1}{2m}[p_\mu\star, p_\nu\star p^\nu\star] = 0 = -\frac{\partial G_\tau(p\star)}{\partial(x^\mu\star)}\ ,
\end{aligned} \tag{45}$$

which are exactly

$$\frac{d}{d\tau}\hat{X}^L_\mu = \frac{\partial G_\tau(\hat{P}^L_\nu)}{\partial\hat{P}^{L\mu}}\ , \qquad \frac{d}{d\tau}\hat{P}^L_\mu = -\frac{\partial G_\tau(\hat{P}^L_\nu)}{\partial\hat{X}^{L\mu}}\ , \tag{46}$$

the standard form of Hamilton's equations of motion for the canonical η-Hermitian operator coordinate pairs $\hat{X}^L_\mu$-$\hat{P}^L_\mu$. As usual in a Hamiltonian formulation, the constant, or τ-independent, momentum $\hat{P}^L_\mu$ is obtained from the equations of motion as velocity multiplied by the particle mass m. Here, $-m^2$ is just the constant value of $p_\nu p^\nu$ as $2mG_\tau$.

For the Schrödinger picture, as η-unitary flows on $\mathcal{K}$, we have the equation

$$\frac{d}{ds}\phi = \frac{1}{2i} G_s \star \phi \,, \tag{47}$$

which for $G_\tau\star$ gives the τ-independent solution for ϕ in the exact form of the Klein–Gordon equation, provided that the $G_\tau\star$ eigenvalue is taken to be $-\frac{m}{2}$. Explicitly, in terms of the basic variables p^μ and x^μ, we have

$$G_\tau \star \phi(p,x) = \frac{1}{2m} p_\mu \star p^\mu \star \phi(p,x) = \frac{1}{2m}\left(p^\mu p_\mu - \eta^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} - 2ip^\mu \partial_{x^\mu}\right)\phi(p,x) \,, \tag{48}$$

giving the wavefunctions $\phi(p,x) = e^{i(2k_\mu - p_\mu)x^\mu}$ for eigenvalues $\frac{2k^\mu k_\mu}{m}$. Eigenvalues of the momentum operators $p_\mu\star$ are $2k_\mu$, satisfying $(2k^\mu)(2k_\mu) = -m^2$. The factor of 2 really corresponds to $\hbar$, as in the standard textbook expression. Finally, the τ-dependence is then given by $\frac{d}{d\tau}\phi = -\frac{m}{4i}\phi$, as expected.

6.2. Lorentz to Galilean Contraction

Contraction to Galilean limit has been presented in Section 5 at the kinematical level. In this section, we present the corresponding contraction in the observable algebra given in the WWGM formalism. Recall that the original Krein space under the contraction becomes reducible into a sum of essentially identical irreducible components, each being spanned by the wavefunctions $\phi(p^i, x^i) \equiv \phi(p^i_{(\chi)}, x^i_{(\chi)})$ for a particular value of 'time' $t_{(\chi)}$. A general operator $\alpha(\hat{X}^L_\mu, \hat{P}^L_\mu)$ should then be seen as $\alpha(\hat{X}^L_i, \hat{P}^L_i, \hat{T}^L_{(\chi)}, \hat{H}^L_{(\chi)})$ with $\hat{X}^L_i \equiv \hat{X}^L_{(\chi)i}$ and $\hat{P}^L_i \equiv \hat{P}^L_{(\chi)i}$, from results of Equation (32). Hence, on $\phi(p^i, x^i)$ we have effectively Hermitian actions of operators $\hat{X}^L_i = x_i + i\partial_{p^i}$, $\hat{P}^L_i = p_i - i\partial_{x^i}$, $\hat{T}^L_{(\chi)} \to t_{(\chi)}$, and $\hat{H}^L_{(\chi)} \to e_{(\chi)}$, with the last two reduced to a simple multiplication by the 'variables' $t_{(\chi)}$ and (formally infinite) $e_{(\chi)}$, respectively. All $\alpha(p^\mu\star, x^\mu\star)$ operators on $\phi(p^i, x^i)$ reduce to $\alpha(p^i\star, x^i\star, t_{(\chi)}, e_{(\chi)})$, or rather simply to $\alpha(p^i\star, x^i\star)$ like in the basic quantum mechanics, a unitary representation theory of $H_R(3)$. The $\star$ should now be seen as the one involving only variables p^i and x^i.

The transformations generated by the Hermitian $G_{-x^i}\star, G_{p^i}\star$ and $G_{\omega^{ij}}\star$ obviously do not change. They represent generators of the $H_R(3)$ subgroup of $H_R(1,3)$ to begin with. $\tilde{G}_{-x^i}, \tilde{G}_{p^i}$ and $\tilde{G}_{\omega^{ij}}$ are also unchanged. $G_{-x^0}\star$ and $G_{p^0}\star$, representing $\hat{P}^L_{(\varsigma)0}$ and $\hat{X}^L_{(\varsigma)0}$, are to be replaced under the contraction by $\hat{H}^L_{(\chi)}$ and $\hat{T}^L_{(\chi)}$, respectively, with $V_{\star(-x^0)} = e^{\frac{ix^0}{2}G_{-x^0}}$ and $V_{\star(p^0)} = e^{\frac{-ip^0}{2}G_{p^0}}$ re-expressed as $V_{\star(t)} = e^{-\frac{it}{2}G_t}$ and $V_{\star(e)} = e^{\frac{ie}{2}G_{-e}}$, where $G_t\star = \hat{H}^L_{(\chi)}$ and $G_{-e}\star = \hat{T}^L_{(\chi)}$. On the wavefunction $\phi(p^i, x^i)$, we have the infinite $G_t\star = e_{(\chi)}$ and finite $G_{-e}\star = t_{(\chi)}$. We also have $\tilde{G}_t = 2i\partial_{t_{(\chi)}}$ and $\tilde{G}_{-e} = -2i\partial_{e_{(\chi)}}$. None of the four operators are of interest, so long as their action on the observable algebra for an irreducible representation $\phi(p,x)$ is concerned.

The other interesting ones to check are the Lorentz boosts under the contraction. The generator J_{i0} in the Lie algebra is replaced by the finite $K_i = \frac{1}{c}J_{i0}$. The group elements $e^{i\omega^{i0}J_{i0}}$ are to be re-expressed as $e^{i\beta^i K_i}$ with $\beta^i = c\,\omega^{i0}$. In the original representation, the J_{i0} action is given by $G_{\omega^{i0}}\star = \hat{X}^L_{(\varsigma)i}\hat{P}^L_{(\varsigma)0} - \hat{X}^L_{(\varsigma)0}\hat{P}^L_{(\varsigma)i}$, from which follows the action of K_i as

$$G_{\beta^i}\star = \hat{X}^L_i\left(\frac{-1}{c^2}\hat{H}^L_{(\chi)}\right) - (-\hat{T}^L_{(\chi)})\hat{P}^L_i \;\to\; t_{(\chi)}p_i\star = t_{(\chi)}G_{-x^i}\star$$

with $V_{\star(\beta^i)} = e^{\frac{-i\beta^i}{2}G_{\beta^i}}$ (no summation over i), a re-writing of $V_{\star(\omega^{i0})}$ with the new finite parameter β^i. We have seen, in Equation (38) explicitly, that $V_{\star(-x^i)}\star$ gives a translation in the variable x^i of the wavefunction. $V_{\star(\beta^i)}\star$ is then a time variable $t_{(\chi)}$-dependent translation, a Galilean boost exactly as the Lie algebra contraction promised, and is now unitary. Similarly, we have

$$\tilde{G}_{\beta^i} = \frac{1}{c}\tilde{G}_{\omega^{i0}} = -\frac{2i}{c^2}(x_i\partial_{t_{(x)}} + e_{(x)}\partial_{p^i}) + 2i(-t_{(x)}\partial_{x^i} - p_i\partial_{e_{(x)}}) \\ \to -2i(t_{(x)}\partial_{x^i} + p_i\partial_{e_{(x)}}) \; . \tag{49}$$

We keep the $\partial_{e_{(x)}}$ since the $\tilde{G}_{\beta^i}$ may act on the mixed states. We have the newly relevant nonzero commutators involving a G_{β^i}, G_t, or G_{-e}, and a $\tilde{G}_s$ as well as those involving a $\tilde{G}_{\beta^i}$, $\tilde{G}_t$, or $\tilde{G}_{-e}$ and a G_s, all from the generators of the Lie algebra, as

$$\begin{aligned}
&[G_{\beta^i}, \tilde{G}_{\omega^{jk}}] = -2i\Big(\delta_{ij}G_{\beta^k} - \delta_{ik}G_{\beta^j}\Big) \; , \\
&[G_{\omega^{ij}}, \tilde{G}_{\beta^k}] = 2i\Big(\delta_{ik}G_{\beta^j} - \delta_{jk}G_{\beta^i}\Big) \; , \\
&[G_{\beta^i}, \tilde{G}_t] = [\tilde{G}_{\beta^i}, G_t] = -2iG_{-x^i} \; , \\
&[G_{\beta^i}, \tilde{G}_{p^j}] = [\tilde{G}_{\beta^i}, G_{p^j}] = -2i\delta_{ij}G_{-e} \; , \\
&[G_{-e}, \tilde{G}_t] = -[G_t, \tilde{G}_{-e}] = -2i \; .
\end{aligned} \tag{50}$$

Since on the Hilbert space of the contracted theory we have only $\phi(p^i, x^i)$ and the corresponding observable algebra as $\alpha(p_i\star, x_i\star)$, the loss of $p_0\star$ and $x_0\star$, the quantum observables of energy and time, means that the Heisenberg equation of motion, in the form of a differential equation in τ, effectively corresponds to the part of $G_\tau\star$ involving only $p^i\star$. We have

$$\frac{d}{d\tau}\alpha\star = \frac{1}{2i}[\alpha\star, G_\tau\star] = \frac{1}{2i}[\alpha\star, G_t\star] \; , \tag{51}$$

where $G_t = \frac{p_i p^i}{2m}$, giving the right time evolution in the 'non-relativistic', or $H_R(3)$, quantum theory, as expected. At the $c \to \infty$ limit, the proper time is just the Newtonian time. One can also see that the quantum Poisson bracket $\frac{1}{2i}[\cdots, \cdots]$ does suggest that the now multiplicative operators $t_{(x)}$ and $e_{(x)}$, from the original $p_0\star$ and $x_0\star$, are to be dropped from the canonical coordinates of the noncommutative symplectic geometry, in line with the Hilbert space picture. A G_t of the form $\frac{p_i p^i}{2m} + v(x^i)$, i.e., with a nontrivial interaction potential of course cannot be retrieved from a G_τ which does not allow that, so long as the Einstein theory is concerned. If we allow a nontrivial $v(x^\mu)$ in G_τ, however, everything works fine. For the latter G_τ to be taken as a 'relativistic' Hamiltonian, one would have to allow violation of the Einstein relation of $p_\mu p^\mu = -m^2$.

7. Contraction to Classical Theory in Brief

In this section, we look at the corresponding classical theory at the Lorentz covariant level through the contraction along the line of the one performed in the 'non-relativistic', $H_R(3)$, case presented in Ref. [1]. Only a sketch will be presented where the mathematics is essentially the same with the latter. The contraction trivializing the commutators between the position and momentum operators is obtained by rescaling the generators as

$$X^c_\mu = \frac{1}{k_x}X_\mu \qquad \text{and} \qquad P^c_\mu = \frac{1}{k_p}P_\mu \; , \tag{52}$$

and taking the limit $k_x, k_p \to \infty$. The only important difference between k_x and k_p parameters is their physical dimensions, giving the X^c_μ and P^c_μ observables with their different classical units. For the corresponding operators we have

$$\begin{aligned} \hat{X}^{cL} &= x^c + i\frac{1}{k_x k_p}\partial_{p^c} \longrightarrow x^c \,, \\ \hat{P}^{cL} &= p^c - i\frac{1}{k_x k_p}\partial_{x^c} \longrightarrow p^c \,, \end{aligned} \tag{53}$$

and the Moyal star-product reduces to a simple commutative product. Functions $\alpha(p\star, x\star)$, representing quantum observables, reduce to multiplicative operators $\alpha(p^c, x^c)$, the classical observables acting on the contracted representation space of the original pure and mixed states.

For the Krein space of pure states, the coherent state basis is taken with the new labels as $|p^c, x^c\rangle$, where $2p^c_\mu$ and $2x^c_\mu$ characterize the expectation values of $\hat{X}^c_\mu$ and $\hat{P}^c_\mu$ operators. We have

$$\begin{aligned} {}_\eta\!\left\langle p'^c_\mu, x'^c_\mu|\hat{X}^c_\mu|p^c_\mu, x^c_\mu\right\rangle &= [(x'^c_\mu + x^c_\mu) - i\frac{k_p}{k_x}(p'^c_\mu - p^c_\mu)]\ {}_\eta\!\left\langle p'^c_\mu, x'^c_\mu \middle| p^c_\mu, x^c_\mu\right\rangle , \\ {}_\eta\!\left\langle p'^c_\mu, x'^c_\mu|\hat{P}^c_\mu|p^c_\mu, x^c_\mu\right\rangle &= [(p'^c_\mu + p^c_\mu) + i\frac{k_x}{k_p}(x'^c_\mu - x^c_\mu)]\ {}_\eta\!\left\langle p'^c_\mu, x'^c_\mu \middle| p^c_\mu, x^c_\mu\right\rangle , \end{aligned} \tag{54}$$

with ${}_\eta\!\left\langle p'^c_\mu, x'^c_\mu \middle| p^c_\mu, x^c_\mu\right\rangle$ at the contraction limit going to zero for two distinct states. Note that the k_p-k_x ratio is, at the contraction limit, a constant with physical dimension and it is showing up in the above equations only to take care of the difference in physical units for p^c and x^c. The Krein space, as a representation for the contracted symmetry, as well as a representation of the now commutative algebra of observables, reduces to a direct sum of one-dimensional representations of the ray spaces of each $\left|p^c_\mu, x^c_\mu\right\rangle$. The only admissible pure states are the exact coherent states, and not any linear combinations. The obtained coherent states can be identified as classical states, on the space of which the $\tilde{G}_s$-type operators act as generators of symmetries. $G_s\star$-type operators, as general $\alpha\star$ in the original observable algebra, contract to commuting multiplicative operators corresponding to classical observables. Results suggest that the projective Krein space, the true quantum phase space, in classical limit gives exactly the classical phase space with p^c_μ and x^c_μ coordinates. The Krein space, or Schrödinger, picture at the classical limit serves rather as the Koopman–von Neumann formulation in a broader setting of mixed state, i.e., statistical mechanics. We do not intend to explore that aspect further in this article. The observable algebra, or Heisenberg picture, gives a much more direct way of examining the full dynamical theory at that contraction limit. It also gives a direct and intuitive picture of the phase space geometry too. The original position and momentum operators, $x_\mu\star$ and $p_\mu\star$, can be seen as noncommutative coordinates of the noncommutative symplectic geometry of the phase space [12]. The contracted versions as x^c_μ and p^c_μ are the classical phase space coordinates with no noncommutativity left.

Let us turn to the noncommutative Hamiltonian transformations. As mentioned above, at the quantum level, a $G_s\star = G_s(p_\mu\star, x_\mu\star)$ operator is a Hamiltonian function of the phase space coordinates $p\star$ and $x\star$, and the corresponding $\frac{-1}{2i}\tilde{G}_s$ is the Hamiltonian vector field. It is, of course, well known since Dirac that what has now been identified as a quantum Poisson bracket $\frac{1}{2i}[\cdot,\cdot]$ [12,13] (and see references therein) reduces exactly to a classical Poisson bracket, which works in our formulation, explicitly shown in Ref. [1]; i.e.,

$$G_s(p_\mu\star, x_\mu\star) \to G^c_s(p^c_\mu, x^c_\mu)\,, \qquad \frac{-1}{2i}\tilde{G}_s = \frac{1}{2i}[\cdot,\cdot] \to \{\cdot, G^c_s\} = \frac{-1}{2i}\tilde{G}^c_s\,.$$

The explicit expressions are in exactly the same form as those of the quantum case, namely

$$\tilde{G}^c_{\omega^{\mu\nu}} = \tilde{G}_{\omega^{\mu\nu}} = -2i(x^c_\mu \partial_{x^{c\nu}} - p^c_\nu \partial_{p^{c\mu}}) - (\mu \leftrightarrow \nu) \; , \\ \tilde{G}_{-x^{c\mu}} = -2i\partial_{x^{c\mu}} \; , \qquad\qquad \tilde{G}_{p^{c\mu}} = 2i\partial_{p^{c\mu}} \; . \tag{55}$$

Note their independence on the contraction parameter k (or k_p and k_x), even before the $k \to \infty$ limit is explicitly taken. In conclusion, from the quantum Poisson bracket in terms of the Moyal bracket, or the Hamiltonian vector field given in terms of $\tilde{G}_s$, we retrieve the Hamiltonian flow equation

$$\frac{d}{ds}\alpha(p^c, x^c) = \{\alpha(p^c, x^c), G^c_s\} = \frac{-1}{2i}\tilde{G}^c_s \alpha(p^c, x^c) \tag{56}$$

for any classical observable $\alpha(p^c, x^c)$ as a function of basic observables $x^{c\mu}$ and $p^{c\mu}$, which also serve as canonical coordinates for the phase space, with the standard expression for the classical Poisson bracket. The Hamilton's Equations (46), as a specific example, become

$$\frac{d}{d\tau}x^c_\mu = \frac{\partial G^c_\tau}{\partial p^{c\mu}} = \frac{p^c_\mu}{m} \qquad\qquad \frac{d}{d\tau}p^c_\mu = -\frac{\partial G^c_\tau}{\partial x^{c\mu}} = -\frac{\partial v(x^{c\nu})}{\partial x^{c\mu}} \; . \tag{57}$$

$G^c_\tau = \frac{p^{c\mu} p^c_\mu}{2m}$ is the covariant classical Hamiltonian.

8. Conclusions

We presented a formulation of covariant quantum mechanics as an irreducible component of the regular representation of the $H_R(1,3)$ (quantum) relativity symmetry, with a pseudo-unitary inner product essentially obtained from an earlier study of the covariant harmonic oscillator problem identified as a representation of the same symmetry [22]. The pseudo-Hermitian nature of operators in the observable algebra is emphasized, with a metric operator $\hat{\eta}$ as the exact quantum manifestation of the Minkowski metric for the classical spacetime. The natural wavefunction representation $\phi(p^\mu, x^\mu)$ is the one in a coherent state basis. The Fock states as eigenstate solutions to the covariant harmonic oscillator Hamiltonian are a great orthonormal basis for the Krein space as a representation space. Actually, the overcomplete set of coherent states and the position and momentum operators as differential operators all have the usual form exactly as in the otherwise unitary representation, completely hiding the incompatibility of the latter with the Fock state system assuming an invariant $n = 0$ state. That seems to have made the incompatibility to escape the attention of the previous authors. Our different starting perspective [21] and a careful analysis, especially in the language of pseudo-Hermitian quantum mechanics allowing a general metric operator $\hat{\eta}$, and hence a general metric/inner product on the space of state vectors, illustrate the proper mathematical description well. In particular, we obtain explicit form of the nondegenerate but indefinite inner product for the wavefunctions $\phi(p^\mu, x^\mu)$, with a nontrivial integration measure, to go along with the η-Hermitian nature of the position and momentum operators. Though the wavefunctions for the Fock states are divergent at timelike infinity, the 'probability amplitude' is finite over any parameter interval. As a complete solution to the covariant harmonic oscillator problem, our results have all the desirable features which have not otherwise been fully available.

To retrieve the standard probability interpretation for the formulation of Lorentz covariant quantum mechanics, one can simply project the theory onto the Lorentz invariant subspace of positively normed states. The very nice properties of the full theory under the Lorentz transformations, again well illustrated in terms of the Fock basis, assure the projection does not lead to any undesirable feature.

Our study is a part of our fully quantum relativity group-theoretically-based program. The constructed quantum mechanics is the 'relativistic' version of the so-called 'non-relativistic' theory based on the $H_R(3)$ group, or on the $\tilde{G}(3)$ group, a $U(1)$ central extension of the Galilean group. $H_R(3)$ is a subgroup of the $H_R(1,3)$ group, while together with

$\tilde{G}(3)$ they are both subgroups of the $c \to \infty$ approximation of the $H_R(1,3)$, obtained as a symmetry, or Lie algebra, contraction. The study here successfully completes the full picture from 'relativistic' quantum mechanics down to the 'relativistic' classical and the 'non-relativistic' quantum and classical theories as successive contractions/approximations.

Some comments on the relation of the work to the noncommutative geometric perspective of quantum physics and quantum spacetime may be in order. Our idea of having the pseudo-unitary metric on the space of states comes mostly from the intuition on the need to take the pseudo-unitary Minkowski metric seriously as a quantum notion, for the noncommutative position and momentum operators as coordinates for the space [21]. The quantum phase space, exactly the projective Hilbert space for the 'non-relativistic' theory, has been shown to serve as the quantum model of the physical space [1,14]. Well known as an infinite dimensional symplectic manifold, a noncommutative geometric picture of it has been presented [12] with the position and momentum operators as coordinates. A new conceptual notion of noncommutative values for the quantum observables [37] has been introduced to achieve a consistent interpretation of the values of the six coordinates being able to specify a point in the phase space otherwise described by an infinite number of real number coordinates. The noncommutative value of an observable for a state carries the full information the mathematical formulation of the theory actually contains for that. The notion gives an intuitive, but noncommutative, picture of quantum physics [13]; perhaps also a noncommutative notion of 'local realism'. The current Lorentz covariant theory is what would allow an analogous picture with $\hat{X}^\mu$ and $\hat{P}^\mu$ as coordinates bearing the Minkowski nature. Our background group theoretical framework has a stable symmetry with X-X and P-P type noncommutativity to which the $H_R(1,3)$ symmetry is a contraction limit [16,17]. Other forms of covariant X-X noncommutative geometry with or without consideration of gravitation have been available in the literature(see for examples Refs. [38–40]). We share the belief that the proper theory of quantum gravity has to be a geometrodynamics of quantum, noncommutative, spacetime. Our framework has the unique feature that one has to take the 'phase space' as the model for the spacetime, as one irreducible representation.

The work focuses only on the formulation aspects, establishing such a theory that has all the nice properties mentioned and can successfully address the various concerns raised on such covariant theories. It may be considered mathematically involved, but unfortunately quite necessary. The only practical physical problem we have addressed is the free particle case, and arguably the covariant harmonic oscillator problem, with the latter being of great theoretical importance. However, practical application of 'relativistic' quantum mechanics is generally tricky [41]. Even with dynamics for the electron, quantum field theory is usually preferred. Applying the usual Dirac equation at the particle dynamics level has to confront the tricky issue of the negative energy solutions and the related zitterbewegung. There is also the fact that it gives $\frac{d\hat{X}^i}{dt} = -\frac{c}{2}\gamma^0\gamma^i \neq \frac{\hat{P}^i}{m}$, (here, $\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and 2 for $\hbar$). Otherwise, the 'nonrelativistic' and classical limits of the theory are well analyzed (see for examples Refs. [42,43]). On the other hand, our group theoretical framework naturally gives the dynamical theories in a covariant symplectic formalism with dynamical evolution to be described through an invariant parameter, like τ. The equation of motions, spin zero and probably also for the higher spin cases, are really in the form initiated by Stückelberg early the 1940s (see Refs. [44,45] and references therein). The studies stick with unitary representations of the Poincaré symmetry, though having to live with a non-Hermitian $\gamma^\mu \hat{P}_\mu$ for a Dirac spinor. Yet there are some nice theoretical features from the theory, including good position operators $\hat{X}^\mu$. Modifying the spin $\frac{1}{2}$ theory to our pseudo-Hermitian setting has a good chance of improving its physics picture better. We plan on taking the theory more seriously along the line of our $H_R(1,3)$ representation framework in terms the formulation before going into studies of practical systems. We hope to report on the results in the near future.

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Appendix A. Illustration of Problems in Unitary Formulation of Covariant Harmonic Oscillator

In this appendix, we summarize the standard approach to covariant harmonic oscillator problem, which attempts to construct a unitary Fock space, assuming the position and momentum operators being Hermitian. A special attention will be given to its problems [18,20], here especially as seen in the $\phi(p^\mu, x^\mu)$ wavefunction picture, which are all avoided in our pseudo-unitary representation. The difference in the two representations is in the inner product, which is simply

$$\langle\phi|\phi'\rangle = \frac{1}{\pi^4}\int d^4p\, d^4x\; \bar{\phi}(p^\mu, x^\mu)\, \phi'(p^\mu, x^\mu)\;,$$

for the unitary case. The first sign of the problem arises already in the abstract vector space.

The ladder operators are given essentially in the same way, $\hat{a}^\mu = \eta^{\mu\nu}(\hat{X}_\nu + i\hat{P}_\nu)$, $\hat{a}^\dagger_\mu = \hat{X}_\mu - i\hat{P}_\mu$, where we drop the corresponding trivial $\hat{\eta}$. As illustrated in the main text, the abstract algebraic analysis is not sensitive to the nature of the metric $\hat{\eta}$. The same conclusion of $\langle m|n\rangle = (-1)^{n_0}\delta_{mn}$ cannot be avoided [20]. Therefore, the Fock space is still the same Krein space, which then cannot be the Hilbert space of the unitary representation of the $H_R(1,3)$ symmetry. The only way to avoid that is to take a $|0\rangle$ state that is not Lorentz invariant [20], meaning that the Lorentz symmetry is spontaneously broken in the system, which hardly sounds like the quantum version of the classical covariant harmonic oscillator system or anything we may have a good reason to be interested in. The key thing is that the noncompact nature of $SO(1,3)$ gives no finite-dimensional unitary representation. Since the Hamiltonian for the problem, or the operator $\hat{N}$, is Lorentz invariant, the n-level subspaces are likewise invariant and hence can only be infinite-dimensional, so long as unitary representations are concerned. The states on a fixed n-level do not transform as symmetric Minkowski n-tensors. That is the key issue behind the incompatibility of the latter and the kind of nice physics picture one would like to have for the system [21], which our pseudo-unitary formulation successfully retrieved.

Now, let us turn to the wavefunction representation. The Fock state wavefunctions are eigenfunctions of

$$\hat{N} = \frac{1}{4}\left(x_\mu x^\mu + p_\mu p^\mu - \frac{\partial^2}{\partial p_\mu \partial p^\mu} - \frac{\partial^2}{\partial x_\mu \partial x^\mu} + 2ix^\mu\partial_{p^\mu} - 2ip^\mu\partial_{x^\mu}\right) - 2$$

operator. One can easily check that

$$\phi_n(p^\mu, x^\mu) = e^{-\frac{x_\mu x^\mu + p_\mu p^\mu}{2}} \prod_{\mu=0}^{3} (x_\mu - ip_\mu)^{n_\mu}$$

are solutions for the eigenvalue $n_0 + n_1 + n_2 + n_3$, with ϕ_o corresponding to $|0\rangle$ state. To stick to the probability interpretation with the trivial measure in the integral inner product, one has to restrict the domain of the wavefunctions to spacelike region of p^μ and x^μ.

In order to normalize the wavefunction, we need to calculate the integral

$$\int \frac{d^4x d^4p}{\pi^4} \bar{\phi}_n(p^\mu, x^\mu)\phi_n(p^\mu, x^\mu) = \int \frac{d^4x}{\pi^2} \int \frac{d^4p}{\pi^2} e^{-x_\mu x^\mu} e^{-p_\mu p^\mu} \prod_{\mu=0}^{3} \left(x_\mu^2 + p_\mu^2\right)^{n_\mu}$$

over the parameter domain. Without the domain restriction the integral surely diverges. Let us focus on the parts of the integral for the $\prod_{\mu=0}^{3} x_\mu^{2n_\mu}$ term, and similarly the $\prod_{\mu=0}^{3} p_\mu^{2n_\mu}$ term, from the expansion of the last factor. Specifically, we have an integral of the form $\int \frac{d^4x}{\pi^2} e^{-x_\mu x^\mu} \prod_{\mu=0}^{3} x_\mu^{2n_\mu}$ to deal with. The integral can be evaluated with coordinates in a polar form as in Ref. [18], by defining $r^2 = x_\mu x^\mu$, $\rho = x^0/\sqrt{x_i x^i}$, and the spatial angular coordinate of which we skip the details. We obtain

$$\mathcal{I}_0 = \frac{1}{\pi^2} \int d\Omega \int_0^\infty dr\, r^3 e^{-r^2} \int_{-1}^{1} \frac{d\rho}{(1-\rho^2)^2} = \frac{2}{\pi} \int_{-1}^{1} \frac{d\rho}{(1-\rho^2)^2} \tag{A1}$$

for the $n = 0$ case, with Ω denoting the spatial solid angle. The ρ-integral is still divergent. The integrand for the specific case is ρ-independent. The divergence is simply due to the infinite range of the boost parameter (ρ being its hyperbolic tangent). Hence, it has been suggested to define the integral with 'the infinite volume factor' absorbed [18,20]. However, there is really no sensible way to do that so long as the Fock state wavefunctions are concerned. The corresponding ρ-integral of ϕ_n for a nonzero n_0 has an extra factor of $\frac{\rho^{2n_0}}{(1-\rho^2)^n}$ from the integrand, giving a higher order divergence for each larger n value. From the structure of the full integral inner product, it is clear that such contributing terms of the higher order divergence stay. That is to say, none of Fock state wavefunctions are really normalizable under the unitary formulation. This is not an artifact of the coherent state framework. The usual Schrödinger wavefunctions $\psi(x^\mu)$ for the Fock states have the same problem.

There is an alternative approach of taking a timelike, instead of spacelike, parameter restriction for the domain of the wavefunctions, really corresponding to defining $|0\rangle$ as satisfying $a^\dagger|0\rangle = 0$. Similar problems persist.

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