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# A Change of Scale Formula for Wiener Integrals about the First Variation on the Product Abstract Wiener Space

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**Abstract:** We shall prove the existence of the Wiener integral and the analytic Wiener and Feynman integral and we obtain those relationships and later, we prove the change of scale formula for the Wiener integral about the first variation of a function defined on the product abstract Wiener space. Later, we obtain those relationships in the Fresnel class as it's corollaries.

**Keywords:** abstract Wiener space; Fourier–Feynman transform; change of scale formula

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## 1. Introduction

It has been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [1] and under translation [2]. However, Cameron, R.H. and Storvick, D.A. [3] proved that an analytic Feynman integral was expressed as a limit of Wiener integrals for a rather larger class of functionals on the Wiener space. They found a nice change of scale formula for Wiener integrals on the Wiener space [4]. Yoo, I. and Skoug, D. and Yoon, G. J. extended those results to an abstract Wiener space and Yeh–Wiener space in [5,6].

In [2,7,8], Cameron, R.H. and Martin, W.T. investigated the behavior of Wiener integrals on transformations and translations in 1945 and in 1947. In [1], they developed the behavior of measures and measurability under the change of scale on the Wiener space in 1947. In [9], Cameron, R.H. and Storvick, D.A. introduced a Banach algebra $S$ of functionals on $C_0[0,T]$. In [10–12], Kim, Y.S. proved the change of scale formula for the Wiener integral on the abstract Wiener space about the cylinder functions and the unbounded cylinder function and proved the relationship between the analytic Feynman integral and the Wiener integral and the first variation. In [13], Kim, Ahn, Chang, and Yoo prove the change of scale formula for the Wiener integral on the product abstract Wiener space in the class $\mathcal{F}(B^\nu)$.

In this paper, we prove the existence of the analytic Feynman integral and prove the change of scale formula about the first variation for functions in $\mathcal{F}(B^\nu)$ of the form : $G(\vec{x}) = \int_H \exp\{i \sum_{j=1}^{\nu} (h, x_j)^\sim\} d\mu(h)$, $\mu \in \mathcal{M}(H)$ on the product abstract Wiener space. We show that the first variation of the analytic Wiener integral and the analytic Feynman integral of $G(\vec{x})$ can be perfectly expressed as the limit of the sequence of Wiener integrals of the first variation. Finally, we prove that for $\nu = 1$, a change of scale formula for the Wiener integral about the first variation of $F(x)$ in the Fresnel class $\mathcal{F}(B)$ holds as its corollaries.

## 2. Definitions and Preliminaries

Let $H$ be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $||\cdot||_0$ be a measurable norm on $H$ with respect to the Gauss measure $\mu$. Let $B$ denote the completion of $H$ with respect to $||\cdot||_0$. Let $i$ denote the natural injection from $H$ into $B$. The adjoint operator $i^*$ of $i$ is one-to-one and maps $B^*$ continuously onto a dense subset of $H^*$, where $H^*$ and $B^*$ are topological duals of $H$ and $B$,

respectively. By identifying $H$ with $H^*$ and $B^*$ with $i^*B^*$, we have a triplet $(B^*, H, B)$ such that $B^* \subset H^* \equiv H \subset B$ and $\langle h, x\rangle = (h, x)$ for all $h$ in $B^*$ and $x$ in $H$, where $(\cdot, \cdot)$ denotes the natural dual pairing between $B^*$ and $B$. By a well-known result of Gross [14], $\mu \cdot i^{-1}$ has a unique countably additive extension $m$ to the Borel $\sigma$-algebra $\mathcal{B}(B)$ on $B$. The triplet $(H, B, m)$ is called an abstract Wiener space and $m$ is called a Wiener measure. We denote the Wiener integral of a functional $F$ by $\int_B F(x)dm(x)$. For more details, see [15–17]. Let $\{e_j\}_{j=1}^{\infty}$ denote a complete orthonormal system in $H$ such that $e_j$s are in $B^*$. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(\cdot, \cdot)^{\sim}$ between $H$ and $B$ as follows:

$$(h, x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (e_j, x) \ , & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is known in [16] that for every $h \in H$, $(h, x)^{\sim}$ exists for $m-$a.e.$x \in B$ in and it has a Gaussian distribution with mean zero and variance $|h|^2$. Furthermore, it is easy to show that $(h, x)^{\sim} = \langle h, x\rangle$ for $m-$a.e. $x$ in $B$ if $h \in B^*$, $(h, x)^{\sim}$ is essentially independent of the choice of the complete orthonormal system used in its definition, and finally that if $\{h_1, \cdots, h_k\}$ is an orthonormal set of elements in $H$, then $(h_1, x)^{\sim}, \cdots, (h_k, x)^{\sim}$ are independent Gaussian functionals with mean zero and variance one. Note that if both $h$ and $x$ are in $H$, then $(h, x)^{\sim} = \langle h, x\rangle$.

Let $\nu$ be a positive integer and let $m^{\nu}$ denote $\nu$-dimensional abstract Wiener measure and let $B^{\nu} = B \times B \times \cdots \times B(\nu$ times). We denote the Wiener integral of a function $F$ defined on $B^{\nu}$ by $\int_{B^{\nu}} F(\vec{x})dm^{\nu}(\vec{x})$. A subset $E$ of $B^{\nu}$ is said to be scale-invariant measurable if $\rho E$ is Wiener measurable for each $\rho > 0$ and a scale-invariant measurable set $N$ is scale-invariant null provided $m^{\nu}(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s.a.e.). See [18] for more about the scale-invariant measurability in abstract Wiener space.

Throughout this paper, let $\mathbf{R}^n$ denote the $n$-dimensional Euclidean space and let C, $\mathrm{C}_+$, and $C_+^{\sim}$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. Let $\Gamma_0 = \{(z_1, z_2, \cdots, z_{\nu}) \in C^{\nu} : z_j \neq 0, Re(z_j) \geq 0, 1 \leq j \leq \nu\}$ and let $\Gamma = \{(z_1, z_2, \cdots, z_{\nu}) \in C^{\nu} : z_j \neq 0, Re(z_j) > 0, 1 \leq j \leq \nu\}$.

**Definition 1.** *Let $G$ be a complex-valued measurable function on $B^{\nu}$ such that the integral*

$$J(G; \vec{\lambda}) = \int_{B^{\nu}} G(\lambda_1^{-\frac{1}{2}} x_1, \lambda_2^{-\frac{1}{2}} x_2, \cdots, \lambda_{\nu}^{-\frac{1}{2}} x_{\nu})\, dm^{\nu}(\vec{x}) \tag{2}$$

*exists for all $\vec{\lambda}$, with $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{\nu}) \in \mathbf{R}^{\nu}, \lambda_j > 0, 1 \leq j \leq \nu$. If there exists a function $J^*(G; \vec{z})$ analytic on $\Omega$ such that $J^*(G; \vec{\lambda}) = J(G; \vec{\lambda})$ for all $\vec{\lambda}$, with $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{\nu}) \in \mathbf{R}^{\nu}, \lambda_j > 0, 1 \leq j \leq \nu$, then we define $J^*(G; \vec{z})$ to be the analytic Wiener integral of $G$ over $B^{\nu}$ with parameter $\vec{z}$, and for each $\vec{z} \in \Omega$, we write*

$$\int_{B^{\nu}}^{anw_{\vec{z}}} G(\vec{x})\, dm^{\nu}(\vec{x}) \ = \ J^*(G; \vec{z}). \tag{3}$$

*Let $\vec{q} = (q_1, q_2, \cdots, q_{\nu}) \in \mathbf{R}^{\nu}$, where $q_j$, $1 \leq j \leq \nu$, is a nonzero real number and let $G$ be a function on $B^{\nu}$ whose analytic Wiener integral exists for each $\vec{z}$ in $\Gamma$. If the following limit exists, then we call it the analytic Feynman integral of $G$ over $B^{\nu}$ with parameter $\vec{q}$, and we write*

$$\int_{B^{\nu}}^{anf_{\vec{q}}} G(\vec{x})\, dm^{\nu}(\vec{x}) \ = \ \lim_{\vec{z} \to -i\vec{q}} \int_{B^{\nu}}^{anw_{\vec{z}}} G(\vec{x})\, dm^{\nu}(\vec{x}), \tag{4}$$

*where $\vec{z}$ approaches $-i\vec{q}$ through $\Gamma$ and $i^2 = -1$.*

Let $\mathcal{M}(H)$ be the space of complex-valued countably additive measures defined on $\mathcal{B}(H)$ , the Borel class of $H$. A complex-valued countably additive measure $\mu$ necessarily has finite total variation $||\mu||$. Under the norm $||\cdot||$ and with convolution as multiplication, $\mathcal{M}(H)$ is a commutative Banach algebra with identity.

Now, we will introduce $\mathcal{F}(B^\nu)$ in [13].

**Definition 2.** *The class $\mathcal{F}(B^\nu)$ is defined as the class of functions $G : B^\nu \to C$ defined on $B^\nu$ of the form*

$$G(\vec{x}) = \int_H \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j)^\sim \right\} d\mu(h) \ , \tag{5}$$

*where $\vec{x} = (x_1, x_2, \cdots, x_\nu) \in B^\nu$, $\mu \in \mathcal{M}(H)$.*

**Remark 1.** *For $\nu = 1$, $\mathcal{F}(B)$ is a Fresnel class of functions $F(x) = \int_H \exp\left\{ i\,(h,x)^\sim \right\} d\mu(h)$, which is a stochastic Fourier transform of a measure $\mu \in \mathcal{M}(H)$.*

In [19], R.H. Cameron introduced the first variation theory of an indefinite Wiener integral on the Wiener space. We define the first variation on $\mathcal{F}(B^\nu)$.

**Definition 3.** *Let $G$ be a function on $\mathcal{F}(B^\nu)$. Then, for $\vec{u} \in B^\nu$, the function defined by*

$$\delta G(\vec{x}|\vec{u}) = \frac{\partial}{\partial h} G(\vec{x} + h\vec{u})|_{h=0} \tag{6}$$

*is called the first variation of $G$ in the direction $\vec{u}$ (if it exists).*

The next theorem, which is quoted from [14], is necessary for proving theorems in $\mathcal{F}(\mathcal{B}^\succeq)$ :

**Theorem 1.** *Let $U$ be an open subset of $C^k$, where $C^K = \times_{j=1}^k C$ is the product of $k$-copies of the complex plane $C$. Assume that $f : U \to C$ is continuous and analytic in each variable separately. That is, for each $j$, $1 \le j \le k$, and for each point $(z_1, z_2, \cdots, z_{j-1}, z_{j+1}, \cdots, z_k) \in C^{k-1}$ such that $U_j = \{z_j \in C : (z_1, z_2, \cdots, z_j, \cdots, z_k) \in U\}$ is non-empty, the function $h(z_j) = f(z_1, z_2, \cdots, z_j, \cdots, z_k)$ is analytic in $U$. Then $h$ is analytic as a function of $k$ complex variables in $U$. If $U$ is connected and contains the set $U^+ = \{(z_1, \cdots, z_k) \in C^k : Re(z_j) > 0, 1 \le j \le k\}$, then $h$ is uniquely determined by its restriction to $U^+$.*

In the next section, we will use the following well-known integration formula:

$$\int_{\mathbf{R}} \exp\left\{ -cu^2 + idu \right\} du = \sqrt{\frac{\pi}{c}} \exp\left\{ -\frac{d^2}{4c} \right\}, \tag{7}$$

where $c$ is a complex number with $Re(c) > 0$, $d$ is a real number, and $i^2 = -1$.

## 3. Main Results

Let $G \in \mathcal{F}(B^\nu)$ be of the form (5). Let $\vec{u} = (u_1, u_2, \cdots, u_\nu)$ with $u_j \in H$ and $|u_j| < \infty$, $1 \le j \le \nu$. Assume that $\int_H |h|\, d|\mu|(h) < \infty$ .

**Lemma 1.**

$$\delta G(\vec{x}|\vec{u}) = \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j)^\sim \right\} d\mu(h). \tag{8}$$

**Proof.** By Definition 3 of the first variation,

$$\begin{aligned} & \delta G(\vec{x}|\vec{u}) \\ = & \frac{\partial}{\partial l} G(\vec{x}+l\vec{u})|_{l=0} \\ = & \frac{\partial}{\partial l} \int_H \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j + l\, u_j)^{\sim} \right\} d\mu(h) \,|_{l=0} \\ = & \int_H \frac{\partial}{\partial l} \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j + l\, u_j)^{\sim} \right\} d\mu(h) \,|_{l=0} \\ = & \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^{\sim} \right] \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j)^{\sim} \right\} d\mu(h). \end{aligned} \tag{9}$$

For $u_j \in H$, $(h, u_j)^{\sim} = \langle h, u_j \rangle$ and $|\langle h, u_j \rangle| < |h| \cdot |u_j|$ for $1 \le j \le \nu$. Therefore,

$$\begin{aligned} & |\, \delta G(\vec{x}|\vec{u})\, | \\ = & \left| \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^{\sim} \right] \exp\{ i \sum_{j=1}^{\nu} (h, x_j)^{\sim} \right\} d\mu(h) \,\Big| \\ \le & \sum_{j=1}^{\nu} \int_H \sum_{j=1}^{\nu} |\, (h, u_j)^{\sim}\, | d|\mu|(h) \\ \le & \left( \sum_{j=1}^{\nu} |u_j| \right) \int_H |h|\, d|\mu|(h) < \infty. \end{aligned} \tag{10}$$

And so $\delta G(\vec{x}|\vec{u})$ exists. □

We can deduce the following result in $\mathcal{F}(B)$ on the abstract Wiener space $(H, B, m)$ :

**Corollary 1.** *For $G \in \mathcal{F}(B)$ and for s.a.e. $u \in B$,*

$$\delta G(x|u) = \int_H \left[ i\, (h, u)^{\sim} \right] \exp\left\{ i\, (h, x)^{\sim} \right\} d\mu(h) \tag{11}$$

Now, we have the analytic Wiener integral and the analytic Feynman integral for the first variation of functions in $\mathcal{F}(B^{\nu})$.

**Theorem 2.** *(1). For every $\vec{z} = (z_1, z_2, \cdots, z_{\nu}) \in \Omega_0$,*

$$\int_{B^{\nu}}^{anw_{\vec{z}}} \delta G(\vec{x}\,|\vec{u})\; dm^{\nu}(\vec{x}) = \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^{\sim} \right] \exp\left\{ -\,|h|^2 \sum_{j=1}^{\nu} \frac{1}{2z_j} \right\} d\mu(h),$$

*where $\Omega_0 = \{(z_1, z_2, \cdots, z_{\nu}) \in C^{\nu} : z_j \neq 0\,,\, Re(z_j) \ge 0\,,\, 1 \le j \le \nu\}$.*

*(2).*

$$\int_{B^{\nu}}^{anf_{\vec{q}}} \delta G(\vec{x}\,|\vec{u}) dm^{\nu}(\vec{x}) = \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^{\sim} \right] \exp\left\{ -\,|h|^2 \sum_{j=1}^{\nu} \frac{i}{2q_j} \right\} d\mu(h), \tag{12}$$

*$\{z_{j,k}\}_k \to -i\, q_j$ through $\Omega$ for each $j = 1, 2, \cdots, \nu$.*

**Proof.** For $\vec{\lambda} = (\lambda_1, \lambda_1, \cdots, \lambda_{\nu}) \in \mathbf{R}^{\nu}$ with $\lambda_j > 0\,,\, 1 \le j \le \nu$,

$$\begin{aligned}
&\int_{B^\nu}^{anw_{\vec{\lambda}}} \delta G(\vec{x}\,|\vec{u})\; dm^\nu(\vec{x}) \\
&= \int_{B^\nu} \delta G(\lambda_1{}^{-\frac{1}{2}}x_1 + y_1, \cdots, \lambda_\nu{}^{-\frac{1}{2}}x_\nu|\vec{u})\, dm^\nu(\vec{x}) \\
&= \int_{B^\nu} \left[\int_H \left[ i \sum_{j=1}^{\nu} (h,u_j)^\sim \right] \exp\left\{ i \sum_{j=1}^{\nu} (h,\lambda_j{}^{-\frac{1}{2}}x_j)^\sim \right\} d\mu(h) \right] dm^\nu(\vec{x}) \\
&= \int_H \left[ i \sum_{j=1}^{\nu} (h,u_j)^\sim \right] \left[ \int_{B^\nu} \exp\left\{ i \sum_{j=1}^{\nu} (h,\lambda_j{}^{-\frac{1}{2}}x_j)^\sim \right\} dm^\nu(\vec{x}) \right] d\mu(h) \\
&= \int_H \left[ i \sum_{j=1}^{\nu} (h,u_j)^\sim \right] \exp\left\{ -|h|^2 \sum_{j=1}^{\nu} \tfrac{1}{2\lambda_j} \right\} d\mu(h).
\end{aligned} \tag{13}$$

By analytic continuation in $\vec{z} = (z_1, z_2, \cdots, z_\nu) \in \Omega_0$ to $\Omega = \{(z_1, z_2, \cdots, z_\nu) \in C^\nu : z_j \neq 0, Re(z_j) > 0, 1 \leq j \leq \nu\}$, we obtain (12).

Now let $\{z_{j,k}\}_k$ be the sequence in $\Omega$, with $\{z_{j,k}\}_k \to -i\,q_j$ whenever $k \to \infty$ for each $j = 1, 2, \cdots, \nu$. Then

$$\begin{aligned}
&\int_{B^\nu}^{anf_{\vec{q}}} \delta G(\vec{x}\,|\vec{u})\, dm^\nu(\vec{x}) \\
&= \lim_{k\to\infty} \int_{B^\nu}^{anw_{\vec{z}_k}} \delta G(\vec{x}\,|\vec{u})\, dm^\nu(\vec{x}) \\
&= \lim_{k\to\infty} \int_H \left[ i \sum_{j=1}^{\nu} (h,u_j)^\sim \right] \exp\left\{ -|h|^2 \sum_{j=1}^{\nu} \tfrac{1}{2z_{j,k}} \right\} d\mu(h) \\
&= \int_H \left[ i \sum_{j=1}^{\nu} (h,u_j)^\sim \right] \exp\left\{ -|h|^2 \sum_{j=1}^{\nu} \tfrac{i}{2q_j} \right\} d\mu(h),
\end{aligned} \tag{14}$$

whenever $\{z_{j,k}\}_k \to -i\,q_j$ through $\Omega$ for each $j = 1, 2, \cdots, \nu$. □

We can deduce the following result in $\mathcal{F}(B)$ on the abstract Wiener space $(H, B, m)$ :

**Corollary 2.** *(1). For $G \in \mathcal{F}(B)$ and for $z \in C^+$ and for s.a.e. $u \in B$,*

$$\int_B^{anw_z} \delta G(x|u) dm(x) = \int_H \left[ i\ (h,u)^\sim \right] \exp\left\{ -\frac{1}{2z}|h|^2 \right\} d\mu(h), \tag{15}$$

*where $C^+ = \{z \in C : Re(z) > 0\}$.*

*(2).*

$$\int_B^{anf_q} \delta G(x|u) dm(x) = \int_H \left[ i\,(h,u)^\sim \right] \exp\left\{ -\frac{i}{2q}|h|^2 \right\} d\mu(h) \tag{16}$$

To expand the main result, let $\{e_j\}, j = 1, 2, \cdots, n$, be an orthonormal set in H and let $h \in H$.

We need the following Lemma in [5] to prove properties in the next set of results.

**Lemma 2** ([5]). *Let $z \in C$ with $Re(z) > 0$. Then,*

$$\begin{aligned}
&\int_B \exp\left\{ \tfrac{1-z}{2} \sum_{j=1}^{n} (e_j,x)^2 + i(h,x)^\sim \right\} dm(x) \\
&= z^{-\frac{n}{2}} \exp\left\{ \tfrac{z-1}{2z} \sum_{j=1}^{n} [\langle e_j, h\rangle]^2 - \tfrac{1}{2}|h|^2 \right\}.
\end{aligned} \tag{17}$$

**Theorem 3.** *For every $\vec{z} \in \Omega$,*

$$\exp\left\{ \sum_{j=1}^{\nu} \left[ \frac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 \right] \right\} \delta G(\vec{x}\,|\vec{u})$$

*is a Wiener integrable function of $\vec{x} \in B^\nu$ and*

$$\begin{aligned} & \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 \right] \right\} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}) \\ = \; & \prod_{j=1}^{\nu} (z_j)^{-\frac{n}{2}} \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{z_j - 1}{2z_j} \sum_{k=1}^{n} |\langle e_k, h\rangle|^2 \right] - \tfrac{\nu}{2}|h|^2 \right\} d\mu(h). \end{aligned} \tag{18}$$

**Proof.** By (17),

$$\begin{aligned} & \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 \right] \right\} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}) \\ = \; & \int_H \left[ \int_{B^\nu} \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \sum_{j=1}^{\nu} [\tfrac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 + i(h, x_j)^\sim] \right\} dm^\nu(\vec{x}) \right] d\mu(h) \\ = \; & \prod_{j=1}^{\nu} (z_j)^{-\frac{n}{2}} \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{z_j - 1}{2z_j} \sum_{k=1}^{n} |\langle e_k, h\rangle|^2 \right] - \tfrac{\nu}{2}|h|^2 \right\} d\mu(h). \end{aligned} \tag{19}$$

The last formula in (19) has a finite value for $\vec{z} \in \Omega$. □

**Corollary 3.** *For $G \in \mathcal{F}(B)$ and for s.a.e. $u \in B$,*

$$\exp\{\frac{1-z}{2} \sum_{j=1}^{n} [(e_j, x)^\sim]^2\} \delta G(x|u)$$

*is a Wiener integrable function of $x \in B$, and*

$$\begin{aligned} & \int_B \exp\left\{ \tfrac{1-z}{2} \sum_{j=1}^{n} [(e_j, x)^\sim]^2 \right\} \delta G(x|u) dm(x) \\ = \; & z^{-\frac{n}{2}} \int_H \left[ i(h, u)^\sim \right] \exp\left\{ \left[ \tfrac{z-1}{2z} \sum_{k=1}^{n} |\langle e_k, h\rangle|^2 \right] - \tfrac{1}{2}|h|^2 \right\} d\mu(h). \end{aligned} \tag{20}$$

Now, we prove that the analytic Wiener integral of the first variation in $\mathcal{F}(B^\nu)$ can be perfectly expressed as the limit of Wiener integrals on $(H, B^\nu, m)$.

**Theorem 4.** *For every $\vec{z} \in \Omega$,*

$$\begin{aligned} & \int_{B^\nu}^{anw_{\vec{z}}} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}) \\ = \; & \lim_{n\to\infty} \left( \prod_{j=1}^{\nu} z_j \right)^{\frac{n}{2}} \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \tfrac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 \right\} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}). \end{aligned} \tag{21}$$

**Proof.** By the bounded convergence theorem and Parseval's relation, we have

$$\begin{aligned} & \lim_{n\to\infty} \left( \prod_{j=1}^{\nu} z_j \right)^{\frac{n}{2}} \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \tfrac{(1-z_j)}{2} \sum_{k=1}^{n} (e_k, x_j)^2 \right\} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}) \\ = \; & \lim_{n\to\infty} \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{(z_j - 1)}{2z_j} \sum_{k=1}^{n} |\langle e_k, h\rangle|^2 \right] - \tfrac{\nu}{2}|h|^2 \right\} d\mu(h) \\ = \; & \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \sum_{j=1}^{\nu} \left[ \tfrac{(z_j - 1)}{2z_j} \sum_{k=1}^{\infty} |\langle e_k, h\rangle|^2 \right] - \tfrac{\nu}{2}|h|^2 \right\} d\mu(h) \\ = \; & \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ \left[ \sum_{j=1}^{\nu} \tfrac{(z_j - 1)}{2z_j} |h|^2 \right] - \tfrac{\nu}{2}|h|^2 \right\} d\mu(h) \\ = \; & \int_H \left[ i \sum_{j=1}^{\nu} (h, u_j)^\sim \right] \exp\left\{ -\tfrac{1}{2} \sum_{j=1}^{\nu} \tfrac{1}{z_j} |h|^2 \right\} d\mu(h) \\ = \; & \int_{B^\nu}^{anw_{\vec{z}}} \delta G(\vec{x}|\vec{u}) dm^\nu(\vec{x}). \end{aligned} \tag{22}$$

□

**Corollary 4.** *For $G \in \mathcal{F}(B)$ and for $z \in C^+$ and for s.a.e.u $\in B$,*

$$\int_B^{anw_z} \delta F(x|u)dm(x) = \lim_{n\to\infty} z^{\frac{n}{2}} \int_B \exp\left\{\frac{1-z}{2}\sum_{j=1}^{n}\left[(e_j,x)^{\sim}\right]^2\right\}\delta G(x|u)dm(x), \quad (23)$$

*whenever $z \to -i\,q$ through $C^+$.*

Now, we prove the change of scale formula for Wiener integrals under the first variation of $G \in \mathcal{F}(B^\nu)$.

**Theorem 5.** *For real $\rho_j > 0, j = 1, 2, \cdots, \nu$,*

$$\begin{aligned} &\int_{B^\nu} \delta G(\rho_1 x_1, \cdots, \rho_\nu x_\nu|\vec{u})dm^\nu(\vec{x}) \\ = \;& \lim_{n\to\infty}\left(\prod_{j=1}^{\nu}\rho_j^{-n}\right)\int_{B^\nu}\exp\left\{\sum_{j=1}^{\nu}\left[\frac{\rho_j^2-1}{2\rho_j^2}\sum_{k=1}^{n}(e_k,x_j)^2\right]\right\}\delta G(\vec{x}|\vec{u})dm^\nu(\vec{x}), \end{aligned} \quad (24)$$

*where $dm^\nu(\vec{x}) = dm(x_1)dm(x_2)\cdots dm(x_\nu)$.*

**Proof.** In Theorem 4, we have that for real $z_j > 0, j = 1, 2, \cdots, \nu$,

$$\begin{aligned} &\int_{B^\nu} \delta G({z_1}^{-\frac{1}{2}} x_1, \cdots, {z_\nu}^{-\frac{1}{2}} x_\nu|\vec{u})dm^\nu(\vec{x}) \\ = \;& \lim_{n\to\infty}\left(\prod_{j=1}^{\nu} z_j\right)^{\frac{n}{2}}\int_{B^\nu}\exp\left\{\sum_{j=1}^{\nu}\left[\frac{(1-z_j)}{2}\sum_{k=1}^{n}(e_k,x_j)^2\right]\right\}\delta G(\vec{x}|\vec{u})dm^\nu(\vec{x}). \end{aligned} \quad (25)$$

Taking $z_j = \rho_j^{-2}, j = 1, 2, \cdots, \nu$ in (25), we have (24). □

Now, we have a change of scale formula for Wiener integrals under the first variation in $\mathcal{F}(B^\nu)$.

**Corollary 5.** *For real $\rho > 0$ and for $\vec{u} \in B^\nu$,*

$$\begin{aligned} &\int_{B^\nu} \delta G(\rho x_1, \cdots, ,, \rho x_\nu|\vec{u})dm^\nu(\vec{x}) \\ = \;& \lim_{n\to\infty}\rho^{-\nu n}\int_{B^\nu}\exp\left\{\sum_{j=1}^{\nu}\left[\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}(e_k,x_j)^2\right]\right\}\delta G(\vec{x}|\vec{u})dm^\nu(\vec{x}). \end{aligned} \quad (26)$$

**Proof.** Taking $z_j = s^{-2}$ for all $j = 1, 2, \cdots, \nu$ in (26), we have the result. □

Using (27) for $\nu = 1$, we have the the change of scale formula for Wiener integrals under the first variation in $\mathcal{F}(B)$:

**Corollary 6.** *For $G \in \mathcal{F}(B)$ and for real $\rho > 0$ and for s.a.e.u $\in B$,*

$$\int_B \delta G(\rho x|u)dm(x) = \rho^{-n}\int_B \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{j=1}^{n}\left[(e_j,x)^{\sim}\right]^2\right\}\delta G(x|u)dm(x). \quad (27)$$

Finally, we show that the analytic Feynman integral of the first variation in $\mathcal{F}(B^\nu)$ can be successfully expressed as the limit of a sequence of Wiener integrals of the first variation on $(H, B^\nu, m)$.

**Theorem 6.** *Let $\{z_{k,m}\}_m$ be a sequence of complex numbers from $\Omega$ such that $\{z_{k,m}\}_m \to -i\,q_k$ $(q_k \neq 0)$ through $\Omega$ as $m \to \infty$ for $k = 1, 2, \cdots, \nu$, where $\vec{q} = (q_1, q_2, \cdots, q_\nu) \in C^\nu$. Then,*

$$\begin{aligned} & \int_{B^\nu}^{anf_q} \delta G(\vec{x}\,|\vec{u})\ dm^\nu(\vec{x}) \\ = \ & \lim_{m\to\infty}\left[\lim_{n\to\infty}\left(\prod_{j=1}^{\nu} z_{j,m}\right)^{\frac{n}{2}} \int_{B^\nu} \exp\left\{\sum_{j=1}^{\nu}\frac{(1-z_{j,m})}{2}\sum_{k=1}^{n}\left[(e_k,x_j)^{\sim}\right]^2\right\}\delta G(\vec{x}|\vec{u})dm^\nu(\vec{x})\right] \end{aligned} \tag{28}$$

*where* $\prod_{j=1}^{\nu} z_{j,m} = z_{1,m}\, z_{2,m}\, \cdots z_{\nu,m}$ *and* $dm^\nu(\vec{x}) = dm(x_1)dm(x_2)\cdots dm(x_\nu)$.

**Proof.** By Definition 1,

$$\begin{aligned} & \int_{B^\nu}^{anf_q} \delta G(\vec{x}|\vec{u})dm^\nu(\vec{x}) \\ = \ & \lim_{m\to\infty}\int_{B^\nu}^{anw_{z_m}} \delta G(\vec{x}|\vec{u})dm^\nu(\vec{x}) \\ = \ & \lim_{m\to\infty}\int_{B^\nu} \delta G((z_{1,m})^{-\frac{1}{2}}x_1,\cdots,(z_{\nu,m})^{-\frac{1}{2}}x_\nu|\vec{u})dm^\nu(\vec{x}) \\ = \ & \lim_{m\to\infty}\left[\lim_{n\to\infty}(\prod_{j=1}^{\nu} z_{j,m})^{\frac{n}{2}}\int_{B^\nu}\exp\left\{\sum_{j=1}^{\nu}\frac{(1-z_{j,m})}{2}\sum_{k=1}^{n}\left[(e_k,x_j)^{\sim}\right]^2\right\}\delta G(\vec{x}|\vec{u})dm^\nu(\vec{x})\right] \end{aligned} \tag{29}$$

whenever $\{z_{k,m}\}_m \to -iq_k(q_k \neq 0)$ through $\Omega$ as $m\to\infty$ for $k = 1,2,\cdots,\nu$. □

**Corollary 7.** *Let* $\{z_n\}$ *be the sequence of complex numbers from* $C^+$ *such that* $\{z_n\}\to -iq(q\neq 0)$ *through* $C^+$. *Then for* $G\in\mathcal{F}(B)$ *and for s.a.e.*$u\in B$,

$$\int_B^{anf_q}\delta G(x|u)dm(x) = \lim_{n\to\infty} z_n^{\frac{n}{2}}\int_B \exp\left\{\frac{1-z_n}{2}\sum_{j=1}^{n}\left[(e_j,x)^{\sim}\right]^2\right\}\delta G(x|u)dm(x), \tag{30}$$

*whenever* $\{z_n\}\to -iq$ *through* $C_+$.

**Remark 2.** *In future works, we will try to prove rather nice formulas than the change of scale formula for the Wiener integral on the Wiener space and the abstract Wiener space and the product abstract Wiener space.*

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