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Geometrical Formulation for Adjoint-Symmetries of Partial Differential Equations

Stephen C. Anco *,† and **Bao Wang** †,‡

Department of Mathematics and Statistics, Brock University, St. Catharines, ON L2S3A1, Canada; wangbao@lsec.cc.ac.cn

* Correspondence: sanco@brocku.ca
† These authors contributed equally to this work.
‡ Current address: Department of Mathematics, Ningbo University, Ningbo 315211, China.

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Abstract: A geometrical formulation for adjoint-symmetries as one-forms is studied for general partial differential equations (PDEs), which provides a dual counterpart of the geometrical meaning of symmetries as tangent vector fields on the solution space of a PDE. Two applications of this formulation are presented. Additionally, for systems of evolution equations, adjoint-symmetries are shown to have another geometrical formulation given by one-forms that are invariant under the flow generated by the system on the solution space. This result is generalized to systems of evolution equations with spatial constraints, where adjoint-symmetry one-forms are shown to be invariant up to a functional multiplier of a normal one-form associated with the constraint equations. All of the results are applicable to the PDE systems of interest in applied mathematics and mathematical physics.

Keywords: adjoint-symmetry; one-form; symmetry; vector field; geometrical formulation

1. Introduction

Symmetries are a fundamental coordinate-free structure of a partial differential equation (PDE). In geometrical terms, an infinitesimal symmetry is an evolutionary (vertical) vector field that is tangent to the solution space of a PDE, where the components of the vector field are the solutions of the linearization of the PDE on its solution space (see, e.g., [1–4]).

Knowledge of the symmetries of a PDE can be used to map given solutions into other solutions, find invariant solutions, detect and find mappings in a target class of PDEs, detect integrability, and find conservation laws through Noether's theorem when a PDE has a variational (Lagrangian) structure.

Solutions of the adjoint linearization of a PDE on its solution space are known as adjoint-symmetries. This terminology was first introduced and explored for ordinary differential equations (ODEs) in [5–8] and then generalized to PDEs in [9,10] (see [11] for a recent overview for PDEs). When a PDE lacks a variation structure, then its adjoint-symmetries will differ from its symmetries.

Knowledge of the adjoint-symmetries of a PDE can be used for several purposes just as symmetries can. Specifically, solutions of the PDE can be found analogously to the invariant surface condition associated with a symmetry; mappings into a target class of PDEs can be detected and found analogously to characterizing the symmetry structure of the target class; integrability can be detected analogously to the existence of higher order symmetries; and conservation laws can be determined analogously to symmetries that satisfy a variational condition. In particular, the counterpart of variational symmetries for a general PDE is provided by multipliers, which are well known to be adjoint-symmetries that satisfy a Euler–Lagrange condition.

However, a simple geometrical meaning (apart from abstract formulations) for adjoint-symmetries has yet to be developed in general for PDEs. Several significant new steps toward this goal will be taken in the present paper.

Firstly, for general PDE systems, adjoint-symmetries will be shown to correspond to evolutionary (vertical) one-forms that functionally vanish on the solution space of the system. This formulation has two interesting applications. It will provide a geometrical derivation of a well-known formula that generates a conservation law from a pair consisting of a symmetry and an adjoint-symmetry [9,12]. It also will yield three different actions of symmetries on adjoint-symmetries from Cartan's formula for the Lie derivative, providing a geometrical formulation of some recent work that used an algebraic viewpoint [13].

Secondly, for evolution systems, these adjoint-symmetry one-forms will be shown to have the structure of a Lie derivative of a simpler underlying one-form, utilizing the flow generated by the system. As a result, adjoint-symmetries of evolution systems will geometrically correspond to one-forms that are invariant under the flow on the solution space of the system. This directly generalizes the geometrical meaning of adjoint-symmetries known for ODEs [8].

Thirdly, a bridge between the preceding results for general PDE systems and evolution systems will be developed by considering evolution systems with spatial constraints. These systems are ubiquitous in applied mathematics and mathematical physics, for example: Maxwell's equations, incompressible fluid equations, magnetohydrodynamical equations, and Einstein's equations. For such systems, invariance of the adjoint-symmetry one-form under the constrained flow will be shown to hold up to a functional multiple of the normal one-form associated with the constraint equations.

Throughout, the approach will be concrete, rather than abstract, so that the results can be readily understood and applied to specific PDE systems of interest in applied mathematics and mathematical physics.

The rest of the paper is organized as follows. Section 2 discusses the evolutionary form of vector fields and its counterpart for one-forms in the mathematical framework of calculus in jet space, which will underlie all of the main results. Section 3 reviews the geometrical formulation of symmetries and presents the counterpart geometrical formulation of adjoint-symmetries. In addition, some examples of adjoint-symmetries of physically interesting PDE systems are discussed. Section 4 gives the two applications of adjoint-symmetry one-forms. Section 5 develops the main results for adjoint-symmetries of evolution systems and extends these results to constrained evolution systems. Some concluding remarks are made in Section 6.

2. Vector Fields, One-Form Fields, and Their Evolutionary Form

To begin, some essential tools [3,11,14] from calculus in jet space will be reviewed. This will set the stage for a discussion of the evolutionary form of vector fields and its counterpart for one-forms, as needed for the main results in the subsequent sections.

Independent variables are denoted x^i, $i = 1, \ldots, n$, and dependent variables are denoted u^α, $\alpha = 1, \ldots, m$. Derivative variables are indicated by subscripts employing a multi-index notation: $I = \{i_1, \ldots, i_N\}$, $u^\alpha_I = u^\alpha_{i_1 \cdots i_N} := \partial_{x^{i_1}} \cdots \partial_{x^{i_N}} u^\alpha$, $|I| = N$; $I = \emptyset$, $u^\alpha_I := u^\alpha$, $|I| = 0$. Some useful notation is as follows: $\partial^k u$ will denote the set $\{u^\alpha_I\}_{|I|=k}$ of all derivative variables of order $k \geq 0$; $u^{(k)}$ will denote the set $\{u^\alpha_I\}_{0 \leq |I| \leq k}$ of all derivative variables of all orders up to $k \geq 0$. The summation convention of summing over any repeated (multi-)index in an expression is used throughout.

Jet space is the coordinate space $\mathrm{J} = (x^i, u^\alpha, u^\alpha_j, \ldots)$. A smooth function $u^\alpha = \phi^\alpha(x) : \mathbb{R}^n \to \mathbb{R}^m$ determines a point in J: at any $x^i = (x_0)^i$; the values $(u_0)^\alpha := \phi^\alpha(x_0)$ and the derivative values $(u_0)^\alpha_J := \partial_{j_1} \cdots \partial_{j_N} \phi^\alpha(x_0)$ for all orders $N \geq 1$ give a map,

$$u^\alpha = \phi^\alpha(x) \xrightarrow{x_0} ((x_0)^i, (u_0)^\alpha, (u_0)^\alpha_j, \ldots) \in \mathrm{J}. \tag{1}$$

In jet space, the primitive geometric objects consist of partial derivatives ∂_{x^i}, $\partial_{u^\alpha_J}$, and differentials $\mathrm{d}x^i$, $\mathrm{d}u^\alpha_J$. They are related by duality (hooking) relations:

$$\partial_{x^i} \rfloor \mathrm{d}x^j = \delta^j_i, \tag{2}$$

$$\partial_{u^\alpha_I} \rfloor \mathrm{d}u^\beta_J = \delta^\beta_\alpha \delta^I_J. \tag{3}$$

It will be useful to also introduce the geometric contact one-forms:

$$\Theta^\alpha_I = \mathrm{d}u^\alpha_I - u^\alpha_{Ii}\mathrm{d}x^i. \tag{4}$$

Under the evaluation map (1), the pull back of a contact one-form vanishes.

Total derivatives are given by $D_i = \partial_{x^i} + u^\alpha_{iJ}\partial_{u^\alpha_J}$, which corresponds to the chain rule under the evaluation map (1). Higher total derivatives are defined by $D_J = D_{j_1}\cdots D_{j_N}$, $J = \{j_1,\ldots,j_N\}$, $|J| = N$. For $J = \emptyset$, $D_\emptyset = \mathrm{id}$ is the identity operator, where $|\emptyset| = 0$. In particular, $D_J u^\alpha = u^\alpha_J$, and $D_J \mathrm{d}u^\alpha = \mathrm{d}u^\alpha_J$.

A differential function is a function $f(x,u^{(k)})$ defined on a finite jet space $\mathrm{J}^{(k)} = (x^i, u^\alpha, u^\alpha_j, \ldots, u^\alpha_{j_1\cdots j_k})$ of order $k \geq 0$. The Frechet derivative of a differential function f is given by

$$f' = f_{u^\alpha_I} D_I \tag{5}$$

which acts on (differential) functions F^α. The adjoint-Frechet derivative of a differential function f is given by

$$(f'^*)_\alpha = (-1)^{|I|} D_I f_{u^\alpha_I} \tag{6}$$

which acts on (differential) functions F, where the right-hand side is viewed as a composition of operators.

The Frechet second-derivative is given by

$$f''(F_1,F_2) = f_{u^\alpha_I u^\beta_J}(D_I F^\alpha_1)(D_J F^\beta_2). \tag{7}$$

This expression is symmetric in the pair of functions (F^α_1, F^α_2).

The commutator of two differential functions f_1 and f_2 is given by $[f_1,f_2] = f_2'(f_1) - f_1'(f_2)$.

The Euler operator (variational derivative) is given by

$$E_{u^\alpha} = (-1)^{|I|} D_I \partial_{u^\alpha_I}. \tag{8}$$

It characterizes total divergence expressions: $E_{u^\alpha}(f) = 0$ holds identically iff $f = D_i F^i$ for some differential vector function $F^i(x,u^{(k)})$. The product rule takes the form:

$$E_{u^\alpha}(f_1 f_2) = f_1'^*(f_2)_\alpha + f_2'^*(f_1)_\alpha. \tag{9}$$

The higher Euler operators

$$E^I_{u^\alpha} = \tbinom{I}{J}(-1)^{|J|} D_J \partial_{u^\alpha_{IJ}} \tag{10}$$

characterize higher order total derivative expressions: $E^I_{u^\alpha}(f) = 0$ holds identically iff $f = D_{i_1}\cdots D_{i_{|I|}} F^{i_1\ldots i_{|I|}}$ for some differential tensor function $F^{i_1\ldots i_{|I|}}(x,u^{(k)})$.

The Frechet derivative is related to the Euler operator by:

$$f'(F) = F^\alpha E_{u^\alpha}(f) + D_i\Gamma^i(F;f), \quad \Gamma^i(F;f) = (D_J F^\alpha) E_{u^\alpha_{iJ}}(f). \tag{11}$$

The Frechet derivative and its adjoint are related by

$$F_2 f'(F_1) - F_1^\alpha f'^*(F_2)_\alpha = D_i \Psi^i(F_1, F_2; f), \quad \Psi^i(F_1, F_2; f) = (D_K F_2)(D_J F_1^\alpha) E^K_{u^\alpha_{iJ}}(f). \tag{12}$$

Evolutionary Vector Fields and One-Form Fields

A vector field in jet space is defined as the geometric object,

$$P^i \partial_{x^i} + P^\alpha_I \partial_{u^\alpha_I} \tag{13}$$

whose components are differential functions. Similarly, a one-form field in jet space is defined as the geometric object,

$$Q_i \mathrm{d}x^i + Q^I_\alpha \mathrm{d}u^\alpha_I \tag{14}$$

whose components are differential functions. Total derivatives $D_i = \partial_{x^i} + u^\alpha_{iI}\partial_{u^\alpha_I}$ represent trivial vector fields that annihilate contact one-forms: $D_i \rfloor \Theta^\alpha_J = 0$.

Geometric counterparts of partial derivatives $\partial_{u^\alpha_J}$ are evolutionary (vertical) differentials $\mathrm{d}u^\alpha_J$, where d is the evolutionary version of d: $\mathrm{d}^2 = 0$, $\mathrm{d}x^i = 0$. They satisfy the duality (hooking) relation:

$$\partial_{u^\alpha_I} \rfloor \mathrm{d}u^\beta_J = \delta^\beta_\alpha \delta^I_J. \tag{15}$$

An evolutionary (vertical) vector field is the geometric object

$$P^\alpha_I \partial_{u^\alpha_I} \tag{16}$$

whose components are differential functions. Every vector field $\mathbf{X} = P^i \partial_{x^i} + P^\alpha_I \partial_{u^\alpha_I}$ has a unique evolutionary form $\hat{\mathbf{X}} = \mathbf{X} - P^i D_i = \hat{P}^\alpha_I \partial_{u^\alpha_I}$ given by the components $\hat{P}^\alpha_I = P^\alpha_I - P^i u^\alpha_{iI}$. Its dual counterpart is an evolutionary (vertical) one-form field,

$$Q^I_\alpha \mathrm{d}u^\alpha_I \tag{17}$$

whose components are differential functions.

For later developments, it will be useful to define the functional pairing relation,

$$\langle P^\alpha_I \partial_{u^\alpha_I}, Q^I_\alpha \mathrm{d}u^\alpha_I \rangle = \int P^\alpha_I Q^I_\alpha \, dx \tag{18}$$

between evolutionary vector fields and evolutionary one-form fields. In the local form, this pairing is given by the expression:

$$P^\alpha_I Q^I_\alpha \text{ mod total } D. \tag{19}$$

Two evolutionary one-forms will be considered functionally equivalent iff their pairings with an arbitrary evolutionary vector field agree,

$$\langle P^\alpha_I \partial_{u^\alpha_I}, Q1^J_\beta \mathrm{d}u^\beta_J \rangle = \langle P^\alpha_I \partial_{u^\alpha_I}, Q2^J_\beta \mathrm{d}u^\beta_J \rangle, \tag{20}$$

or in the local form,

$$P^\alpha_I (Q1^I_\alpha - Q2^I_\alpha) = 0 \text{ mod total } D. \tag{21}$$

The functional equivalence of one-forms is closely related to the notion of functional one-forms in the variational bi-complex. See [3] for details.

3. Geometric Formulation of Symmetries and Adjoint-Symmetries

Consider a general PDE system of order N consisting of M equations,

$$G^A(x, u^{(N)}) = 0, \quad A = 1, \dots, M \tag{22}$$

where x^i, $i = 1, \dots, n$, are the independent variables and u^α, $\alpha = 1, \dots, m$, are the dependent variables. The space of formal solutions $u^\alpha(x)$ of the PDE system will be denoted $\mathcal{E}$.

There are many equivalent starting points for the formulation of infinitesimal symmetries. For the present purpose, the most useful one is given by evolutionary vector fields and utilizes only the Frechet derivative. A symmetry is a vector field,

$$\mathbf{X}_P = P^\alpha(x, u^{(k)})\partial_{u^\alpha} \tag{23}$$

whose component functions $P^\alpha(x, u^{(k)})$ are non-singular on $\mathcal{E}$ and satisfy the linearization of the PDE system on $\mathcal{E}$,

$$(\mathrm{pr}\mathbf{X}_P G^A)|_{\mathcal{E}} = G'(P)^A|_{\mathcal{E}} = 0. \tag{24}$$

This is the *symmetry determining equation*, and the functions P^α are called the characteristic of the symmetry.

In this setting, an *adjoint-symmetry* consists of functions $Q_A(x, u^{(l)})$ that are non-singular on $\mathcal{E}$ and that satisfy the adjoint linearization of the PDE system on $\mathcal{E}$,

$$G'^*(Q)_\alpha|_{\mathcal{E}} = 0. \tag{25}$$

This is the *adjoint-symmetry determining equation*.

In particular, the two determining equations (24) and (25) are formal adjoints of each other. They coincide only in two cases: either $G' = G'^*$, which is the necessary and sufficient condition for a PDE system to be a Euler–Lagrange equation (namely, possess a variational structure) [1,3,11]; or $G' = -G'^*$, which is the necessary and sufficient condition for a PDE system to be a linear, constant-coefficient system of odd order [10].

Since P^α has the geometrical status as the components of the vector field (23), a natural question is whether Q_A has any status given by the components of some other geometrical object [11,12].

It will be useful to work with a coordinate-free description of the PDE system (22) in jet space. Such a system of equations $(G^1(x, u^{(N)}), \dots, G^M(x, u^{(N)})) = 0$ describes a set of M surfaces in the finite space $\mathrm{J}^{(N)}(x, u, \partial u, \dots, \partial^N u)$. Total derivatives of these equations, $(D_I G^1(x, u^{(N)}), \dots, D_I G^M(x, u^{(N)})) = 0$, correspondingly describe sets of surfaces in the higher derivative finite spaces $\mathrm{J}^{(N+|I|)}(x, u, \partial u, \dots, \partial^{N+|I|} u)$. Altogether, the set comprised by the equations and the derivative equations for all orders $|I| \geq 0$ corresponds to an infinite set of surfaces in jet space, which can be identified with the solution space $\mathcal{E}$.

As is well known, symmetry vector fields geometrically describe tangent vector fields with respect to $\mathcal{E}$. To see this explicitly, first consider the identities:

$$\mathrm{d}G^A = (G^A)_{u^\alpha_I} \mathrm{d}u^\alpha_I, \tag{26}$$

$$G'(P)^A = \mathrm{pr}\mathbf{X}_P G^A = \mathrm{pr}\mathbf{X}_P \rfloor \, \mathrm{d}G^A. \tag{27}$$

Now, observe that $\mathrm{d}G^A$ is the normal one-form to the surfaces $G^A = 0$. The symmetry determining equation (24) then shows that the prolonged vector field $\mathrm{pr}\mathbf{X}_P$ is annihilated by the normal one-form and hence is tangent to these surfaces iff $\mathbf{X}_P$ is a symmetry of the PDE system.

This normal one-form (26) provides a natural way to associate a one-form to an adjoint-symmetry via:

$$\varpi_Q = Q_A(x, u^{(l)})\mathrm{d}G^A. \tag{28}$$

A functionally equivalent one-form is obtained through integration by parts:

$$Q_A \mathrm{d}G^A = Q_A (G^A)'(\mathrm{d}u) = G'^*(Q)_\alpha \mathrm{d}u^\alpha \text{ mod total } D. \tag{29}$$

Evaluating this one-form on the solution space $\mathcal{E}$ then gives

$$\varpi_Q|_\mathcal{E} = 0 \text{ mod total } D. \tag{30}$$

Thus, a one-form ϖ_Q functionally vanishes on the surfaces $\mathcal{E}$ iff its components Q_A are an adjoint-symmetry.

This establishes a main geometrical result.

Theorem 1. *Adjoint-symmetries describe evolutionary one-forms $Q_A \mathrm{d}G^A$ that functionally vanish on the solution space $\mathcal{E}$ of a PDE system* (22)*.*

These developments have used evolutionary (vertical) vector fields and evolutionary one-forms. It is straightforward to reformulate everything in terms of full vector fields and full one-forms.

First, consider the normal one-form

$$\begin{aligned} \mathrm{d}G^A &= (G^A)_{x^i}\mathrm{d}x^i + (G^A)'(\mathrm{d}u) \\ &= (G^A)'(\Theta) + ((G^A)_{x^i} + (G^A)'(u_i))\mathrm{d}x^i \\ &= (G^A)'(\Theta) + D_i G^A \mathrm{d}x^i \end{aligned} \tag{31}$$

which yields the relation

$$\mathrm{d}G^A|_\mathcal{E} = (G^A)'(\Theta)|_\mathcal{E}. \tag{32}$$

Then, observe:

$$\begin{aligned} Q_A \mathrm{d}G^A|_\mathcal{E} &= Q_A (G^A)'(\Theta)|_\mathcal{E} \\ &= (G^A)'^*(Q_A)_\alpha|_\mathcal{E}\Theta^\alpha \text{ mod total } D. \end{aligned} \tag{33}$$

As a consequence, $Q_A \mathrm{d}G^A|_\mathcal{E}$ vanishes mod total D iff Q_A satisfies the adjoint-symmetry determining Equation (25). Moreover, the determining equation itself can be expressed directly in terms of the one-form $Q_A \mathrm{d}G^A|_\mathcal{E}$ by $E_{\Theta^\alpha}(Q_A \mathrm{d}G^A)|_\mathcal{E} = (G^A)'^*(Q_A)|_\mathcal{E} = 0$.

Proposition 1. *The adjoint-symmetry determining Equation* (25) *can be expressed geometrically as:*

$$E_{\Theta^\alpha}(Q_A \mathrm{d}G^A)|_\mathcal{E} = 0. \tag{34}$$

Examples of Adjoint-Symmetries

To illustrate the results, some examples of PDEs that possess non-trivial adjoint-symmetries will be given.

The Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{35}$$

for shallow water waves is an example of an evolutionary wave equation. Its symmetries $\mathbf{X} = P\partial_u$ are the solutions of the determining equation

$$G'(P)|_\mathcal{E} = (D_t P + D_x(uP) + D_x^3 P)|_\mathcal{E} = 0, \tag{36}$$

with $G' = D_t + D_x u + D_x^3$ being the Frechet derivative of the KdV equation, where P is a non-singular function of t, x, u, and derivatives of u on the space of KdV solutions $\mathcal{E}$. The determining equation for adjoint-symmetries $\varpi = QG'(\mathrm{d}u)$ is the adjoint equation

$$G'^{*}(Q)|_{\mathcal{E}} = (-D_t Q - u D_x Q - D_x^3 Q)|_{\mathcal{E}} = 0, \tag{37}$$

where Q is a non-singular function of t, x, u, and derivatives of u on $\mathcal{E}$.

KdV adjoint-symmetries up to first-order $Q(t, x, u, u_t, u_x)$ are given by [9] the span of

$$Q^{(1)} = 1, \quad Q^{(2)} = u, \quad Q^{(3)} = tu - x. \tag{38}$$

The first two are part of a hierarchy of higher order adjoint-symmetries generated by a recursion operator $\mathcal{R} = D_x^2 + \frac{1}{3}u + \frac{1}{3}D_x^{-1}uD_x$ applied to $Q = u$. The third one along with all of the ones in the hierarchy are related to symmetries of the KdV equation through the Hamiltonian operator $\mathcal{H} = D_x$. If a linear combination of the lowest order adjoint-symmetries is used like an invariant surface condition, $c_1 + c_2(tu - x) + c_3 u = 0$, then this yields $u = (c_2 x - c_1)/(c_2 t + c_3)$, which is a similarity solution of the KdV equation.

An example of a non-evolutionary equation is,

$$\Delta\phi_t + \phi_x \Delta\phi_y - \phi_y \Delta\phi_x = 0 \tag{39}$$

which governs the vorticity $\Omega = \Delta\phi$ for incompressible inviscid fluid flow in two spatial dimensions, where the fluid velocity has the components $\vec{v} = (-\phi_y, \phi_x)$. The symmetries $\mathbf{X} = P\partial_\phi$ of this equation are the solutions of the determining equation,

$$G'(P)|_{\mathcal{E}} = (D_t \mathbf{\Delta} P + \phi_x D_y \mathbf{\Delta} P + \Delta\phi_y D_x P - \phi_y D_x \mathbf{\Delta} P - \Delta\phi_x D_y P)|_{\mathcal{E}} = 0, \tag{40}$$

where P is a non-singular function of t, x, y, ϕ, and derivatives of ϕ on the space of vorticity solutions $\mathcal{E}$, with $G' = D_t \mathbf{\Delta} + \phi_x D_y \mathbf{\Delta} + \Delta\phi_y D_x - \phi_y D_x \mathbf{\Delta} - \Delta\phi_x D_y$ being the Frechet derivative of the vorticity equation given in terms of the total Laplacian operator $\mathbf{\Delta} = D_x^2 + D_y^2$. The determining equation for adjoint-symmetries $\boldsymbol{\varpi} = QG'(\mathrm{d}\phi)$ is the adjoint equation,

$$G'^{*}(Q)|_{\mathcal{E}} = -(D_t \mathbf{\Delta} Q + D_y \mathbf{\Delta}(\phi_x Q) + D_x(\Delta\phi_y Q) - D_x \mathbf{\Delta}(\phi_y Q) - D_y(\Delta\phi_x Q)|_{\mathcal{E}} = 0, \tag{41}$$

where Q is a non-singular function of t, x, y, ϕ, and derivatives of ϕ on $\mathcal{E}$.

The first-order adjoint-symmetries $Q(t, x, y, \phi, \phi_t, \phi_x, \phi_y)$ are given by [13] the span of,

$$Q^{(1)} = x^2 + y^2, \quad Q^{(2)} = \phi, \quad Q^{(3)} = f(t), \quad Q^{(4)} = xf(t), \quad Q^{(5)} = yf(t), \tag{42}$$

where $f(t)$ is an arbitrary smooth function. If a linear combination of these adjoint-symmetries is used like an invariant surface condition, $c_1(x^2 + y^2) + c_2\phi + c_3 f(t) + c_4 x f(t) + c_5 y f(t) = 0$, then taking $c_2 = -1$ gives $\phi = c_1(x^2 + y^2) + (c_3 + c_4 x + c_5 y)f(t)$, which is a constant vorticity solution, with $\Omega = 2c_1$ and $\vec{v} = (-2c_1 y + c_5 f(t), 2c_1 x + c_4 f(t))$.

Maxwell's equations in free space are an example of an evolution system with spatial constraints:

$$\vec{E}_t - \nabla \times \vec{B} = 0, \quad \vec{B}_t + \nabla \times \vec{E} = 0, \quad \nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0 \tag{43}$$

(in relativistic units with the speed of light set to one). The symmetries $\mathbf{X} = \vec{P}^E \cdot \partial_{\vec{E}} + \vec{P}^B \cdot \partial_{\vec{B}}$ of this system are the solutions of the determining equations

$$G' \begin{pmatrix} \vec{P}^E \\ \vec{P}^B \end{pmatrix} \Bigg|_{\mathcal{E}} = \begin{pmatrix} (D_t \vec{P}^E - \mathbf{\nabla} \times \vec{P}^B)|_{\mathcal{E}} \\ (D_t \vec{P}^B + \mathbf{\nabla} \times \vec{P}^E)|_{\mathcal{E}} \\ (\mathbf{\nabla} \cdot \vec{P}^E)|_{\mathcal{E}} \\ (\mathbf{\nabla} \cdot \vec{P}^B)|_{\mathcal{E}} \end{pmatrix} = 0, \tag{44}$$

where $\vec{P}^E$ and $\vec{P}^B$ are non-singular vector functions of t, x, y, z, $\vec{E}$, $\vec{B}$, and derivatives of $\vec{E}$, $\vec{B}$ on the space of Maxwell solutions $\mathcal{E}$, with $G' = \begin{pmatrix} D_t & -\nabla\times \\ \nabla\times & D_t \\ \nabla\cdot & 0 \\ 0 & \nabla\cdot \end{pmatrix}$ being the Frechet derivative of the system in terms of the total derivative operator $\nabla = (D_x, D_y, D_z)$. The determining equation for adjoint-symmetries $\varpi = \begin{pmatrix} \vec{Q}^E & \vec{Q}^B & Q^E & Q^B \end{pmatrix} G' \begin{pmatrix} \mathrm{d}\vec{E} \\ \mathrm{d}\vec{B} \end{pmatrix}$ is the adjoint equation

$$G'^* \begin{pmatrix} \vec{Q}^E & \vec{Q}^B & Q^E & Q^B \end{pmatrix}\Big|_{\mathcal{E}} = \begin{pmatrix} (-D_t\vec{Q}^E + \nabla\times\vec{Q}^B - \nabla Q^E)|_{\mathcal{E}} \\ (-D_t\vec{Q}^B - \nabla\times\vec{Q}^E - \nabla Q^B)|_{\mathcal{E}} \end{pmatrix} = 0, \tag{45}$$

where the vectors $\vec{Q}^E$, $\vec{Q}^B$, and the scalars Q^E, Q^B, are non-singular functions of t, x, y, z, $\vec{E}$, $\vec{B}$, and derivatives of $\vec{E}$, $\vec{B}$ on $\mathcal{E}$. Note that the adjoint $*$ here includes a matrix transpose applied to the row matrix comprising the adjoint-symmetry vector and scalar functions.

Because Maxwell's equations are a linear system and contain constraints, it possesses three types of adjoint-symmetries [15,16]: elementary adjoint-symmetries such that $\vec{Q}^E$, $\vec{Q}^B$, Q^E, Q^B are functions only of t, x, y, z; gauge adjoint-symmetries given by $\vec{Q}^E = \nabla\chi^E$, $\vec{Q}^B = \nabla\chi^B$, $Q^E = -D_t\chi^E$, $Q^B = -D_t\chi^B$ in terms of scalars χ^E and χ^B that are arbitrary non-singular functions of t, x, y, z, $\vec{E}$, $\vec{B}$, and derivatives of $\vec{E}$, $\vec{B}$ on $\mathcal{E}$; and a hierarchy of linear adjoint-symmetries. The linear adjoint-symmetries of zeroth order are given by the span of

$$\vec{Q}^E = \vec{\xi}\times\vec{B} + \zeta\vec{E}, \quad \vec{Q}^B = -\vec{\xi}\times\vec{E} + \zeta\vec{B}, \quad Q^E = \vec{\xi}\cdot\vec{E}, \quad Q^B = \vec{\xi}\cdot\vec{B} \tag{46}$$

and

$$\vec{Q}^E = \vec{\xi}\times\vec{E} - \zeta\vec{B}, \quad \vec{Q}^B = \vec{\xi}\times\vec{B} + \zeta\vec{E}, \quad Q^E = -\vec{\xi}\cdot\vec{B}, \quad Q^B = \vec{\xi}\cdot\vec{E} \tag{47}$$

where

$$\begin{aligned} \vec{\xi} &= \vec{a}_0 + \vec{a}_1\times\vec{x} + \vec{a}_2 t + a_3\vec{x} + a_4 t\vec{x} + (\vec{a}_5\cdot\vec{x})\vec{x} - \tfrac{1}{2}\vec{a}_5(\vec{x}\cdot\vec{x} + t^2), \\ \zeta &= a_0 + \vec{a}_2\cdot\vec{x} + a_3 t + \tfrac{1}{2}a_4(\vec{x}\cdot\vec{x} + t^2) + (\vec{a}_5\cdot\vec{x})t, \end{aligned} \tag{48}$$

in terms of arbitrary constant scalars a_0, a_3, a_4 and arbitrary constant vectors $\vec{a}_0$, $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_5$, with $\vec{x} = (x, y, z)$. The pair $(\vec{\xi}, \zeta)$ represents a conformal Killing vector in Minkowski space $\mathbb{R}^{3,1}$.

These two zeroth-order adjoint-symmetries are related by the duality symmetry $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$. The linear first-order adjoint-symmetries are more complicated and involve conformal Killing–Yano tensors. All higher order adjoint-symmetries can be obtained from the zeroth and first-order adjoint-symmetries by taking Lie derivatives with respect to conformal Killing vectors. Their explicit description can be found in [15,16]. An unexplored question is whether the lowest order adjoint-symmetries can be used like an invariant surface condition to produce solutions of Maxwell's equations.

4. Some Applications

Two geometrical applications of Theorem 1 will be presented. The first application is a geometrical derivation of a well-known formula that generates a conservation law from a pair consisting of a symmetry and an adjoint-symmetry. This derivation will use the functional pairing (18). The second application is a geometrical derivation of three actions of symmetries on adjoint-symmetries. These symmetry actions have been obtained in recent work using an algebraic point of view [13]. They will be shown here to arise from Cartan's formula for the Lie derivative of an adjoint-symmetry one-form (28).

It will be useful to work with the determining equations for symmetries and adjoint-symmetries off of the solution space $\mathcal{E}$ of a given PDE system (22). More precisely, the determining equations will be expressed in the full jet space containing $\mathcal{E}$.

Remark 1. *A PDE system* (22) *will be assumed to be regular [11], so that Hadamard's lemma holds: a differential function f satisfies $f|_{\mathcal{E}} = 0$ iff $f = R_f(G)$, where R_f is a linear differential operator whose coefficients are non-singular on $\mathcal{E}$.*

Consequently, for symmetries, $G'(P)^A|_{\mathcal{E}} = 0$ holds iff

$$G'(P)^A = R_P(G)^A, \tag{49}$$

and likewise for adjoint-symmetries, $G'^*(Q)_\alpha|_{\mathcal{E}} = 0$ holds iff

$$G'^*(Q)_\alpha = R_Q(G)_\alpha, \tag{50}$$

where R_P and R_Q are linear differential operators whose coefficients are non-singular on $\mathcal{E}$.

4.1. Conservation Laws from Symmetries and Adjoint-Symmetries

The functional pairing (18) between a symmetry vector field (23) and an adjoint-symmetry one-form (28) is given by,

$$\langle \mathrm{pr}\mathbf{X}_P, \varpi_Q \rangle = \langle \mathrm{pr}P^\alpha \partial_{u^\alpha}, Q_A \mathrm{d}G^A \rangle = \int Q_A G'(P)^A \, dx \tag{51}$$

from identity (27). This pairing in local form (19) is the expression,

$$Q_A G'(P)^A \text{ mod total } D. \tag{52}$$

There are two different ways to evaluate it.

First, since $\mathbf{X}_P$ is a symmetry, $Q_A G'(P)^A = Q_A R_P(G)^A$. Second, since ϖ_Q is an adjoint-symmetry, $Q_A G'(P)^A = G'^*(Q)_\alpha P^\alpha + D_i \Psi^i(P,Q)_G = P^\alpha R_Q(G)_\alpha + D_i \Psi^i(P,Q;G)$, where

$$\Psi^i(P,Q;G) = (D_K Q_A)(D_J P^\alpha) E^K_{u^\alpha_{iJ}}(G^A). \tag{53}$$

Hence, on $\mathcal{E}$, $Q_A G'(P)^A|_{\mathcal{E}} = D_i \Psi^i(P,Q)_G|_{\mathcal{E}} = 0$, which is equivalent to $\langle \mathrm{pr}\mathbf{X}_P, \varpi_Q \rangle|_{\mathcal{E}} = 0$. This establishes the following conservation law.

Theorem 2. *Vanishing of the functional pairing* (51) *for any symmetry* (23) *and any adjoint-symmetry* (28) *corresponds to a conservation law*

$$D_i \Psi^i(P,Q;G)|_{\mathcal{E}} = 0 \tag{54}$$

holding for the PDE system $G^A = 0$, where the conserved current $\Psi^i(P,Q;G)$ is given by expression (53)*.*

4.2. Action of symmetries on adjoint-symmetries

For any PDE system (22), its set of adjoint-symmetries is a linear space, and as shown in [13], symmetries of the PDE system have three different actions on this space.

The primary symmetry action can be derived from the Lie derivative of an adjoint-symmetry one-form with respect to a symmetry vector field.

Proposition 2. *If ϖ_Q is an adjoint-symmetry one-form* (28)*, namely $\varpi_Q|_{\mathcal{E}} = 0$ (mod total D), then its Lie derivative with respect to any symmetry vector $\mathbf{X}_P = P^\alpha \partial_{u^\alpha}$ yields an adjoint-symmetry one-form,*

$$\mathcal{L}_{\mathbf{X}_P}\varpi_Q|_{\mathcal{E}} = \varpi_{S_P(Q)}|_{\mathcal{E}} = 0 \text{ (mod total } D) \tag{55}$$

where

$$S_P(Q)_A = Q'(P)_A + R_P^*(Q)_A \tag{56}$$

are its components.

Here and throughout, R_P and R_Q are the linear differential operators determined by Equations (49) and (50). The adjoints of these operators are denoted R_P^* and R_Q^*.

Proof. Recall that the Lie derivative has the following properties: it acts as a derivation; it commutes with the differential d; it reduces to the Frechet derivative when acting on a differential function.

By the use of these properties,

$$\begin{aligned} \mathcal{L}_{\mathbf{X}_P}\varpi_Q &= \mathcal{L}_{\mathbf{X}_P}(Q_A \mathrm{d}G^A) \\ &= (\mathcal{L}_{\mathbf{X}_P}Q_A)\mathrm{d}G^A + Q_A\mathcal{L}_{\mathbf{X}_P}(\mathrm{d}G^A) \\ &= Q'(P)_A\mathrm{d}G^A + Q_A\mathrm{d}(G'(P)^A) \\ &= Q'(P)_A\mathrm{d}G^A + Q_A\mathrm{d}(R_P(G)^A). \end{aligned} \tag{57}$$

The last term can be simplified on $\mathcal{E}$: $Q_A\mathrm{d}(R_P(G)^A)|_{\mathcal{E}} = Q_A R_P(\mathrm{d}G)^A|_{\mathcal{E}} = R_P^*(Q)_A\mathrm{d}G^A$ (mod total D). This yields

$$\mathcal{L}_{\mathbf{X}_P}\varpi_Q|_{\mathcal{E}} = ((Q'(P)_A + R_P^*(Q)_A)\mathrm{d}G^A)|_{\mathcal{E}} \text{ (mod total } D) \,, \tag{58}$$

completing the derivation. □

There is an elegant formula, due to Cartan, for the Lie derivative in terms of the operations d and $\rfloor$. This formula gives rise to two additional symmetry actions.

Theorem 3. *The terms in Cartan's formula*

$$\mathcal{L}_{\mathbf{X}_P}\varpi_Q = \mathrm{d}(\mathrm{pr}\mathbf{X}_P\rfloor\,\varpi_Q) + \mathrm{pr}\mathbf{X}_P\rfloor\,(d\varpi_Q) \tag{59}$$

evaluated on $\mathcal{E}$ each yield an action of symmetries on adjoint symmetries. The action produced by the Lie derivative term has the components (56), and the actions produced by the differential term and the hook term respectively have the components

$$S_{1\,P}(Q) = R_P^*(Q)_A - R_Q^*(P)_A, \tag{60}$$
$$S_{2\,P}(Q) = Q'(P)_A + R_Q^*(P)_A. \tag{61}$$

Proof. Consider the first term on right-hand side in the formula (59). It can be evaluated in two different ways. Firstly, $\mathrm{pr}\mathbf{X}_P\rfloor\,(Q_A\mathrm{d}G^A) = Q_A G'(P)^A = Q_A R_P(G)^A$ yields

$$\mathrm{d}(\mathrm{pr}\mathbf{X}_P\rfloor\,(Q_A\mathrm{d}G^A))|_{\mathcal{E}} = \mathrm{d}(Q_A R_P(G)^A)|_{\mathcal{E}} = (Q_A R_P(\mathrm{d}G^A))|_{\mathcal{E}} = (R_P^*(Q)_A\mathrm{d}G^A)|_{\mathcal{E}}. \tag{62}$$

Secondly, $Q_A\mathrm{d}G^A = R_Q(G)_\alpha\Theta^\alpha + Q_A(D_iG^A)\mathrm{d}x^i$ (mod total D) gives $\mathrm{pr}\mathbf{X}_P\rfloor\,(Q_A\mathrm{d}G^A) = \mathrm{pr}\mathbf{X}_P\rfloor\,(R_Q(G)_\alpha\Theta^\alpha + Q_A(D_iG^A)\mathrm{d}x^i$ (mod total D) $) = R_Q(G)_\alpha P^\alpha$ (mod total D). This yields

$$\begin{aligned} \mathrm{d}(\mathrm{pr}\mathbf{X}_P\rfloor\,(Q_A\mathrm{d}G^A))|_{\mathcal{E}} &= \mathrm{d}(R_Q(G)_\alpha P^\alpha \text{ (mod total } D)\,)|_{\mathcal{E}} \\ &= (R_Q(\mathrm{d}G)_\alpha P^\alpha \text{ (mod total } D)\,)|_{\mathcal{E}} \\ &= (R_Q^*(P)_A\mathrm{d}G^A \text{ (mod total } D)\,)|_{\mathcal{E}}. \end{aligned} \tag{63}$$

Then, equating expressions (62) and (63) leads to the result:

$$((R_P^*(Q)_A - R_Q^*(P)_A)\mathrm{d}G^A)|_{\mathcal{E}} = 0 \text{ (mod total } D)|_{\mathcal{E}}. \tag{64}$$

This equation shows that the symmetry action (60) produces an adjoint-symmetry.

Now, consider the second term on the right-hand side in formula (59). Similarly to the first term, it can be evaluated in two different ways. Firstly, $\mathrm{d}\varpi_Q = \mathrm{d}Q_A \wedge \mathrm{d}G^A$ yields

$$\mathrm{pr}\mathbf{X}_P \rfloor (\mathrm{d}Q_A \wedge \mathrm{d}G^A) = Q'(P)_A \mathrm{d}G^A - G'(P)^A \mathrm{d}Q_A = Q'(P)_A \mathrm{d}G^A - R_P(G)^A \mathrm{d}Q_A. \tag{65}$$

Hence, on $\mathcal{E}$,

$$(\mathrm{pr}\mathbf{X}_P \rfloor (\mathrm{d}Q_A \wedge \mathrm{d}G^A))|_{\mathcal{E}} = (Q'(P)_A \mathrm{d}G^A)|_{\mathcal{E}}. \tag{66}$$

Secondly, $\mathrm{d}\varpi_Q = \mathrm{d}(R_Q(G)_\alpha \Theta^\alpha + Q_A(D_i G^A)\mathrm{d}x^i)$ (mod total D) gives

$$\mathrm{d}\varpi_Q|_{\mathcal{E}} = (R_Q(\mathrm{d}G)_\alpha \wedge \Theta^\alpha + Q_A(D_i \mathrm{d}G^A) \wedge \mathrm{d}x^i)|_{\mathcal{E}} \text{ (mod total } D). \tag{67}$$

This yields

$$\begin{aligned} &(\mathrm{pr}\mathbf{X}_P \rfloor (R_Q(\mathrm{d}G)_\alpha \wedge \Theta^\alpha + Q_A(D_i \mathrm{d}G^A) \wedge \mathrm{d}x^i))|_{\mathcal{E}} \\ &= (R_Q(G'(P))_\alpha \Theta^\alpha - P^\alpha R_Q(\mathrm{d}G)_\alpha + Q_A(D_i G'(P)^A)\mathrm{d}x^i)|_{\mathcal{E}} \\ &= -(R_Q^*(P)_A \mathrm{d}G^A)|_{\mathcal{E}} \text{ (mod total } D). \end{aligned} \tag{68}$$

Equating expressions (66) and (68) then gives the equation

$$((Q'(P)_A + R_Q^*(P)_A)\mathrm{d}G^A)|_{\mathcal{E}} = 0 \text{ (mod total } D)|_{\mathcal{E}}, \tag{69}$$

showing that the symmetry action (61) produces an adjoint-symmetry. □

Observe that the three actions (56), (60) and (61) are related by:

$$S_{1\,P}(Q) + S_{2\,P}(Q) = S_P(Q). \tag{70}$$

Each action is a mapping on the linear space of adjoint-symmetries Q_A. The algebraic properties of these actions can be found in [13].

5. Geometrical Adjoint-Symmetries of Evolution Equations

A general system of evolution equations of order N has the form

$$u_t^\alpha = g^\alpha(x, u, \partial_x u, \ldots, \partial_x^N u) \tag{71}$$

where t is the time variable, x^i, $i = 1, \ldots, n$, are now the space variables, and u^α, $\alpha = 1, \ldots, m$, are the dependent variables. The space of formal solutions $u^\alpha(t, x)$ of the system will be denoted $\mathcal{E}$.

The developments for general PDE systems can be specialized to evolution systems, with $G^\alpha = u_t^\alpha - g^\alpha$ via identifying the indices $A = \alpha$ ($M = m$). On $\mathcal{E}$, since u_t^α can be eliminated through the evolution equations, the components of symmetries and adjoint-symmetries can be assumed to contain only u^α and its spatial derivatives in addition to t and x^i. Hereafter, multi-indices will refer to spatial derivatives.

A symmetry is thereby an evolutionary vector field,

$$\mathbf{X}_P = P^\alpha(t, x, \partial_x u, \ldots, \partial_x^k u)\partial_{u^\alpha} \tag{72}$$

satisfying the linearization of the evolution system on $\mathcal{E}$:

$$(\mathrm{pr}\mathbf{X}_P(u_t^\alpha - g^\alpha))|_{\mathcal{E}} = (D_t P^\alpha - g'(P)^\alpha)|_{\mathcal{E}} = 0. \tag{73}$$

Off of $\mathcal{E}$, $D_t P^\alpha = (P_t + P'(g))^\alpha + P'(G)^\alpha$, whereby $R_P = P'$. Consequently, the symmetry determining equation (73) can be expressed simply as:

$$(P_t + [g, P])^\alpha = 0. \tag{74}$$

The determining equation for adjoint-symmetries $Q_\alpha(t, x, \partial_x u, \ldots, \partial_x^l u)$ is given by the adjoint linearization of the evolution system on $\mathcal{E}$:

$$(-D_t Q - g'^*(Q))_\alpha|_\mathcal{E} = 0. \tag{75}$$

Similar to the symmetry case, here, $R_Q = -Q'$ off of $\mathcal{E}$, and the adjoint-symmetry determining equation simply becomes

$$(Q_t + Q'(g) + g'^*(Q))_\alpha = 0. \tag{76}$$

These two determining equations have a geometrical formulation given by a Lie derivative defined in terms of a flow arising from the evolution system, similar to the situation for ODEs [8]. Specifically, observe that $D_t u^\alpha|_\mathcal{E} = g^\alpha$, and hence, $D_t f|_\mathcal{E} = f_t + f'(g)$ for any differential function f. This motivates introducing the flow vector field,

$$\mathbf{Y} = \partial_t + g^\alpha \partial_{u^\alpha} \tag{77}$$

which is related to the total time derivative by prolongation,

$$\mathrm{pr}\mathbf{Y} = D_t|_\mathcal{E} = \partial_t + (D_I g^\alpha)\partial_{u_I^\alpha}. \tag{78}$$

Associated with this flow vector field is the Lie derivative

$$\mathcal{L}_t := \mathcal{L}_{\mathrm{pr}\mathbf{Y}} \tag{79}$$

which acts on differential functions by $\mathcal{L}_t f = \mathrm{pr}\mathbf{Y}(f) = D_t f|_\mathcal{E}$. On evolutionary vector fields (72), this Lie derivative acts in the standard way as a commutator:

$$\begin{aligned} \mathcal{L}_t \mathrm{pr}\mathbf{X}_P &= \mathrm{pr}((\mathrm{pr}\mathbf{Y}(P) - \mathrm{pr}\mathbf{X}_P(g))^\alpha \partial_{u^\alpha}) \\ &= \mathrm{pr}((P_t + P'(g) - g'(P))^\alpha \partial_{u^\alpha}) \\ &= \mathrm{pr}((P_t + [g, P])^\alpha \partial_{u^\alpha}). \end{aligned} \tag{80}$$

Thus, the symmetry determining equation (74) can be formulated as the vanishing of the Lie derivative expression (80). This establishes the following well-known geometrical result.

Proposition 3. *A symmetry of an evolution system* (71) *is an evolutionary vector field* (72) *that is invariant under the associated flow* (79).

In particular, the resulting Lie-derivative vector field

$$\mathcal{L}_t \mathbf{X}_P = (P_t + [g, P])^\alpha \partial_{u^\alpha} \tag{81}$$

vanishes iff the functions P_α are the components of a symmetry.

A similar characterization will now be given for adjoint-symmetries, based on viewing the adjoint relation between the determining equations (74) and (76) as a duality relation between vectors and one-forms.

Introduce the evolutionary one-form:

$$\omega_Q = Q_\alpha(t, x, \partial_x u, \ldots, \partial_x^l u) \mathrm{d}u^\alpha. \tag{82}$$

Its Lie derivative is given by

$$\begin{aligned}\mathcal{L}_t\omega_Q &= (\mathcal{L}_t Q_\alpha)\mathrm{d}u^\alpha + Q_\alpha\mathcal{L}_t(\mathrm{d}u^\alpha)\\ &= (Q_t + Q'(g))_\alpha \mathrm{d}u^\alpha + Q_\alpha \mathrm{d}(\mathcal{L}_t u^\alpha)\\ &= (Q_t + Q'(g))_\alpha \mathrm{d}u^\alpha + Q_\alpha \mathrm{d}g^\alpha\\ &= (Q_t + Q'(g) + g'^*(Q))_\alpha \mathrm{d}u^\alpha \text{ (mod total } D).\end{aligned} \tag{83}$$

This shows that the adjoint-symmetry determining equation (76) can be formulated as the functional vanishing of the Lie derivative expression (83).

Theorem 4. *An adjoint-symmetry of an evolution system* (71) *is an evolutionary one-form* (82) *that is functionally invariant under the associated flow* (79)*.*

In particular, the resulting Lie-derivative one-form

$$\mathcal{L}_t\omega_Q = (Q_t + Q'(g) + g'^*(Q))_\alpha \mathrm{d}u^\alpha \text{ (mod total } D) \tag{84}$$

functionally vanishes iff the functions Q_α are the components of an adjoint-symmetry. This one-form (84) is functionally equivalent to the adjoint-symmetry one-form (28) introduced for a general PDE system. To see the relationship in detail, observe that:

$$\begin{aligned}\varpi_Q = Q_\alpha \mathrm{d}G^\alpha &= Q_\alpha \mathrm{d}(u^\alpha_t - g^\alpha)\\ &= Q_\alpha(D_t(\mathrm{d}u^\alpha) - g'(\mathrm{d}u)^\alpha)\\ &= -(D_t Q_\alpha + g'^*(Q)_\alpha)\mathrm{d}u^\alpha \text{ (mod total } D)\\ &= -\mathcal{L}_t\omega_Q \text{ (mod total } D).\end{aligned} \tag{85}$$

An interesting question is how to extend this relationship to more general PDE systems.

Evolution Equations with Spatial Constraints

A wide generalization of evolution systems occurring in applied mathematics and mathematical physics is given by systems comprised of evolution equations with spatial constraints. Some notable examples are Maxwell's equations, incompressible fluid equations, magnetohydrodynamical equations, and Einstein's equations.

The constraints in such systems in general consist of spatial equations

$$C^\Upsilon(x, u, \partial_x u, \ldots, \partial_x^{N'} u) = 0, \quad \Upsilon = 1, \ldots, M' \tag{86}$$

that are compatible with the evolution equation (71). Compatibility means that the time derivative of the constraints vanishes on the solution space $\mathcal{E}$ of the whole system, $(D_t C^\Upsilon)|_{\mathcal{E}} = 0$. For systems that are regular [11], Hadamard's lemma implies that the system obeys a differential identity,

$$D_t C^\Upsilon = C'(G)^\Upsilon + \mathcal{D}(C)^\Upsilon \tag{87}$$

where $G^\alpha = u^\alpha_t - g^\alpha$ denotes the evolution equation (71), and where $\mathcal{D}$ is a linear differential spatial operator whose coefficients are non-singular on $\mathcal{E}$. Equivalently, the constraints must obey the identity $C'(g)^\Upsilon = \mathcal{D}(C)^\Upsilon$. A comparison of the differential order of each side of this identity shows that $\mathcal{D}$ is of the same order N as the evolution equations, namely:

$$\mathcal{D} = \sum_{0\leq|I|\leq N} R^{I\Upsilon}_{\Lambda} D_I. \tag{88}$$

The full system consists of $n + M'$ equations $G^\alpha = 0$, $C^\Upsilon = 0$. Note that, in the previous notation (22), $(G^\alpha, C^\Upsilon) = (G^A)$ with $A = (\alpha, \Upsilon)$.

The symmetry determining equation is given by the linearization of the full system on $\mathcal{E}$, which is comprised by the evolution part (73) and the constraint part

$$(\mathrm{pr}\mathbf{X}_P C^\Upsilon)|_\mathcal{E} = C'(P)^\Upsilon|_\mathcal{E} = 0. \tag{89}$$

Off of $\mathcal{E}$, $C'(P)^\Upsilon = R_C(C)^\Upsilon$, where R_C is a linear differential spatial operator whose coefficients are non-singular on $\mathcal{E}$. Hence, the determining equations (73) and (89) can be stated as:

$$(P_t + [g, P])^\alpha|_{\mathcal{E}_C} = 0, \quad C'(P)^\Upsilon|_{\mathcal{E}_C} = 0 \tag{90}$$

where $\mathcal{E}_C$ denotes the solution space of the spatial constraint equation (86).

The adjoint-symmetry determining equation is given by the adjoint linearization of the full system on $\mathcal{E}$, which comprises evolution terms and additional constraint terms:

$$(-D_t Q - g'^*(Q) + C'^*(q))_\alpha|_\mathcal{E} = 0. \tag{91}$$

Here, the components of an adjoint-symmetry consist of

$$(Q_\alpha(t, x, \partial_x u, \ldots, \partial_x^l u), q_\Upsilon(t, x, \partial_x u, \ldots, \partial_x^{l'} u)) \tag{92}$$

with Q_α being associated with the evolution equations as before, while q_Υ is associated with the constraint equations. Similar to the symmetry case, the determining equation can be stated as:

$$(Q_t + Q'(g) + g'^*(Q) - C'^*(q))_\alpha|_{\mathcal{E}_C} = 0. \tag{93}$$

These determining equations for symmetries and adjoint-symmetries have a geometrical formulation in terms of a constrained flow (77), generalizing the previous formulation for evolution systems as follows.

Theorem 5. *A symmetry of a constrained evolution system* (71) *and* (86) *is an evolutionary vector field* (72) *that is invariant under the associated constrained flow* (79) *and that preserves the constraints.*

The proof of this result is simply the observation that, first, the determining Equation (89) corresponds to the constraints being preserved, and second, the Lie derivative of the symmetry vector field (81) along the flow vanishes on the constraint solution space.

Theorem 6. *An adjoint-symmetry of a constrained evolution system* (71) *and* (86) *is an evolutionary one-form* (82) *that is functionally invariant under the associated constrained flow* (79)*, up to a functional multiple of the normal one-form* $\mathrm{d}C^\Upsilon$ *arising from the constraints.*

The proof is given by the earlier computation (84) for the Lie derivative of the adjoint-symmetry one-form. This computation shows that the adjoint-symmetry determining Equation (93) now can be expressed as:

$$\mathcal{L}_t \omega_Q|_{\mathcal{E}_C} = (C'^*(q)_\alpha \mathrm{d}u^\alpha)|_{\mathcal{E}_C} = (q_\Upsilon \mathrm{d}C^\Upsilon)|_{\mathcal{E}_C} \text{ (mod total } D) \tag{94}$$

where $\mathrm{d}C^\Upsilon$ is the normal one-form given by the constraints viewed as surfaces in jet space.

The Lie-derivative one-form (94) is functionally equivalent to the adjoint-symmetry one-form (28) introduced for a general PDE system. In the present notation, the full system of evolution and constraint equations (71) and (86) consists of $(G^\alpha, C^\Upsilon) = 0$, and the corresponding one-form associated with this system is given by $\varpi_{Q,q} = Q_\alpha \mathrm{d}G^\alpha + q_\Upsilon \mathrm{d}C^\Upsilon$. Now, using the relation (85), observe that:

$$\varpi_{Q,q} = q_\Upsilon \mathrm{d}C^\Upsilon - \mathcal{L}_t \omega_Q \text{ (mod total } D). \tag{95}$$

There is a class of adjoint-symmetries arising from the summed product of arbitrary functions $\chi_\Upsilon(t,x)$ and the components of the the differential identity (87). This yields, after integration by parts,

$$\begin{aligned} 0 &= \chi_\Upsilon(D_t C^\Upsilon - C'(G)^\Upsilon - \mathcal{D}(C)^\Upsilon) \\ &= D_t(\chi_\Upsilon C^\Upsilon) + D_i \Psi^i(\chi, G; C) - D_i \Phi^i(\chi, C; R) - (D_t\chi + \mathcal{D}^*(\chi))_\Upsilon C^\Upsilon - C'^*(\chi)_\alpha G^\alpha \end{aligned} \tag{96}$$

where $\Phi^i(\chi, C; R) = \sum_{0 \le |I| \le N-1} (-1)^{|J|} D_J(\chi_\Upsilon R^{iI\Upsilon}_{\Lambda}) D_{I/J} C^\Lambda$ from expression (88). Hence,

$$D_t(\chi_\Upsilon C^\Upsilon) + D_i(\Psi^i(\chi, G; C) - \Phi^i(\chi, C; R)) = C'^*(\chi)_\alpha G^\alpha + (D_t\chi + \mathcal{D}^*(\chi))_\Upsilon C^\Upsilon \tag{97}$$

has the form of a conservation law off $\mathcal{E}$, with $(C'^*(\chi)_\alpha, (D_t\chi + \mathcal{D}^*(\chi))_\Upsilon)$ being the multiplier. As is well known, every multiplier for a regular PDE system is an adjoint-symmetry [1,3,11,17,18]. This can be proven here by applying the Euler operator E_{u^α} and using its product rule. Consequently,

$$Q_\alpha = C'^*(\chi)_\alpha, \quad q_\Upsilon = (D_t\chi + \mathcal{D}^*(\chi))_\Upsilon \tag{98}$$

are components of an adjoint-symmetry, involving the arbitrary functions $\chi_\Upsilon(t,x)$. Such adjoint-symmetries are a counterpart of gauge symmetries, and accordingly are called gauge adjoint-symmetries [11].

The corresponding gauge adjoint-symmetry one-form is given by

$$\omega_\chi = C'^*(\chi)_\alpha \mathrm{d}u^\alpha = \chi_\Upsilon \mathrm{d}C^\Upsilon \text{ (mod total } D) \tag{99}$$

and satisfies the geometrical relation

$$\mathcal{L}_t \omega_\chi|_{\mathcal{E}_C} = ((D_t\chi + \mathcal{D}^*(\chi))_\Upsilon \mathrm{d}C^\Upsilon)|_{\mathcal{E}_C} \text{ (mod total } D). \tag{100}$$

This establishes the following geometrical result.

Theorem 7. *A gauge adjoint-symmetry* (98) *is functionally equivalent to a normal one-form ω_χ associated with the constraint equation* (86)*. Under the evolution flow, it is mapped into another normal one-form.*

The preceding developments for general systems of evolution equations with spatial constraints have used the classical notion of symmetries and adjoint-symmetries. It would be interesting to extend the formulation and the results by considering a notion of conditional symmetries and corresponding conditional adjoint-symmetries based on the spatial constraints.

Specifically, on the solution space of the full system, consider a symmetry given by an evolutionary vector field (72) that satisfies

$$(P_t + [g, P])^\alpha|_{\mathcal{E}_C} = 0 \tag{101}$$

where $\mathcal{E}_C$ denotes the solution space of the spatial constraint Equation (86). Such conditional symmetries (101) differ from classical symmetries (90) by relaxing the condition that the constraints are preserved. Their natural adjoint counterpart is given by an evolutionary one-form (82) satisfying

$$(Q_t + Q'(g) + g'^*(Q))_\alpha|_{\mathcal{E}_C} = 0. \tag{102}$$

which is the adjoint of the determining Equation (101). Such conditional adjoint-symmetries (102) differ from classical adjoint-symmetries (93) by excluding the terms arising from the spatial constraints.

This notion of conditional symmetries and adjoint-symmetries is more general than the classical notion because the conditional determining equations hold on $\mathcal{E}_C$ instead of the whole jet space.

6. Concluding Remarks

The main results showing how adjoint-symmetries correspond to evolutionary one-forms with certain geometrical properties provides a first step towards giving a fully geometrical interpretation for adjoint-symmetries. In particular, for systems of evolution equations, adjoint-symmetries can be geometrically described as one-forms that are invariant under the flow generated by the system on the solution space. This interesting result has a straightforward generalization to systems of evolution equations with spatial constraints. Consequently, the results presented here are applicable to all PDE systems of interest in applied mathematics and mathematical physics.

One direction for future work will be to translate and generalize these results into the abstract geometrical setting of secondary calculus [2,19] developed by Vinogradov and Krasil'shchik and their co-workers.

It will also be interesting to fully develop the use of adjoint-symmetries in the study of specific PDE systems, as outlined in the Introduction: finding exact solutions, detecting and finding mappings into a target class of PDEs, and detecting integrability, which are the counterparts of some important uses of symmetries. Another use of adjoint-symmetries, which has been introduced very recently [20], is for finding pre-symplectic operators.

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References

1. Bluman, G.W.; Cheviakov, A.; Anco, S.C. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA, 2009.
2. Krasil'shchik, I.S.; Vinogradov, A.M. (Eds.) *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*; Translations of Math. Monographs 182; American Mathematical Society: Providence, RI, USA, 1999.
3. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
4. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
5. Bluman, G.W.; Anco, S.C. *Symmetry and Integration Methods for Differential Equations*; Springer: New York, NY, USA, 2002.
6. Sarlet, W.; Cantrijn, F.; Crampin, M. Pseudo-symmetries, Noether's theorem and the adjoint equation. *J. Phys. A Math. Gen.* **1987**, *20*, 1365–1376. [CrossRef]
7. Sarlet, W.; Bonne, J.V. REDUCE procedures for the study of adjoint symmetries of second-order differential equations. *J. Symb. Comput.* **1992**, *13*, 683–693. [CrossRef]
8. Sarlet, W. Construction of adjoint symmetries for systems of second-order and mixed first- and second-order ordinary differential equations. *Math. Comput. Model.* **1997**, *25*, 39–49. [CrossRef]
9. Anco, S.C.; Bluman, G. Direct construction of conservation laws from field equations. *Phys. Rev. Lett.* **1997**, *78*, 2869–2873. [CrossRef]
10. Anco, S.C.; Bluman, G. Direct construction method for conservation laws of partial differential equations Part II: General treatment. *Eur. J. Appl. Math.* **2002**, *41*, 567–585. [CrossRef]
11. Anco, S.C. Generalization of Noether's theorem in modern form to non-variational partial differential equations. In *Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science*; Springer: New York, NY, USA, 2017; Volume 79, pp. 119–182,

12. Anco, S.C. On the incompleteness of Ibragimov's conservation law theorem and its equivalence to a standard formula using symmetries and adjoint-symmetries. *Symmetry* **2017**, *9*, 33. [CrossRef]
13. Anco, S.C.; Wang, B. Algebraic structures for adjoint-symmetries and symmetries of partial differential equations. *arXiv* **2020**, arXiv:2008.07476.
14. Nestruev, J. *Smooth Manifolds and Observables*; Graduate Texts in Mathematics 220; Springer: Berlin, Germany, 2002.
15. Anco, S.C.; Pohjanpelto, J. Classification of local conservation laws of Maxwell's equations. *Acta Appl. Math.* **2001**, *69*, 285–327. [CrossRef]
16. Anco, S.C.; Pohjanpelto, J. Symmetries and currents of massless neutrino fields, electromagnetic and graviton fields. In *CRM Proceedings and Lecture Notes (Workshop on Symmetry in Physics)*; American Mathematical Society: Providence, RI, USA, 2004; Volume 34, pp. 1–12.
17. Vinogradov, A.M. The C-spectral sequence, Lagrangian formalism, and conservation laws I. The linear theory. *J. Math. Anal. Appl.* **1984**, *100*, 1–40. [CrossRef]
18. Vinogradov, A.M. The C-spectral sequence, Lagrangian formalism, and conservation laws II. The nonlinear theory. *J. Math. Anal. Appl.* **1984**, *100*, 41–129. [CrossRef]
19. Vinogradov, A.M. Introduction to Secondary Calculus. In *Proceedings of the Conference Secondary Calculus and Cohomology Physics*; Henneaux, M., Krasil'shchik, I.S., Vinogradov, A.M., Eds.; Contemporary Mathematics; American Mathematical Society: Providence, RI, USA, 1998.
20. Anco, S.C.; Wang, B. A formula for symmetry recursion operators from non-variational symmetries of partial differential equations. *arXiv* **2020**, arXiv:2004.03743.