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An Extended VIKOR Method Based on q-Rung Orthopair Shadowed Set and Its Application to Multi-Attribute Decision Making

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Abstract: In the multi-attribute decision making (MADM) process, the attribute values are sometimes provided by experts or the public in the form of words. To model the linguistic evaluation more accurately, this paper proposes the q-rung orthopair shadowed set (q-ROSS) to represent attribute values and extends the VIKOR (Visekriterijumska optimizacija i KOmpromisno Resenje) method to solve MADM problems in the q-ROSS context. First, we propose the q-ROSS to express evaluation information. Some basic operation rules and distance measures are investigated accordingly. When the amount of data is large, the left and right endpoints of the collected interval numbers will obey symmetric normal distribution. Secondly, based on the normal distribution assumption, the collected data intervals are mapped to shadowed sets through a data processing approach. Furthermore, we extend the VIKOR model to tackle the MADM problem where the evaluation values are expressed by q-rung orthopair shadowed numbers. A location selection problem verifies the practicability of our method, and the effectiveness and superiority of the presented approach are reflected through comparative analysis.

Keywords: shadowed set; q-rung orthopair shadowed set; distance measure; VIKOR; multi-attribute decision making

1. Introduction

Multi-attribute decision-making (MADM) is a process focusing on selecting the best alternative or ranking alternatives in a finite discrete decision-making space based on attribute weights and evaluation values. It is applied to various fields such as economy, management and engineering technology. With the increasing complexity of social environment and decision-making problem, it is more difficult for decision makers (DMs) to give precise evaluation values directly and accurately. Uncertainty and fuzziness become common phenomena in decision-making problems. In order to solve this problem, Zadeh [1] proposed fuzzy sets (FSs), which use the membership degree to describe the level to of an element belonging to a fuzzy set. Subsequently, FSs has been expanded to different types from various research angles. Atanassov [2] proposed the intuitionistic fuzzy sets (IFSs) considering membership and non-membership degrees expressed as μ and v , respectively, thus the choice of support and objection in MADM is evaluated. It is noteworthy that IFS also has its limitations in describing decision-making information with the limitation that the sum of the two parameters is less than or equal to 1. For example, in some situations, if the μ of an element to the set is 0.8 and v is 0.3, obviously, the sum of them exceeds 1, then the decision making information cannot be represented by IFSs. Then Yager [3,4] proposed the concept of a Pythagorean fuzzy set (PFS), where the sum of squares of the μ and v is less than or equal to 1. In order to make up for the deficiency of IFSs and PFSs,

Yager [5,6] further defined q -rung orthopair fuzzy sets (q -ROFSs), which is featured by the sum of the q powers of μ and v not exceeding 1. Apparently, when $q = 1$, q -ROFS degrades to IFS; when $q = 2$, q -ROFS becomes PFS; Figure 1 shows their relationship. Considering the better ability of q -ROFS in depicting fuzzy information, q -ROFS has been extensively studied from different perspectives. For example, some extended forms have been investigated such as q -rung orthopair hesitant fuzzy set [7], q -rung orthopair uncertain linguistic set [8], q -rung orthopair normal fuzzy set [9], etc. Based on q -ROFSs, Liu et al. [10] explored consensus reaching process combined with fuzzy behavioral TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method. Peng and Liu [11] investigated the systematic transformation between the distance measure, the similarity measure, the entropy, and the inclusion measure for q -ROFSs. Considering the impact of the hesitation degree and the different shapes constructed by different q th order, Adem Pinar and Fatih Emre Boran [12] proposed a new distance measure to solve the supplier selection issue. Liang et al. [13] developed q -rung orthopair fuzzy sets-based decision-theoretic rough sets for three-way decisions in the context of group decision making. To effectively aggregate information, Peng et al. [14] derived the q -rung orthopair fuzzy weighted exponential aggregation operator. Within the q -rung orthopair uncertain linguistic context, a family of the extended Muirhead mean operators are utilized by Wang [15] to cope with the multicriteria decision-making problem. Liu et al. [7] presented the distance measures between q -ROHFSs based on the concept of a “multiple fuzzy set”.

Figure 1. The relationship between intuitionistic fuzzy set (IFS), Pythagorean fuzzy set (PFS), and q -rung orthopair fuzzy set (q -ROFS).

Different from fuzzy sets, shadowed sets consist of three parts according to membership grade. The membership grade in the core area is 1, which means they can fully reflect the collected evaluation information. Oppositely, the membership degree of the elements in the exclusion area is 0, which means they do not represent the collected evaluation information at all. The membership degree in the shadowed area is undetermined, which means there is indeterminacy whether the elements in the shadowed area can represent the evaluation information [16]. Compared with fuzzy sets, it reduces the computational complexity. Shadowed sets are usually generated by optimizing the threshold selection or by shadowed clustering algorithm [16–18]. Tahayori et al. [19] presented a new algorithm by applying defuzzification of the gradual number of fuzziness to determine the thresholds. Zhang et al. [20] optimized the selection of thresholds in the light of the game theory between elevation and reduction errors. Yao et al. [21] give a new decision-theoretic formulation of shadowed sets according to the principle of uncertainty invariance. In the multi-attribute decision-making problem involving public evaluation, shadowed sets can be directly constructed by a data-driven method from the collected data [22,23].

The motivation and purpose of our study are summarized as follows:

1. As the evaluated objects in MADM problems, for example, some infrastructure projects, are closely related to public interest, the public's opinions are vital and more reasonable in real situation. The more the public participate in the evaluation, the more reliable and extensive the evaluation

information is. In most cases, it is much easier for people to assign words rather than numbers to each attribute value. To process the linguistic evaluation collected in the form of interval numbers, in this paper, an encoding method that transforms interval numbers to shadowed sets is constructed.

2. In q-ROFS, as q increases, the space of possible orthopairs (u, v) increases, which has less limitation of the value description and hence provides the DMs more freedom to express their evaluations. Inspired by this, we propose q-rung shadowed sets (q-ROSSs) and establish the corresponding basic operations.
3. VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) method is a powerful and effective tool with merits of considering the group utility value and the individual regret value simultaneously, thus an extended VIKOR method to solve the MADM problem with q-ROSS decision information is introduced. A site selection problem is adopted to verify the validity of the approach.

The rest of this paper is organized as follows. In Section 2, the concepts of q-rung orthopair fuzzy set and shadowed set are briefly reviewed. Section 3 defines the q-rung orthopair shadowed set, some properties and operations are derived and proved, and two aggregated operators are also proposed, namely q-rung orthopair shadowed set weighted average operator (q -SNWA) and the q-rung orthopair shadowed set weighted geometric operator (q -SNWG). In Section 4, the shadowed set transformation from interval numbers is achieved by a data-driven method. Section 5 builds the q-rung orthopair shadowed VIKOR model for MADM problems. The feasibility of the proposed extend model is demonstrated through a site selection problem in Section 6. Finally, in Section 7, the conclusions and future researches are presented.

2. Preliminaries

2.1. Q-Rung Orthopair Fuzzy Set (q-ROFS)

Definition 1 ([6]). Suppose $X = \{x_1, x_2, x_3, \dots, x_n\}$ is a fixed set. A q-ROFS on X is given by:

$$A = \left\{ \langle x_i, (u_A(x_i), v_A(x_i)) \rangle_q \mid x_i \in X \right\} \quad q \geq 1 \quad (1)$$

where $u_A(x_i)$ and $v_A(x_i) \in [0, 1]$ represent the membership and non-membership degree of $x_i \in X$, respectively. The restrictive condition is that $0 \leq (u_A(x_i))^q + (v_A(x_i))^q \leq 1 (q \geq 1)$. The hesitancy degree $\pi_A(x_i) = \left(1 - (u_A(x_i))^q - (v_A(x_i))^q\right)^{\frac{1}{q}}$. Obviously, $0 \leq \pi_A(x_i) \leq 1 (x_i \in X)$. For convenience, we call $\alpha = (u_\alpha, v_\alpha)$ a q-rung orthopair fuzzy number (q-ROFN), where $u_\alpha, v_\alpha \in [0, 1]$, $0 \leq (u_\alpha)^q + (v_\alpha)^q \leq 1$ and $\pi_\alpha = \left(1 - (u_\alpha)^q - (v_\alpha)^q\right)^{\frac{1}{q}}$.

Definition 2 ([24]). Suppose $\alpha = (u_\alpha, v_\alpha)$ is a q-ROFN, the score function S of α is defined as $S(\alpha) = \frac{1}{2} \left(1 + (u_\alpha)^q - (v_\alpha)^q\right)$, and the accuracy function H of α is defined as $H(\alpha) = (u_\alpha)^q + (v_\alpha)^q$, where $q \geq 1$.

Based on the above score function and the accuracy function, a comparison method between q-ROFNs is put forward, which is elaborated as follows.

Definition 3 ([24]). Suppose $\alpha = (u_\alpha, v_\alpha)$ and $\beta = (u_\beta, v_\beta)$ are any two q-ROFNs, the score of α and β are S_α and S_β , respectively, then

- (1) If $S_\alpha > S_\beta$, $\alpha > \beta$
- (2) If $S_\alpha = S_\beta$, then

If $H_\alpha > H_\beta$, $\alpha > \beta$;

If $H_\alpha = H_\beta$, $\alpha = \beta$.

2.2. Shadowed Set

Definition 4 ([16,25]). A shadowed set S is a set-valued mapping $S : U \rightarrow \{0, [0, 1], 1\}$ with a pair of thresholds (α, β) where $0 \leq \beta \leq \alpha \leq 1$, and U is the universe of discourse. That is:

$$S_{(\alpha, \beta)}(\mu_A(x)) = \begin{cases} 0, & \mu_A(x) \leq \beta, \\ [0, 1], & \beta \leq \mu_A(x) \leq \alpha, \\ 1, & \mu_A(x) \geq \alpha. \end{cases} \quad (2)$$

Given a concept A and an element x in the universe U , if the membership degree $\mu_A(x)$ less than or equal to β , an reduction operation reduces $\mu_A(x)$ to 0, which means that x is excluded from A . The exclusion area is expressed as,

$$\text{excl}(S) = \{x \in U \mid S_{(\alpha, \beta)}(\mu_A(x)) = 0\} \quad (3)$$

If $\mu_A(x)$ is greater than or equal to α , an elevation operation elevates $\mu_A(x)$ to 1, which means that x completely belongs to A . All these elements whose mapping values are equal to 1 compose the core of S , which is denoted as,

$$\text{core}(S) = \{x \in U \mid S_{(\alpha, \beta)}(\mu_A(x)) = 1\} \quad (4)$$

The element x will be arranged in shadowed area if $\mu_A(x)$ is between α and β , which is mapped to the unit interval $[0, 1]$, the shadowed area takes the form of:

$$\text{shad}(S) = \{x \in U \mid S_{(\alpha, \beta)}(\mu_A(x)) = [0, 1]\} \quad (5)$$

Figure 2 shows one specific shadowed set generated by a Gaussian fuzzy set including its core area, shadowed area and exclusion area.

Figure 2. The concept of a shadowed set and its corresponding fuzzy set.

The result of shadowed sets are shadowed numbers. A shadowed number (SN) is defined with the following parameters [22].

$$SN = (a, b, c, d) \quad (6)$$

where the two intervals $[a, b]$, $[c, d]$ are the left and right shadowed areas and $[b, c]$ is the core area.

Definition 5. Suppose $SN_1 = (a_1, b_1, c_1, d_1)$ and $SN_2 = (a_2, b_2, c_2, d_2)$ are any two SNs, some basic operations are defined as follows:

- (1) $SN_1 + SN_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$
- (2) $SN_1 \times SN_2 = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2)$
- (3) $\kappa SN_1 = (\kappa a_1, \kappa b_1, \kappa c_1, \kappa d_1)$
- (4) $SN_1^\kappa = (a_1^\kappa, b_1^\kappa, c_1^\kappa, d_1^\kappa)$

3. q-Rung Orthopair Shadowed Set (q-ROSS)

In this section, we will define the q-rung orthopair shadowed set (q-ROSS) and explore some properties and operations of it.

3.1. q-Rung Orthopair Shadowed Set (q-ROSS)

Definition 6. Let X be a nonempty set, a q-rung orthopair shadowed set Q on X can be defined as follows:

$$Q = \left\{ \langle SN, (u_Q(x), v_Q(x)) \rangle_q \mid x \in X \right\} \quad (7)$$

where $SN = (a, b, c, d)$ is a shadowed number. $\mu_Q(x) \in [0, 1]$ and $v_Q(x) \in [0, 1]$ are the membership degree and the non-membership degree of SN , respectively. $0 \leq (\mu_Q(x))^q + (v_Q(x))^q \leq 1$ ($q \geq 1$), $\pi_Q(x) = (1 - (\mu_Q(x))^q - (v_Q(x))^q)^{\frac{1}{q}}$. For the sake of simplicity, a q-rung orthopair shadowed number (q-SN) can be expressed as $N = \langle SN, (\mu_N(x), v_N(x)) \rangle_q$.

Definition 7. Basic operation rules of q-SNs. Suppose $N_1 = \langle SN_1, (\mu(x_1), v(x_1)) \rangle_q$ and $N_2 = \langle SN_2, (\mu(x_2), v(x_2)) \rangle_q$ are two q-SNs, where $SN_1 = (a_1, b_1, c_1, d_1)$ and $SN_2 = (a_2, b_2, c_2, d_2)$, then some basic operations are defined as follows:

(1)

$$N_1 + N_2 = \left\langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2), \left(\sqrt[q]{(\mu(x_1))^q + (\mu(x_2))^q - (\mu(x_1))^q(\mu(x_2))^q}, v(x_1)v(x_2) \right) \right\rangle_q$$

(2)

$$N_1 \times N_2 = \left\langle (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2), \left(\mu(x_1)\mu(x_2), \sqrt[q]{(v(x_1))^q + (v(x_2))^q - (v(x_1))^q(v(x_2))^q} \right) \right\rangle_q$$

(3)

$$\kappa N_1 = \left\langle (\kappa a_1, \kappa b_1, \kappa c_1, \kappa d_1), \left(\sqrt[q]{1 - (1 - (\mu(x_1))^q)^\kappa}, (v(x_1))^\kappa \right) \right\rangle_q$$

(4)

$$N_1^\kappa = \left\langle (a_1^\kappa, b_1^\kappa, c_1^\kappa, d_1^\kappa), \left((\mu(x_1))^\kappa, \sqrt[q]{1 - (1 - (v(x_1))^q)^\kappa} \right) \right\rangle_q$$

Definition 8. Let $p_j = \langle (a_j, b_j, c_j, d_j), (\mu_j, v_j) \rangle_q$ ($j = 1, 2, \dots, n$) be q-SNs, then the q-SN weighted average operator (q-SNWA) and the q-SN weighted geometric operator (q-SNWG) of q-SNs can be defined as Equations (8) and (9).

$$\begin{aligned} q-SNWA &= (p_1, p_2, \dots, p_n) = \sum_{j=1}^n w_j p_j \\ &= \left\langle \left(\sum_{j=1}^n w_j a_j, \sum_{j=1}^n w_j b_j, \sum_{j=1}^n w_j c_j, \sum_{j=1}^n w_j d_j \right), \left(\sqrt[q]{1 - \prod_{j=1}^n (1 - (\mu_j)^q)^{w_j}}, \prod_{j=1}^n (v_j)^{w_j} \right) \right\rangle_q \end{aligned} \quad (8)$$

$$\begin{aligned} q-SNWG &= (p_1, p_2, \dots, p_n) = \sum_{j=1}^n (p_j)^{w_j} \\ &= \left\langle \left(\prod_{j=1}^n (a_j)^{w_j}, \prod_{j=1}^n (b_j)^{w_j}, \prod_{j=1}^n (c_j)^{w_j}, \prod_{j=1}^n (d_j)^{w_j} \right), \left(\prod_{j=1}^n (\mu_j)^{w_j}, \sqrt[q]{1 - \prod_{j=1}^n (1 - (v_j)^q)^{w_j}} \right) \right\rangle_q \end{aligned} \quad (9)$$

where w_j is the weight vector of p_j ($j = 1, 2, \dots, n$), $0 \leq w_j \leq 1$, $\sum_{j=1}^n w_j = 1$.

3.2. Distance Measure between q -SNs

Definition 9. Suppose $\alpha = (u_\alpha, v_\alpha)$ and $\beta = (u_\beta, v_\beta)$ are any two q -ROFNs, the Euclidean distance measure between α and β is expressed as follows:

$$d_{q\text{-ROFNED}}(\alpha, \beta) = \sqrt{\frac{1}{2} \left(\left| (\mu_\alpha)^q - (\mu_\beta)^q \right|^2 + \left| (v_\alpha)^q - (v_\beta)^q \right|^2 + \left| (\pi_\alpha)^q - (\pi_\beta)^q \right|^2 \right)} \quad (10)$$

Definition 10. Suppose $SN_1 = (a_1, b_1, c_1, d_1)$ and $SN_2 = (a_2, b_2, c_2, d_2)$ are any two SNs, then the distance between A and B is defined as follows:

$$d_{SND}(SN_1, SN_2) = \sqrt{\frac{1}{2} \left((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2 \right)} \quad (11)$$

Definition 11. Suppose $N_1 = \langle SN_1, (\mu(x_1), v(x_1)) \rangle_q$ and $N_2 = \langle SN_2, (\mu(x_2), v(x_2)) \rangle_q$ are two q -SNs, where $SN_1 = (a_1, b_1, c_1, d_1)$ and $SN_2 = (a_2, b_2, c_2, d_2)$, the distance between N_1 and N_2 takes the form of:

$$D(N_1, N_2) = \sqrt{d_{SND}(SN_1, SN_2) d_{q\text{-ROFNED}}(x_1, x_2)} \quad (12)$$

where

$$d_{SND}(SN_1, SN_2) = \sqrt{\frac{1}{2} \left((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2 \right)},$$

$$d_{q\text{-ROFNED}}(x_1, x_2) = \sqrt{\frac{1}{2} \left(\left| (\mu(x_1))^q - (\mu(x_2))^q \right|^2 + \left| (v(x_1))^q - (v(x_2))^q \right|^2 + \left| (\pi(x_1))^q - (\pi(x_2))^q \right|^2 \right)}.$$

Theorem 1. Let $N_1 = \langle SN_1, (\mu(x_1), v(x_1)) \rangle_q$ and $N_2 = \langle SN_2, (\mu(x_2), v(x_2)) \rangle_q$ be any two q -SNs, where $SN_1 = (a_1, b_1, c_1, d_1)$, $SN_2 = (a_2, b_2, c_2, d_2)$. Then the distance measure $D(N_1, N_2)$ satisfies the following properties:

- (1) **Positivity** $D(N_1, N_2) \geq 0$.
- (2) **Symmetry** $D(N_1, N_2) = D(N_2, N_1)$.
- (3) **Triangular inequality** For any q -SN $N_3 = \langle SN_3, (\mu(x_3), v(x_3)) \rangle_q$ where $SN_3 = (a_3, b_3, c_3, d_3)$, $D(N_1, N_2) + D(N_2, N_3) \geq D(N_1, N_3)$.

Proof.

- (1) Because $q \geq 1$, it is obvious that $d_{SND}(SN_1, SN_2)$ and $d_{q\text{-ROFNED}}(x_1, x_2)$ satisfy the positivity,
- (2) therefore $D(N_1, N_2) \geq 0$.

(3) According to Equation (12), the order of N_1 and N_2 does not influence the final result, so the symmetry is satisfied.

$$\begin{aligned}
 & (D(N_1, N_2) + D(N_2, N_3))^2 \\
 &= \left(\frac{1}{2} \sqrt{((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2) \left(|(\mu(x_1))^q - (\mu(x_2))^q|^2 + |(v(x_1))^q - (v(x_2))^q|^2 + |(\pi(x_1))^q - (\pi(x_2))^q|^2 \right)} \right. \\
 &+ \left. \frac{1}{2} \sqrt{((a_2 - a_3)^2 + (b_2 - b_3)^2 + (c_2 - c_3)^2 + (d_2 - d_3)^2) \left(|(\mu(x_2))^q - (\mu(x_3))^q|^2 + |(v(x_2))^q - (v(x_3))^q|^2 + |(\pi(x_2))^q - (\pi(x_3))^q|^2 \right)} \right)^2 \\
 &= \frac{1}{4} ((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2) \left(|(\mu(x_1))^q - (\mu(x_2))^q|^2 + |(v(x_1))^q - (v(x_2))^q|^2 + |(\pi(x_1))^q - (\pi(x_2))^q|^2 \right) \\
 &+ \frac{1}{4} ((a_2 - a_3)^2 + (b_2 - b_3)^2 + (c_2 - c_3)^2 + (d_2 - d_3)^2) \left(|(\mu(x_2))^q - (\mu(x_3))^q|^2 + |(v(x_2))^q - (v(x_3))^q|^2 + |(\pi(x_2))^q - (\pi(x_3))^q|^2 \right) \\
 &+ \frac{1}{2} \sqrt{((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2) \left(|(\mu(x_1))^q - (\mu(x_2))^q|^2 + |(v(x_1))^q - (v(x_2))^q|^2 + |(\pi(x_1))^q - (\pi(x_2))^q|^2 \right)} \\
 &\sqrt{((a_2 - a_3)^2 + (b_2 - b_3)^2 + (c_2 - c_3)^2 + (d_2 - d_3)^2) \left(|(\mu(x_2))^q - (\mu(x_3))^q|^2 + |(v(x_2))^q - (v(x_3))^q|^2 + |(\pi(x_2))^q - (\pi(x_3))^q|^2 \right)} \\
 &\geq \frac{1}{4} ((a_1 - a_2 + a_2 - a_3)^2 + (b_1 - b_2 + b_2 - b_3)^2 + (c_1 - c_2 + c_2 - c_3)^2 + (d_1 - d_2 + d_2 - d_3)^2) \\
 &\left(|(\mu(x_1))^q - (\mu(x_2))^q + (\mu(x_2))^q - (\mu(x_3))^q|^2 + |(v(x_1))^q - (v(x_2))^q + (v(x_2))^q - (v(x_3))^q|^2 + |(\pi(x_1))^q - (\pi(x_2))^q + (\pi(x_2))^q - (\pi(x_3))^q|^2 \right) \\
 &= \frac{1}{4} ((a_1 - a_3)^2 + (b_1 - b_3)^2 + (c_1 - c_3)^2 + (d_1 - d_3)^2) \left(|(\mu(x_1))^q - (\mu(x_3))^q|^2 + |(v(x_1))^q - (v(x_3))^q|^2 + |(\pi(x_1))^q - (\pi(x_3))^q|^2 \right) \\
 &= D^2(N_1, N_3)
 \end{aligned}$$

Therefore, the proposed distance measure satisfies the aforementioned three properties. $\square$

Example 1. Let $N_1 = \langle SN_1, (0.7, 0.5) \rangle_3$ and $N_2 = \langle SN_2, (0.9, 0.5) \rangle_3$ be two q -SNs, where $SN_1 = (1, 2, 5, 7)$ and $SN_2 = (2, 4, 6, 10)$, thus $\pi(N_1) = (1 - 0.7^3 - 0.5^3)^{\frac{1}{3}} = 0.81$, $\pi(N_2) = (1 - 0.9^3 - 0.5^3)^{\frac{1}{3}} = 0.53$, the distance between N_1 and N_2 can be calculated based on the above formulas:

$$D(N_1, N_2) = \sqrt{\frac{1}{2}((1-2)^2 + (2-4)^2 + (5-6)^2 + (7-10)^2) \times \sqrt{\frac{1}{2}(|0.7^3 - 0.9^3|^2 + |0.5^3 - 0.5^3|^2 + |0.8^3 - 0.5^3|^2)}} = 1.06$$

4. Shadowed Set Construction from Interval Data

In the actual MADM process, based on the data-driven idea, different groups are investigated for different attributes due to the difference between attributes in property. After collecting enough evaluation information, the data are preprocessed and calculated, then a more accurate evaluation value of a certain attribute can be obtained in the form of a shadowed number.

4.1. Interval Data Preprocessing

Wu et al. [26] and Liu and Mendel [27] proposed a method for interval data preprocessing which can be applied in the MADM process. In the process of data collection, each individual provides the left and right endpoints of an interval about an evaluation value on a prescribed scale (e.g., 0–10). Suppose that n_0 interval data $[a_i, b_i] (i = 1, 2, \dots, n)$ for the attribute G are collected from n_0 individuals and $L_i = b_i - a_i$ is the interval length, where a_i represents the left endpoint, b_i represents the right endpoint, respectively. The data can be preprocessed by the following four steps:

- (1) **Bad data processing:** This is performed for the purpose of deleting unreasonable results from the surveyed people. The interval endpoints which satisfy the following conditions,

$$\begin{cases} 0 \leq a_i \leq 10 \\ 0 \leq b_i \leq 10, \quad i = 1, \dots, n_0 \\ a_i \leq b_i \end{cases} \quad (13)$$

will be accepted. Otherwise, they will be omitted. After this step, $n_1 \leq n_0$ interval data will remain.

- (2) **Outlier processing:** This step aims to eliminate those outliers which are extremely large or small by using the Box and Whisker test. At first, the left and right endpoints can be processed

by applying outlier tests, respectively. As a result, only the interval endpoints satisfying the following conditions are retained:

$$\begin{cases} Q_a(0.25) - 1.5L_a \leq a_i \leq Q_a(0.75) + 1.5L_a \\ Q_b(0.25) - 1.5L_b \leq b_i \leq Q_b(0.75) + 1.5L_b \end{cases} \quad i = 1, \dots, n_1 \quad (14)$$

where $Q_a(Q_b)$ and $L_a(L_b)$ are the quartiles and interquartile ranges for the remaining n_1 left (right) endpoints, and $Q(0.25), Q(0.75)$ are the first and third quartiles, respectively. This step reduces n_1 interval endpoints to n_2 interval endpoints. Secondly, for the n_2 intervals, only those satisfying the below condition are kept:

$$Q_L(0.25) - 1.5L_L \leq L_i \leq Q_L(0.75) + 1.5L_L, \quad i = 1, \dots, n_2 \quad (15)$$

where $Q_L(0.25)$ and $Q_L(0.75)$ are the first and third quartiles, respectively, and L_L is the interquartile ranges. After this step, $n_3 \leq n_2$ interval data will be left.

- (3) **Tolerance limit processing:** The interval endpoints which meet the following conditions will be remained, otherwise be removed.

$$\begin{cases} m_l - \lambda\sigma_l \leq a_i \leq m_l + \lambda\sigma_l \\ m_r - \lambda\sigma_r \leq b_i \leq m_r + \lambda\sigma_r \\ m_L - \lambda\sigma_L \leq L_i \leq m_L + \lambda\sigma_L \end{cases} \quad i = 1, \dots, n_3 \quad (16)$$

where m and σ represent the mean values and standard deviations of the n_3 reserved data endpoints, respectively, λ is tolerance factor. The value of λ can be obtained from Table 1 (suppose the parameters γ and α are 0.05 and 0.01 in this paper). After this step, $n_4 \leq n_3$ interval data will remain.

- (4) **Reasonable-interval processing:** Intervals that have overly long breadth or have little overlap with others are eliminated in this step. Suppose both the left endpoints and the right endpoints obey Gaussian distribution [28]. Thus, m_l, m_r, σ_l and σ_r are computed based on the remaining n_4 interval endpoints. Consequently, the endpoints of an interval satisfy the below conditions will be kept.

$$\begin{cases} 2m_l - \xi \leq a_i \leq \xi \\ \xi \leq b_i \leq 2m_r - \xi \end{cases}, \quad i = 1, \dots, n_4 \quad (17)$$

where

$$\xi = \frac{(m_r\sigma_l^2 - m_l\sigma_r^2) \pm \sigma_l\sigma_r \sqrt{(m_l - m_r)^2 + 2(\sigma_l^2 - \sigma_r^2) \ln\left(\frac{\sigma_r}{\sigma_l}\right)}}{\sigma_l^2 - \sigma_r^2} \quad (18)$$

After this step, $n_5 \leq n_4$ interval data will remain.

Table 1. Tolerance factor λ .

n_3	$1-\gamma = 0.95$		$1-\gamma = 0.99$	
	$1-\alpha$		$1-\alpha$	
	0.90	0.95	0.90	0.95
10	2.839	3.379	3.582	4.265
15	2.480	2.954	2.945	3.507
20	2.310	2.752	2.659	3.168
30	2.140	2.549	2.358	2.841
50	1.996	2.379	2.162	2.576
100	1.874	2.233	1.977	2.355
1000	1.709	2.036	1.736	2.718
∞	1.645	1.960	1.645	1.960

4.2. Calculations of the Retained Interval Data

After data preprocessing, the distribution of data intervals is shown in Figure 3. The distribution of the left and right endpoints reflects the uncertainty inhered in evaluation information. Hence, the shadowed area of shadowed set is represented by representative intervals of left and right endpoints, and the core of shadowed set is represented by core area. In order to calculate the representative interval, we used the percentile method to calculate the percentages.

Figure 3. A representation of the interval data.

Assume that the left and right endpoints of the remaining interval data are arranged in nondecreasing order:

$$\begin{cases} x_1^l \leq x_2^l \leq \dots \leq x_n^l \\ x_1^r \leq x_2^r \leq \dots \leq x_n^r \end{cases} \quad (19)$$

For a given p ($p < 0.5$), assume that the $100p^{th}$ and $100(1-p)^{th}$ percentiles are denoted as $[T_p, T_{1-p}]$. The $(1-2p) * 100\%$ proportion of the datapoints are included in this interval. In this paper, we suppose $p = 0.1$, that means 80% proportion of the datapoints are contained in left and right endpoints, respectively. For the left endpoints, $100p^{th}$ and $100(1-p)^{th}$ percentiles can be computed as [29]:

$$T_p^l = \begin{cases} \frac{x_{np}^l + x_{np+1}^l}{2}, & np \text{ is an integer} \\ x_{[np]'}^l, & np \text{ is not an integer} \end{cases} \quad (20)$$

$$T_{1-p}^l = \begin{cases} \frac{x_{n(1-p)}^l + x_{n(1-p)+1}^l}{2}, & n(1-p) \text{ is an integer} \\ x_{[n(1-p)]'}^l, & n(1-p) \text{ is not an integer} \end{cases} \quad (21)$$

where $[.]$ denotes the “ceiling function”. For T_p^l , if np is an integer, then the $100p^{th}$ percentile value is the average of x_{np}^l and x_{np+1}^l ; if np is not an integer, the $100p^{th}$ percentile value is $x_{[np]'}^l$, where $[np]'$ means that we take the next largest integer, that is, round up. Similarly, for T_{1-p}^l , if $n(1-p)$ is an

integer, then the $100(1-p)^{th}$ percentile value is the average of $x_{n(1-p)}^l$ and $x_{n(1-p)+1}^l$, or the $100(1-p)^{th}$ percentile value is $x_{[n(1-p)]}^l$ if $n(1-p)$ is not an integer.

In the same way, we can obtain the representative interval of the right endpoints which is denoted by $[T_p^r, T_{1-p}^r]$.

Example 2. For interval numbers (1.25, 7.50), (2.50, 6.84), (3.56, 8.41), (2.86, 7.88), (3.10, 8.50), (2.75, 6.80), (1.37, 8.62), (3.41, 8.56), (2.60, 7.17), (3.25, 7.38), the representative interval can be calculated as ($p = 1, n = 10$): The left endpoints are arranged in nondecreasing order:

$$1.25 < 1.37 < 2.50 < 2.60 < 2.75 < 2.86 < 3.10 < 3.25 < 3.41 < 3.56$$

According to Equations (20) and (21),

$$T_{0.1}^l = \frac{x_1^l + x_2^l}{2} = \frac{1.25 + 1.37}{2} = 1.31; T_{0.9}^l = \frac{x_9^l + x_{10}^l}{2} = \frac{3.41 + 3.55}{2} = 3.48$$

Thus, the representative interval can be obtained: (1.31, 3.48). In the same way, the representative interval of the right endpoints is (6.82, 8.56).

5. The Extended VIKOR Method for q-SNs MADM Problems

Within the context of q-rung orthopair shadowed sets, the q-rung orthopair shadowed VIKOR model can be constructed based on the conventional VIKOR method, to deal with MADM problems, where all the evaluation information is expressed by q-SNs.

Suppose $A = \{A_1, A_2, \dots, A_m\}$ is a finite set of alternatives and $G = \{G_1, G_2, \dots, G_n\}$ is a set of attributes with weighting vector $W = \{w_1, w_2, \dots, w_n\}$, ($0 \leq w_i \leq 1, \sum_{i=1}^n w_i = 1$). The whole decision-making process of q-rung orthopair shadowed VIKOR method consists of the following steps.

Step 1. Establish decision making matrix.

Based on the shadowed sets built from the collected interval data, we can obtain the matrix $X = [SN_{ij}]_{m \times n} = [(a_{ij}, b_{ij}, c_{ij}, d_{ij})], i = 1, \dots, m, j = 1, \dots, n$ as Equation (22). Then the q-rung orthopair shadowed decision-making matrix $R = [\langle SN_{ij}, (\mu_{ij}, v_{ij}) \rangle_q]_{m \times n}, i = 1, \dots, m, j = 1, \dots, n$ can be established according to experts' evaluation.

$$X = \begin{bmatrix} SN_{11} & \cdots & SN_{1j} & \cdots & SN_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ SN_{i1} & \cdots & SN_{ij} & \cdots & SN_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ SN_{m1} & \cdots & SN_{mj} & \cdots & SN_{mn} \end{bmatrix}_{m \times n}; i = 1, \dots, m; j = 1, \dots, n \quad (22)$$

$$R = \begin{bmatrix} \langle SN_{11}, (\mu_{11}, v_{11}) \rangle_q & \cdots & \langle SN_{1j}, (\mu_{1j}, v_{1j}) \rangle_q & \cdots & \langle SN_{1n}, (\mu_{1n}, v_{1n}) \rangle_q \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle SN_{i1}, (\mu_{i1}, v_{i1}) \rangle_q & \cdots & \langle SN_{ij}, (\mu_{ij}, v_{ij}) \rangle_q & \cdots & \langle SN_{in}, (\mu_{in}, v_{in}) \rangle_q \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle SN_{m1}, (\mu_{m1}, v_{m1}) \rangle_q & \cdots & \langle SN_{mj}, (\mu_{mj}, v_{mj}) \rangle_q & \cdots & \langle SN_{mn}, (\mu_{mn}, v_{mn}) \rangle_q \end{bmatrix}_{m \times n}; i = 1, \dots, m; j = 1, \dots, n \quad (23)$$

Step 2. Calculate the best R_{ij} specified as R_j^+ and the worst R_{ij} as R_j^- through the comparison of score function values for each attribute.

For benefit attributes:

$$\begin{cases} R_j^+ = \left\langle (SN_{ij})_{\max_i}, (G_j(A_i))_{\max_i S_{ij}} \right\rangle_q \\ R_j^- = \left\langle (SN_{ij})_{\min_i}, (G_j(A_i))_{\min_i S_{ij}} \right\rangle_q \end{cases} ; \quad i = 1, \dots, m; j = 1, \dots, n \quad (24)$$

For cost attributes:

$$\begin{cases} R_j^+ = \left\langle (SN_{ij})_{\min_i}, (G_j(A_i))_{\min_i S_{ij}} \right\rangle_q \\ R_j^- = \left\langle (SN_{ij})_{\max_i}, (G_j(A_i))_{\max_i S_{ij}} \right\rangle_q \end{cases} ; \quad i = 1, \dots, m; j = 1, \dots, n \quad (25)$$

where $(SN_{ij})_{\max_i} = (\max_i a_i, \max_i b_i, \max_i c_i, \max_i d_i)$, $(SN_{ij})_{\min_i} = (\min_i a_i, \min_i b_i, \min_i c_i, \min_i d_i)$, $G_j(A_i) = (u_{ij}, v_{ij})$ is a q-ROFN and the score value of $G_j(A_i)$ is $S_{ij} = \frac{1}{2}(1 + (u_{ij})^q - (v_{ij})^q)$.

Step 3. Calculate $D(R_j^+, R_{ij})$ and $D(R_j^+, R_j^-)$ according to the distance measure between q-SNs.

Step 4. Calculate the results of the group utility value S_i , the individual regret value T_i and θ_i .

$$S_i = \sum_{j=1}^n w_j \frac{D(R_j^+, R_{ij})}{D(R_j^+, R_j^-)}, \quad i = 1, \dots, m \quad (26)$$

$$T_i = \max_j \left(w_j \frac{D(R_j^+, R_{ij})}{D(R_j^+, R_j^-)} \right), \quad i = 1, \dots, m; j = 1, \dots, n \quad (27)$$

$$\begin{cases} \theta_i = \alpha \times \frac{S_i - S^+}{S^- - S^+} + (1 - \alpha) \frac{T_i - T^+}{T^- - T^+} \\ S^+ = \min_i S_i, S^- = \max_i S_i \\ T^+ = \min_i T_i, T^- = \max_i T_i \end{cases} ; \quad i = 1, \dots, m \quad (28)$$

where α represents the compromise coefficient and $\alpha \in [0, 1]$. When $\alpha \leq 0.5$, experts tend to make decisions by minimize T_i ; when $\alpha \geq 0.5$, experts are inclined to make decisions by maximize S_i , and when $\alpha = 0.5$, experts follow a balanced approach. Without loss of generality, α is set to 0.5.

Step 5. Arrange the S_i , T_i and θ_i in an ascending order to determine the compromise alternatives.

Suppose the arrangement result of alternatives is $A_{(1)}, A_{(2)}, \dots, A_{(m)}$ according to the ascending order of θ_i , $\theta_{(1)} \leq \theta_{(2)} \leq \dots \leq \theta_{(m)}$. We can obtain the optimal alternative $A_{(1)}$ if the following two conditions are met simultaneously [30]:

- (1) Acceptable advantage: $\theta_{(2)} - \theta_{(1)} \geq \frac{1}{m-1}$;
- (2) Acceptable stability: The alternative l must also be the first position ranked by S_i or T_i .

If one of the conditions is not satisfied, we can obtain the compromise solutions:

- (a) If condition (2) is not satisfied, then $A_{(1)}$ and $A_{(2)}$ are both compromise solutions.
- (b) If condition (1) is not satisfied, then $A_{(1)}, A_{(2)}, \dots, A_{(l)}$ are all compromise solutions, where $A_{(l)}$ satisfy $\theta_{(l)} - \theta_{(1)} \leq \frac{1}{m}$ for the maximum l .

6. Numerical Study

In this section, an illustrative example is provided to select the most suitable gas station location for an oil company by using the q-rung shadowed VIKOR method. In addition, comparisons with the extended TOPSIS method with q-SNWA and q-SNWG operators are conducted to manifest the validity of our approach.

6.1. Site Selection Problem

An oil company needs to construct a new gas station, and there are five location alternatives $A_i (i = 1, 2, 3, 4, 5)$ to be selected. The experts evaluate the projects according to four attributes $G_j (j = 1, 2, 3, 4)$: ① G_1 denotes environmental pollution hazards; ② G_2 denotes traffic convenience; ③ G_3 denotes expected profit; ④ G_4 denotes construction cost, and the weighting vector of the four attributes is $w = (0.3, 0.2, 0.4, 0.1)$.

Step 1. Establish decision making matrix.

The attribute evaluation values of each alternatives are collected in the form of interval numbers ranging from 0 to 10 for different groups. After preprocessing and statistical data calculation, a SNs evaluation matrix can be generated as shown in Table 2. Then the evaluation matrix can be transformed into a q-SN decision matrix after expert evaluation as shown in Table 3.

Table 2. Decision matrix with SNs.

	G_1	G_2
A_1	(2.39, 3.28, 3.92, 4.71)	(5.76, 6.48, 7.34, 7.96)
A_2	(3.79, 4.62, 5.73, 6.62)	(7.32, 7.97, 8.56, 8.99)
A_3	(7.51, 7.72, 8.53, 8.84)	(3.83, 4.85, 5.44, 6.21)
A_4	(3.76, 4.62, 5.73, 6.61)	(5.17, 6.22, 7.39, 8.12)
A_5	(5.41, 6.02, 6.97, 7.59)	(2.37, 3.51, 4.03, 4.67)
	G_3	G_4
A_1	(7.01, 7.22, 8.30, 8.49)	(7.53, 7.93, 8.60, 8.92)
A_2	(7.51, 7.66, 8.11, 8.23)	(5.21, 5.77, 6.01, 6.67)
A_3	(8.51, 9.03, 9.52, 9.87)	(3.26, 4.63, 5.49, 6.88)
A_4	(6.99, 7.32, 8.01, 8.21)	(5.03, 5.62, 6.33, 7.02)
A_5	(5.42, 6.13, 7.42, 7.55)	(2.37, 3.31, 3.96, 4.12)

Table 3. Decision matrix with q-rung orthopair shadowed numbers (q-SNs).

	G_1	G_2
A_1	$\langle (2.39, 3.28, 3.92, 4.71), (0.8, 0.2) \rangle_3$	$\langle (5.76, 6.48, 7.34, 7.96), (0.3, 0.5) \rangle_3$
A_2	$\langle (3.79, 4.62, 5.73, 6.62), (0.6, 0.7) \rangle_3$	$\langle (7.32, 7.97, 8.56, 8.99), (0.7, 0.8) \rangle_3$
A_3	$\langle (7.51, 7.72, 8.53, 8.84), (0.7, 0.5) \rangle_3$	$\langle (3.83, 4.85, 5.44, 6.21), (0.9, 0.4) \rangle_3$
A_4	$\langle (3.76, 4.62, 5.73, 6.61), (0.7, 0.6) \rangle_3$	$\langle (5.17, 6.22, 7.39, 8.12), (0.6, 0.7) \rangle_3$
A_5	$\langle (5.41, 6.02, 6.97, 7.59), (0.4, 0.6) \rangle_3$	$\langle (2.37, 3.51, 4.03, 4.67), (0.7, 0.5) \rangle_3$
	G_3	G_4
A_1	$\langle (7.01, 7.22, 8.30, 8.49), (0.8, 0.3) \rangle_3$	$\langle (7.53, 7.93, 8.60, 8.92), (0.4, 0.6) \rangle_3$
A_2	$\langle (7.51, 7.66, 8.11, 8.23), (0.3, 0.9) \rangle_3$	$\langle (5.21, 5.77, 6.01, 6.67), (0.6, 0.1) \rangle_3$
A_3	$\langle (8.51, 9.03, 9.52, 9.87), (0.4, 0.5) \rangle_3$	$\langle (3.26, 4.63, 5.49, 6.88), (0.5, 0.3) \rangle_3$
A_4	$\langle (6.99, 7.32, 8.01, 8.21), (0.4, 0.9) \rangle_3$	$\langle (5.03, 5.62, 6.33, 7.02), (0.1, 0.6) \rangle_3$
A_5	$\langle (5.42, 6.13, 7.42, 7.55), (0.8, 0.3) \rangle_3$	$\langle (2.37, 3.31, 3.96, 4.12), (0.1, 0.6) \rangle_3$

Step 2. Calculate R_j^+ and R_j^- .

Since G_1, G_4 are cost attributes, G_2, G_3 are benefit attributes, the best and worst attribute values can be obtained as follows:

$$R_j^+ = \left\{ \langle (2.39, 3.28, 3.92, 4.71), (0.4, 0.6) \rangle_3, \langle (7.32, 7.97, 8.56, 8.99), (0.9, 0.4) \rangle_3, \langle (8.51, 9.03, 9.52, 9.87), (0.8, 0.3) \rangle_3, \langle (2.37, 3.31, 3.96, 4.12), (0.1, 0.6) \rangle_3 \right\}$$

$$R_j^- = \left\{ \langle (7.51, 7.72, 8.53, 8.84), (0.8, 0.2) \rangle_3, \langle (2.37, 3.51, 4.03, 4.67), (0.7, 0.8) \rangle_3, \langle (5.42, 6.13, 7.42, 7.55), (0.3, 0.9) \rangle_3, \langle (7.52, 7.93, 8.60, 8.92), (0.6, 0.1) \rangle_3 \right\}$$

Step 3. Based on the distance measure between q-SNs, the weighted distance D_{ij}^* can be calculated by $D_{ij}^* = w_j D(R_j^+, R_{ij})$, as shown in Table 4.

Table 4. The value of weighted distance.

	G_1	G_2	G_3	G_4
D_{1j}^*	0.0000	0.2263	0.0000	0.0643
D_{2j}^*	0.2267	0.0000	0.4667	0.0870
D_{3j}^*	0.3821	0.0000	0.0000	0.0646
D_{4j}^*	0.2443	0.1986	0.4721	0.0000
D_{5j}^*	0.000	0.3052	0.2929	0.0000
$D(R_j^+, R_j^-)$	1.5875	1.6486	1.5287	1.2090

Step 4. According to the Equations (25) and (26), the group utility value S_i and the individual regret value T_i are obtained.

$$S_1 = 0.1905, S_2 = 0.5201, S_3 = 0.2941, S_4 = 0.5832, S_5 = 0.3767;$$

$$T_1 = 0.1372, T_2 = 0.3053, T_3 = 0.2407, T_4 = 0.3088, T_5 = 0.1916.$$

Without loss of generality, let $\alpha = 0.5$, on the basis of S_i and T_i , the values of θ_i are obtained.

$$\theta_1 = 0.0000, \theta_2 = 0.9095, \theta_3 = 0.4335, \theta_4 = 1.0000, \theta_5 = 0.3956.$$

Step 5. Sort S_i, T_i and θ_i in ascending order according to the results. The minimum value of θ_i is θ_1 , followed by θ_5 , and $\theta_5 - \theta_1 = 0.3956 \geq \frac{1}{5-1}$. Simultaneously, S_1 and T_1 are the minimum values of S_i and T_i , respectively. Thus, we can obtain the optimal alternative A_1 . According to the value of θ_i , the ranking order of A_2 to A_5 is $A_5 > A_3 > A_2 > A_4$.

Therefore, we can get the ranking order of the alternatives as: $A_1 > A_5 > A_3 > A_2 > A_4$.

6.2. Comparative Analysis

TOPSIS is a classical and widely used MADM method. VIKOR and TOPSIS share some similarities by both considering the closeness of alternatives and ideal solutions. Therefore, the extended VIKOR method is compared with the TOPSIS combined with the q-SNWA operator and the q-SNWG operator to verify its validity. Firstly, we denote the normalized $R = \left[\langle SN_{ij}, (\mu_{ij}, v_{ij}) \rangle_q \right]_{m \times n}$, $i = 1, \dots, m, j = 1, \dots, n$ as $\bar{R} = \left[\langle \bar{SN}_{ij}, (\bar{\mu}_{ij}, \bar{v}_{ij}) \rangle_q \right]_{m \times n}$, $i = 1, \dots, m, j = 1, \dots, n$, which can be obtained by Equations (29) and (30) [31]:

For benefit attributes:

$$\bar{R}_{ij} = \left\langle \left(\frac{a_{ij}}{\sqrt{\sum_{i=1}^m (d_{ij})^2}} \right), \left(\frac{b_{ij}}{\sqrt{\sum_{i=1}^m (c_{ij})^2}} \right), \left(\frac{c_{ij}}{\sqrt{\sum_{i=1}^m (b_{ij})^2}} \right), \left(\frac{d_{ij}}{\sqrt{\sum_{i=1}^m (a_{ij})^2}} \right), (\mu_{ij}, v_{ij}) \right\rangle_q \quad (29)$$

For cost attributes:

$$\bar{R}_{ij} = \left\langle \left(\frac{\frac{1}{d_{ij}}}{\sqrt{\sum_{i=1}^m \left(\frac{1}{a_{ij}} \right)^2}} \right), \left(\frac{\frac{1}{c_{ij}}}{\sqrt{\sum_{i=1}^m \left(\frac{1}{b_{ij}} \right)^2}} \right), \left(\frac{\frac{1}{b_{ij}}}{\sqrt{\sum_{i=1}^m \left(\frac{1}{c_{ij}} \right)^2}} \right), \left(\frac{\frac{1}{a_{ij}}}{\sqrt{\sum_{i=1}^m \left(\frac{1}{d_{ij}} \right)^2}} \right), (v_{ij}, \mu_{ij}) \right\rangle_q \quad (30)$$

Then, we can obtain the best and the worst $\bar{R}_j$, respectively.

$$\begin{cases} \bar{R}_j^+ = \left\langle \max_i \bar{S}N_{ij}, (\bar{G}_j(A_i))_{\max S_{ij}} \right\rangle_q \\ \bar{R}_j^- = \left\langle \min_i \bar{S}N_{ij}, (\bar{G}_j(A_i))_{\min S_{ij}} \right\rangle_q \end{cases} ; i = 1, \dots, m; j = 1, \dots, n \quad (31)$$

The aggregated values can be calculated by the two aggregation operators based on the attributes' weights which are listed in Table 5, where A^+ and A^- represent the normalized comprehensive positive solution and negative solution, respectively.

Table 5. The aggregated values by q-SNWG and q-SNWA aggregation operators.

	Aggregated Values by q-SNWG	Aggregated Values by q-SNWA
A₁	$\langle (0.33, 0.42, 0.55, 0.62), (0.42, 0.61) \rangle_3$	$\langle (0.34, 0.43, 0.57, 0.65), (0.65, 0.46) \rangle_3$
A₂	$\langle (0.34, 0.40, 0.53, 0.59), (0.41, 0.81) \rangle_3$	$\langle (0.35, 0.40, 0.53, 0.60), (0.58, 0.75) \rangle_3$
A₃	$\langle (0.28, 0.36, 0.45, 0.53), (0.49, 0.57) \rangle_3$	$\langle (0.30, 0.37, 0.46, 0.55), (0.66, 0.53) \rangle_3$
A₄	$\langle (0.30, 0.38, 0.51, 0.58), (0.51, 0.80) \rangle_3$	$\langle (0.31, 0.38, 0.51, 0.58), (0.54, 0.64) \rangle_3$
A₅	$\langle (0.24, 0.31, 0.42, 0.50), (0.66, 0.46) \rangle_3$	$\langle (0.25, 0.32, 0.44, 0.53), (0.67, 0.40) \rangle_3$
A⁺	$\langle (0.41, 0.51, 0.66, 0.76), (0.76, 0.45) \rangle_3$	$\langle (0.41, 0.42, 0.66, 0.78), (0.80, 0.35) \rangle_3$
A⁻	$\langle (0.23, 0.27, 0.36, 0.41), (0.28, 0.84) \rangle_3$	$\langle (0.22, 0.28, 0.37, 0.42), (0.45, 0.81) \rangle_3$

Calculate the relative closeness Q_i of each alternative by Equation (32), and the results are shown in Table 6.

$$Q_i = \frac{D_i^-}{D_i^+ + D_i^-} \quad (32)$$

where D_i^+ and D_i^- represent the weighted distance between alternative A_i and ideal solutions A^+ and A^- , respectively.

Table 6. The relative closeness.

	q-SNWG Method	q-SNWA Method
A₁	0.5640	0.6503
A₂	0.2828	0.3859
A₃	0.4369	0.5059
A₄	0.3240	0.4640
A₅	0.4571	0.4564

The bigger the relative closeness Q_i , the better the choice. Therefore, the order of all alternatives can be obtained, as shown in Table 7.

Table 7. The ranking order.

	Order
q-SNWG	$1 > 5 > 3 > 4 > 2$
q-SNWA	$1 > 3 > 4 > 5 > 2$
q-SN VIKOR Model	$1 > 5 > 3 > 2 > 4$

Analyze the results in Table 7, we can draw the following conclusions.

- (1) As can be seen from Table 7, though the ranking for other alternatives is slightly different, the three methods have obtained the same optimal alternative A_1 . This manifested the rationality of our proposed method.
- (3) It is worth noting that the aggregation operation may cause information loss or distortion when using the q-SNWG or q-SNWA combined with TOPSIS to aggregate decision information. At the same time, the degree of discrimination between alternatives is also smaller using TOPSIS.
- (3) The extended VIKOR method for q-SNs has the remarkable features of considering the trade-off between maximizing group utility and minimizing individual regret, which is more meaningful to MADM problems.

7. Conclusions

With the prominent advantages of q-rung orthopair fuzzy set and shadowed set, a new MADM method named as q-SN VIKOR is proposed. The q-rung orthopair shadowed set is defined and their basic operation rules are introduced. A data-driven method is offered to produce shadowed numbers. In addition, we define the distance measure between the two q-SNs based on the calculation principle of Euclidean distance. An extended q-SN VIKOR method is constructed to solve the location selection problem of a gas station where the decision information values take the form of q-SNs. Finally, the comparative analysis verifies the practicability and rationality of the proposed method.

In the near future, we will explore the combination of q-rung orthopair fuzzy sets with other types of shadowed sets, such as left-shoulder, right-shoulder and non-cored.

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