



Article Ulam Stability of a Second Linear Differential Operator with Nonconstant Coefficients

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Abstract: In this paper, we obtain a result on Ulam stability for a second order differential operator acting on a Banach space. The result is connected to the existence of a global solution for a Riccati differential equation and some appropriate conditions on the coefficients of the operator.

Keywords: Ulam stability; linear differential operator; Riccati equation; Banach spaces

MSC: 34A30, 34D10, 39B82

1. Introduction

Ulam stability is one of the main topics in functional equation theory. The starting point of Ulam stability was the question formulated by S.M. Ulam during a talk given at Madison University, Wisconsin, and concerns the approximate homomorphisms of a metric group (see [1]). The topic was intensively developed and studied in the last 50 years. For some results, methods, extensions and generalizations of the notion we refer the reader to [1,2].

The first results on Ulam stability for differential equations were obtained by M. Obloza [3] and C. Alsina and R. Ger [4]. T. Miura, S. Miyajima and S. Takahasi proved that the linear differential operator of *n*-th order with constant coefficients is stable in Ulam sense if and only if its characteristic equation has no roots on the imaginary axis [5]. This result was generalized by D. Popa and I. Raşa who obtained a sharp estimate of Ulam constant [6]. Recently, A.R. Baias and D. Popa obtained the best Ulam constant for a second order linear differential operator with constant coefficients [7]. Recall also the result obtained by M. Onitsuka et al. on the best constant of linear differential equation of the first order [8]. For more details on Ulam stability of differential equations and differential operators, we refer the reader to [9–12].

To this moment there are few results on Ulam stability for the linear differential operator of higher order with nonconstant coefficients. This is indeed the reason for which in this paper we deal with Ulam stability of a second order linear differential operator with nonconstant coefficients.

In what follows, let *X* be a Banach space over \mathbb{R} . Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$, $f \in C^1(I, \mathbb{R})$, $g \in C(I, \mathbb{R})$. By $C^n(I, X)$, we denote as usual the set of all *n*-times continuously differentiable functions defined on *I* with values in the Banach space *X*.

Definition 1. [13] Let A be a linear space over the field **R**. A function $\rho_A : A \to [0, +\infty]$ is called a gauge on A if:

1. $\rho_A(x) = 0$ if and only if x = 0;

2.
$$\rho_A(\lambda x) = |\lambda| \rho(x)$$
 for all $x \in A, \lambda \in \mathbf{R}, \lambda \neq 0$.

Let us consider the linear differential operator with nonconstant coefficients $D : C^2(I, X) \to C(I, X)$ given by

$$D(y) = y'' + fy' + gy, \ y \in C^2(I, X).$$
(1)

Let $\varphi \in C^n(I, X)$, $n \in \mathbb{N}$ and denote

$$||\varphi||_{\infty} = \sup \{||\varphi(x)|| : x \in I\}$$

Then $||\varphi||_{\infty}$ is a gauge. We suppose also that $C^n(I, X)$, $n \in \mathbb{N}$, are endowed with the same gauge $||\cdot||_{\infty}$ for n = 0, 1, 2.

Definition 2. *The operator D is called Ulam stable if there exists a nonnegative constant L such that for every* $\varepsilon > 0$ *and every* $y \in C^2(I, X)$ *satisfying*

$$|D(y)||_{\infty} \le \varepsilon \tag{2}$$

there exists $y_0 \in C^2(I, X)$ such that $D(y_0) = 0$ and

$$||y - y_0||_{\infty} \le L \cdot \varepsilon. \tag{3}$$

A function *y* satisfying (2) for some $\varepsilon > 0$ is called an *approximate solution* of the equation D(y) = 0. The number *L* in the relation (3) is called an *Ulam constant* of *D*.

2. Main Result

The main result on Ulam stability for the operator D defined by the relation (1) is contained in the next two theorems.

Theorem 1. Suppose that there exists K > 0 such that

$$g(x) \geq \frac{1}{K}, x \in I$$

and the Riccati equation

$$u' = -u^2 + fu + f' - g (4)$$

has a solution $u \in C^1(I, \mathbb{R})$ with u(a) = f(a). Then for every $y \in C^2(I, X)$ satisfying

$$||D(y)||_{\infty} \le \varepsilon, \tag{5}$$

there exists $z \in C^2(I, X)$, such that D(z) = 0 and

$$||y - z||_{\infty} \le K\varepsilon. \tag{6}$$

Proof. With the above *u* satisfying (4), let

$$U(x) := \int_a^x u(t) \mathrm{d}t, \ \varphi(x) := e^{U(x)}, x \in I,$$

and

$$F(x) := \int_{a}^{x} f(t) \mathrm{d}t, \ \psi(x) := e^{F(x) - U(x)}, x \in I$$

Consider the equation

$$y''(x) + f(x)y'(x) + g(x)y(x) = h(x), \ x \in I.$$
(7)

Multiplying it by $\varphi(x)$, by using (4) and having in mind that $\varphi' = u\varphi$, we get

$$(\varphi y' + (f - u)\varphi y)' = \varphi h$$

which leads to

$$\varphi(x) \cdot y'(x) + (f(x) - u(x)) \cdot \varphi(x) \cdot y(x) - y'(a) = \int_a^x \varphi(t)h(t)dt, \ x \in I.$$

Multiplying the previous relation by $\frac{\psi(x)}{\varphi(x)}$, we obtain

$$\psi(x) \cdot y'(x) + \psi(x) \cdot (f(x) - u(x)) \cdot y(x) = \left(y'(a) + \int_a^x \varphi(t)h(t)dt\right) \frac{\psi(x)}{\varphi(x)}, \ x \in I.$$

Since $\psi' = (f - u)\psi$, it follows that

$$(\psi y)'(t) = \left(y'(a) + \int\limits_{a}^{t} \varphi(s)h(s)\mathrm{d}s\right) \cdot \frac{\psi(t)}{\varphi(t)}, \ t \in I,$$

and this yields

$$\psi(x) \cdot y(x) = y(a) + \int_{a}^{x} \left(y'(a) + \int_{a}^{t} \varphi(s)h(s) \mathrm{d}s \right) \frac{\psi(t)}{\varphi(t)} \mathrm{d}t, \ x \in I.$$

To resume, (7) is equivalent to

$$y(x) = \frac{1}{\psi(x)} \left(y(a) + \int_{a}^{x} \left(y'(a) + \int_{a}^{t} \varphi(s)h(s)ds \right) \frac{\psi(t)}{\varphi(t)}dt \right), \ x \in I$$
(8)

In particular, the homogeneous equation

$$z''(x) + f(x)z'(x) + g(x)z(x) = 0, \ x \in I$$
(9)

is equivalent to

$$z(x) = \frac{1}{\psi(x)} \left(z(a) + \int_{a}^{x} z'(a) \frac{\psi(t)}{\varphi(t)} dt \right), \ x \in I.$$
(10)

Now let $\varepsilon > 0$ and $y \in C^2(I)$ satisfying (5). Then y satisfies also (7) and (8) with $|h(x)| \le \varepsilon$, $x \in I$. Let $z \in C^2(I)$, be a solution of (9) with z(a) = y(a) and z'(a) = y'(a). According to (10),

$$z(x) = \frac{1}{\psi(x)} \left(y(a) + \int_{a}^{x} y'(a) \frac{\psi(t)}{\varphi(t)} dt \right), \ x \in I$$

and it satisfies the relation D(z) = 0. So it remains to prove (6). Using (8) and (10), we obtain

$$|y(x) - z(x)| = \frac{1}{\psi(x)} \left| \int_{a}^{x} \left(\int_{a}^{t} \varphi(s)h(s)ds \right) \frac{\psi(t)}{\varphi(t)}dt \right|, \ x \in I.$$
(11)

Suppose that $x \in I$, x > a. Then

$$|y(x) - z(x)| \le \varepsilon \frac{1}{\psi(x)} \int_{a}^{x} \left(\int_{a}^{t} \varphi(s) \mathrm{d}s \right) \frac{\psi(t)}{\varphi(t)} \mathrm{d}t.$$

If $x \in I$, x < a, then by (11) we obtain

$$\begin{aligned} |y(x) - z(x)| &= \frac{1}{\psi(x)} \left| \int_{x}^{a} \left(\int_{a}^{t} \varphi(s)h(s)ds \right) \frac{\psi(t)}{\varphi(t)}dt \right| \\ &\leq \frac{1}{\psi(x)} \int_{x}^{a} \left| \int_{t}^{a} \varphi(s)h(s)ds \right| \frac{\psi(t)}{\varphi(t)}dt \\ &\leq \varepsilon \frac{1}{\psi(x)} \int_{x}^{a} \left(\int_{t}^{a} \varphi(s)ds \right) \cdot \frac{\psi(t)}{\varphi(t)}dt \\ &= \varepsilon \frac{1}{\psi(x)} \int_{a}^{x} \left(\int_{a}^{t} \varphi(s)ds \right) \cdot \frac{\psi(t)}{\varphi(t)}dt. \end{aligned}$$

Therefore, we have

$$|y(x)-z(x)| \leq \varepsilon \frac{1}{\psi(x)} \int_{a}^{x} \left(\int_{a}^{t} \varphi(s) \mathrm{d}s \right) \cdot \frac{\psi(t)}{\varphi(t)} \mathrm{d}t, \ x \in I.$$

To finish the proof it remains to show that

$$Q(x) := \int_{a}^{x} \left(\int_{a}^{t} \varphi(s) \mathrm{d}s \right) \frac{\psi(t)}{\varphi(t)} \mathrm{d}t - K\psi(x) \le 0, \ x \in I.$$
(12)

To this end, let us remark that $Q'(x) = rac{\psi(x)}{\varphi(x)} P(x)$, where

$$P(x) := \int_{a}^{x} \varphi(s) \mathrm{d}s - K\varphi(x)(f(x) - u(x)), \ x \in I.$$

Since $\varphi' = \varphi \cdot u$ we have

$$P'(x) = \varphi(x) - K \cdot \varphi(x) \cdot u(x) \cdot (f(x) - u(x)) - K \cdot \varphi(x) \cdot (f'(x) - u'(x)), x \in I,$$

and using (4) we get

$$P'(x) = \varphi(x) \cdot (1 - Kg(x)) \le 0, x \in I.$$
(13)

Combined with P(a) = 0, this yields $P(x) \ge 0$ for x < a, $x \in I$, and $P(x) \le 0$, for $x > a, x \in I$, respectively.

Accordingly, $Q'(x) \ge 0$ for $x \in I, x < a$ and $Q'(x) \le 0$ for $x \in I, x > a$, respectively. Since Q(a) = -K < 0, we conclude that $Q(x) \le 0$, $x \in I$. Thus, (12) is proved and this finishes the proof of the theorem. \Box

A similar result is presented in the following.

Theorem 2. Suppose that there exists K > 0 such that

$$g(x) \leq -\frac{1}{K}, \ x \in I$$

and the equation

$$u' = -u^2 + fu + f' + g$$

has a solution $u \in C^1(I, \mathbb{R})$, with u(a) = f(a). Then the operator D is Ulam stable with the Ulam constant K.

Proof. The proof is analogous to the proof of Theorem 1. Only (13) should be replaced by $P'(x) = \varphi(x) \cdot (1 + Kg(x)) \le 0, x \in I$. \Box

In what follows we obtain some direct consequences of the Theorems 1 and 2.

Corollary 1. *Suppose that there exists* K > 0 *such that*

$$g(x) \ge \frac{1}{K}, \ x \in I,$$

and the equation

$$u' = -u^2 - g$$

has a solution $u \in C^1(I, \mathbb{R})$, with u(a) = 0 for some $a \in I$. Then the operator $D : C^2(I, X) \to C(I, X)$

$$D(y) = y'' + gy$$

is Ulam stable with the Ulam constant K.

Corollary 2. *Suppose that there exists* K > 0 *such that*

$$g(x) \leq -\frac{1}{K}, \ x \in I$$

and the equation

$$u' = -u^2 + g$$

has a solution $u \in C^1(I, \mathbb{R})$ with u(a) = 0 for some $a \in I$. Then the operator $D : C^2(I, X) \to C(I, X)$

$$D(y) = y'' + gy$$

is Ulam stable with the Ulam constant K.

The proof of the previous corollaries follows immediately by taking f = 0 in the Theorems 1 and 2, respectively.

Remark 1. The particular case $g(x) = \lambda'(x) - \lambda^2(x)$, $\lambda \in C^1(I, \mathbb{R})$ being a periodic function, is studied in [14,15] and corresponds to the Hill's equation.

Remark 2. The stability of the operator *D* defined by (1) depends on the existence of a global solution for the Riccati Equation (4). In the following is given an example where some appropriate conditions on the coefficients

of the Equation (4) lead to the existence of such a solution. For this we will use the following result contained in [16].

Theorem 3. Let $I = [0, +\infty)$ and $a, b, c \in C(I, \mathbb{R})$, $x_0 \in I$ and $y_0 \ge 0$. If $a(x) \le 0$, $c(x) \ge 0$ for all $x \in I$, then the Cauchy problem

$$\begin{cases} y' = a(x)y^2 + b(x)y + c(x), \ x \in I \\ y(x_0) = y_0 \end{cases}$$

has a unique nonnegative solution $y = \varphi(x)$ *on* $[x_0, +\infty)$ *.*

Corollary 3. Let $I = [0, +\infty)$ and suppose that:

(*i*) $f(0) \ge 0;$

(ii) $f'(x) \ge g(x), x \in I;$

(iii) $g(x) \ge \frac{1}{K} > 0, x \in I.$

Then the operator D defined by (1) is Ulam stable with the Ulam constant K.

Proof. The result is a simple consequence of the Theorems 1 and 3. \Box

Analogously, we obtain the following result.

Corollary 4. *Let* $I = [0, +\infty)$ *and suppose that:*

i) $f(0) \ge 0;$ ii) $f'(x) \ge -g(x), x \in I;$ iii) $g(x) \le -\frac{1}{K} < 0, x \in I.$

Then the operator D defined by (1) is Ulam stable with the Ulam constant K.

Example 1. Let $I = [0, +\infty)$, 0 < b < a and suppose that $b \le |g(x)| \le a, x \in I$. Then the operator D defined by the relation

$$D(y)(x) = y''(x) + axy'(x) + g(x)y(x), \ y \in C^2(I), x \in I,$$

is Ulam stable with the Ulam constant $\frac{1}{h}$.

Proof. Let $f(x) = ax, x \in I$. The condition $b \le |g(x)| \le a, x \in I$ and the continuity of g implies $g(x) \in [b, a], x \in I$, or $g(x) \in [-a, -b], x \in I$ (clearly, if g has two values of opposite signs, then g must take the value zero at some point in I).

If $g(x) \in [b, a], x \in I$, then, according to Corollary 3, *D* is Ulam stable with the Ulam constant $\frac{1}{b}$. If $g(x) \in [-a, -b], x \in I$, then *D* is Ulam stable with the Ulam constant $\frac{1}{b}$, in view of Corollary 4. \Box

3. Conclusions

In this paper, we studied the Ulam stability of a linear differential operator of second order acting on a Banach space. We proved the Ulam stability of the equation using the assumption that a certain Riccati differential equation has a global solution. In appropriate conditions of the coefficients of the operators, we obtained an explicit representation of its Ulam constant. Finally, we presented some stability results for particular cases of the operator.

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