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Impacts of Viscous Dissipation and Brownian motion on Jeffrey Nanofluid Flow over an Unsteady Stretching Surface with Thermophoresis

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Abstract: The goal of this investigation is to explore the influence of viscous dissipation and Brownian motion on Jeffrey nanofluid flow over an unsteady moving surface with thermophoresis and mixed convection. Zero mass flux is also addressed at the surface such that the nanoparticles fraction of maintains itself on huge obstruction. An aiding transformation is adopted to renovate the governing equations into a set of partial differential equations which is solved using a new fourth-order finite difference continuation method and various graphical outcomes are discussed in detail with several employed parameters. The spectacular influence of pertinent constraints on velocity and thermal curves are inspected through various plots. Computational data for the heat transfer rate and skin-friction coefficient are also reported graphically. Graphical outcomes indicate that an augmentation in buoyance ratio and thermophoretic parameter leads to diminish the velocity curves and increase the temperature curves. Furthermore, it is inspected that escalating Deborah number exhibits increasing in the skin friction and salient decreasing heat transmission. Increasing magnetic strength leads to a reduction in the skin friction and enhancement in the Nusselt number, whilst a reverse reaction is manifested with mixed convection aspects.

Keywords: Jeffrey nanofluid; MHD; mixed convection; viscous dissipation; moving surface

1. Introduction

In modern times, the novel investigations of nanofluid flow through stretchable surfaces have got valuable attention among study scientists and community because of its abundant practical utilization, in several areas of science and biotechnology. Nanofluids play a momentous role in the mechanism of heat transfer. The typical base fluids, e.g., water, alcohol, ethylene glycol, and oil have a weak capacity to promote the heat transfer rate. However, this complex scenario was resolved by asserting the tiny sized solid fragments in the base fluids. It was basically proposed by Choi [1] that the tendency of the base fluids to embellish the thermal properties can be more effective by adding nanoparticles in these fluids. Nanofluids are spawned by dispersion of nanoparticles along with base fluids and these fluids are the amalgamations of suspended nano-sized pieces (1–100 nm) in base fluids. The constituents of nanoparticles may contain metals, carbides, nitrides, and oxides.

With enhanced and improved thermal mechanism, nanofluids have huge applications containing microwave tubes, microelectronic chips engines, grinder machines, drag reduction, refrigerators, pharmacology, tumor and cancer therapy, gas recovery of boiler exhaust fuel, supersonic and ultrasonic fields, high power lasers, welding cooling, vehicle engine cooling, nuclear system cooling, and thermal storage capacity [2,3]. In a new era of research, the study community has made tremendous efforts to enrich the experimental work, theoretical models, and numerical simulations of magneto-nanofluids. The characteristics of radiation impact on the stretched magneto-flow of non-Newtonian nanofluid were explored by Kumar et al. [4]. The hybrid linearization differential quadrature method was utilized to scrutinize the magneto-nanofluid flow past a radiative surface with streamwise diversity in wall temperature reported by EL-Zahar et al. [5]. Many investigators [6–14] to undertake the nanofluids flow past various configurations and geometries.

Lately, there have been serious developments in studying the non-Newtonian fluid flows. The non-Newtonian fluids are viscoelastic in nature, such as oils, paints, ketchup, and fluid polymers, all of which show some amazing marvels. Improving enthusiasm of numerous researchers has elucidated that these flows are imperative in industry, food and paper production, polymer processing, and technology that these flows are important. The rheological characteristics of fluids are indicated by their purported constitutive equations. The pattern of liquid under examination is Jeffrey liquid. This pattern has a time derivative instead of the convected derivative. Sandeep et al. [15], Dalir et al. [16], Hayat et al. [17], and Shehzad et al. [18] investigated the magneto-Jeffery nanofluid flow with several aspects. Raju et al. [19] communicated the magneto-Jeffery nanofluid flow over a radiated cone with thermophoresis and Brownian impacts. Hayat et al. [20] examined Jeffrey nanoliquid flow and heat transmission through a stretchable cylinder. Nadeem and Saleem [21] addressed analytically the unsteady rotating Jeffrey nanofluid flow on rotating cone by unsteady mixed convection. Zin et al. [22] explained the magneto-rotating Jeffrey nanoliquid through a vertical disk in a permeable medium. Hsiao [23–26] examined significant investigations about magneto-mixed convection of non-Newtonian nanofluid flows.

The major goal of the current work is to model the effectiveness of viscous dissipation and Brownian motion together on Jeffrey-nanofluid flow owing to an unsteady moving surface with thermophoresis. To the researcher's knowledge, there is no discussion of Jeffrey-nanofluid flow across the unsteady stretchable surface with thermophoresis. Furthermore, the roles of nanofluids parameters and Eckert and Deborah numbers have been computationally exhibited.

2. Modeling

Here, the magneto-mixed convection Jeffrey-nanofluid flow is treated cross over an unsteady vertical moving surface. Brownian motion characteristics are addressed through thermophoresis and dissipation effects. Constant magnetic strength, B_0 , is utilized perpendicular to the surface and zero nanoparticles mass flux is assumed. As proposed, the constant temperature at the surface T_w , whilst the ambient temperature field is T_∞ , and the concentration distribution is C_∞ . The flow sketch is exhibited in Figure 1. The flow under these attentions can be evaluated in the ensuing form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = & \frac{v}{(1+\lambda_2)} \left\{ \frac{\partial^2 u}{\partial y^2} + \lambda_1 \left(\frac{\partial^3 u}{\partial y^2 \partial t} + u \frac{\partial^3 u}{\partial y^2 \partial x} - \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^3 u}{\partial y^3} \right) \right\} \\ & + (1 - C_\infty)(T - T_\infty) \tilde{\beta} \rho_{f\infty} g - (C - C_\infty) g (\rho_p - \rho_{f\infty}) - \frac{\sigma B_0^2}{\rho} u \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = & \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{v}{(1+\lambda_2)} \left\{ \left(\frac{\partial u}{\partial y} \right)^2 + \lambda_1 \left(u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial t \partial y} + v \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \right) \right\} \\ & + \varepsilon \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] \end{aligned} \quad (3)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} \quad (4)$$

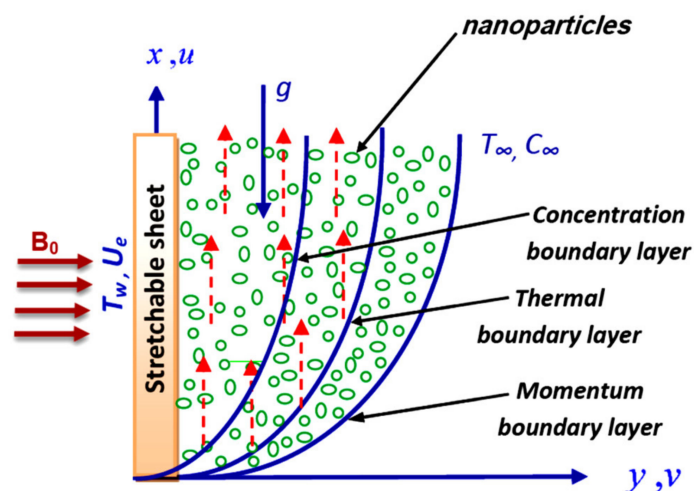


Figure 1. The flow pattern geometry.

The corresponding initial and boundary conditions are:

$$\begin{aligned} \text{For } t < 0 \text{ we have } T &= T_\infty \text{ and } C = C_\infty \text{ at any value of } x, y \\ \text{For } t \geq 0 \text{ we have } u &= U_e, v = 0, T = T_w, \text{ and } D_B \frac{\partial C}{\partial y} + \frac{D_T}{T_\infty} \frac{\partial C}{\partial y} = 0 \text{ at } y = 0, \\ \text{For } u \rightarrow 0 \text{ we have, } T &\rightarrow T_\infty \text{ and } C \rightarrow C_\infty \text{ as } y \rightarrow \infty, \end{aligned} \quad (5)$$

Note that u and v characterize the flow velocities in x - and y -trends respectively while $\nu = \mu/\rho$, μ , ρ describe kinematic viscosity, dynamic viscosity, and density of base liquid respectively. The symbols g are for gravity g , and λ_1 for retardation time; λ_2 for the ratio of relaxation to retardation times; σ symbolizes the electrical conductivity; B_0 symbolizes the magnetic field; T the temperature; $\alpha_m = k/(\rho C)_f$, k , $(\rho C)_f$, $(\rho C)_p$ symbolize the thermal diffusivity, thermal conductivity, liquid heat capacity, and nanoparticles effective heat capacity; $\varepsilon = (\rho C)_p/(\rho C)_f$ is the nanofluid heat capacity ratio, D_B the Brownian diffusion coefficient, D_T for the thermophoretic diffusion coefficient; and T and C for fluid temperature and concentration, respectively.

Launching the dimensionless variables as:

$$\begin{aligned} \psi &= (av_f)^{1/2} \tau^{1/2} x f(\tau, \eta), \eta = (a/v_f)^{1/2} \tau^{-1/2} y, \tau = 1 - e^{-at}, \theta(\tau, \eta) = \frac{T - T_\infty}{T_w - T_\infty}, \\ \phi(\tau, \eta) &= \frac{C - C_\infty}{C_\infty}, u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}. \end{aligned} \quad (6)$$

into Equations (1)–(5) gives us the following dimensionless equations:

$$\begin{aligned} \frac{1}{1+\lambda_2} [\tau - \beta(1-\tau)] f''' + \frac{\beta}{1+\lambda_2} \left\{ \tau(1-\tau) \frac{\partial f'''}{\partial \tau} - \frac{1}{2}(1-\tau) \eta f^{(iv)} + \tau(f''^2 - f f^{(iv)}) \right\} \\ + \frac{1}{2} \tau \eta (1-\tau) f'' - \tau^2 (f'^2 + Ha^2 f' - f f'' - \lambda(\theta - Nr \phi)) = \tau^2 (1-\tau) \frac{\partial f'}{\partial \tau} \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{1}{Pr} \tau \theta'' + \frac{1}{2} \tau (1-\tau) \theta' + \tau^2 f \theta' + \tau (Nb \theta' \phi' + Nt \theta'^2) \\ + \frac{Ec}{1+\lambda_2} \left\{ \tau f''^2 + \beta \left[\tau (f' f''^2 - f f'' f''') - (1-\tau) \left(\frac{1}{2} \eta f'' f''' - \tau f'' \frac{\partial f''}{\partial \tau} + \frac{1}{2} f''^2 \right) \right] \right\} = \tau^2 (1-\tau) \frac{\partial \theta}{\partial \tau} \end{aligned} \quad (8)$$

$$\frac{1}{Sc} \tau \phi'' + \frac{1}{2} \tau (1-\tau) \phi' + \tau^2 f \phi' + \frac{1}{Sc} \frac{Nt}{Nb} \tau \theta'' = \tau^2 (1-\tau) \frac{\partial \phi}{\partial \tau} \quad (9)$$

The transformed initial and boundary conditions become:

$$f(\tau, 0) = 0, f'(\tau, 0) = \theta(\tau, 0) = 1, Nb\phi'(\tau, 0) + Nt\theta'(\tau, 0) = 0, \theta(\tau, \infty) = f'(\tau, \infty) = \phi(\tau, \infty) = 0 \quad (10)$$

where $()'$ designates differentiation concerning the transformed transverse coordinate, η . The parameters in Equations (7)–(10) are defined as follows:

$$Nb = \frac{\varepsilon D_B C_\infty}{v}, Nt = \frac{\varepsilon D_T (T_w - T_\infty)}{T_\infty v}, Nr = \frac{(\rho_p - \rho_{f\infty}) C_\infty}{(1 - C_\infty) \beta (T_w - T_\infty)}, Sc = \frac{v}{D_B}, \beta = \lambda_1 a, \\ Ec = \frac{U_e^2}{C_p (T_w - T_\infty)}, \lambda = Gr_x / Re_x^2, Gr_x = g \beta (T_w - T_\infty) (1 - C_\infty) x^3 / v^2, Re_x = (U_e x / v), \\ Ha = \sqrt{\frac{B_0^2 \sigma}{a \rho}}, Pr = v \rho C_p / k \quad (11)$$

where Nb symbolizes the Brownian motion parameter, Nt symbolizes the thermophoresis parameter, Nr reflects the buoyancy ratio parameter, Sc reflects the Schmidt number, β relates the Deborah number, Ec symbolizes the Eckert number, λ symbolizes the mixed convection parameter, Gr_x and Re_x define the Grashof and Reynolds numbers, respectively, Ha reflects the Hartmann number, and Pr symbolizes the Prandtl number.

The quantities of engineering interest in dimensionless form skin friction and Nusselt number may be defined as:

$$C_{fx} = -\frac{\mu}{(1 + \lambda_2)} \left\{ \frac{\partial u}{\partial y} + \lambda_1 \left[\frac{\partial^2 u}{\partial y \partial t} + u \frac{\partial^2 u}{\partial y \partial x} + v \frac{\partial^2 u}{\partial y^2} \right] \right\}_{y=0} = -Re_x^{-1/2} \tau^{-1/2} \frac{1}{\lambda_2 + 1} \{ (1 + \beta) f''(\tau, 0) \\ - (1 - \tau) \beta \left[\frac{\partial f''(\tau, 0)}{\partial \tau} - \frac{1}{2\tau} f''(\tau, 0) \right] \}, \quad (12) \\ C_f(\tau, 0) = C_{fx} Re_x^{1/2} = -\tau^{-1/2} \frac{1}{\lambda_2 + 1} \{ (1 + \beta) f''(\tau, 0) - \beta (1 - \tau) \left[\frac{\partial f''(\tau, 0)}{\partial \tau} - \frac{1}{2\tau} f''(\tau, 0) \right] \} \\ Nu_x = -\frac{x(\partial T / \partial y)_{y=0}}{(T_w - T_\infty)} = -Re_x^{1/2} \tau^{-1/2} \theta'(\tau, 0), Nu(\tau, 0) = Nu_x Re_x^{-1/2} = -\tau^{-1/2} \theta'(\tau, 0) \quad (13)$$

3. Fourth-Order Finite Difference Continuation Method (FFDCM)

The partial differential equations (PDE) system (7)–(10) has a multi-scale solution behavior over an infinite interval where the solution varies quickly over the boundary layer and away from this layer the solution varies slowly and behaves regularly and that is according to the time constants of the solution components. Moreover, the PDE system (7)–(10) is very sensitive to the initial conditions due to the singularity associated with the highest derivative-term in the system and hence more accurate and efficient adaptive methods are required for solving this class of PDE systems [5,27–32]. System (7)–(10) is converted into a first-order PDE system and discretized using a fourth-order finite difference method in η -orientation and a two-point backward finite difference method [28] in τ -orientation. The solution is obtained through a continuation technique in η -orientation. We will denote the suggested method as fourth-order finite difference continuation method (FFDCM).

By defining $\pi_1 = f, \pi_2 = f', \pi_3 = f'', \pi_4 = f''', \pi_5 = \theta, \pi_6 = \theta', \pi_7 = \phi, \pi_8 = \phi'$ system (7)–(10) can be rewritten as:

$$\left. \begin{aligned}
\pi_1' &= \pi_2, \\
\pi_2' &= \pi_3, \\
\pi_3' &= \pi_4, \\
\frac{\beta(0.5\eta(1-\tau) + \tau y_1)}{1 + \lambda_2} \pi_4' &= \left[\frac{\tau - \beta(1-\tau)}{1 + \lambda_2} \pi_4 + \frac{\beta}{\lambda_2 + 1} \left(\tau(1-\tau) \frac{\partial \pi_4}{\partial \tau} + \tau \pi_3^2 \right) \right. \\
&\quad \left. + \frac{1}{2} \tau \pi_3 \eta(1-\tau) + \tau^2 (\pi_1 \pi_3 - \pi_2^2 - Ha^2 \pi_2 + \lambda(\pi_5 - Nr \pi_7)) - \tau^2(1-\tau) \frac{\partial \pi_2}{\partial \tau} \right], \\
\pi_5' &= \pi_6, \\
\frac{-1}{Pr} \tau \pi_6' &= 0.5\eta\tau(1-\tau)\pi_6 + \tau^2 \pi_1 \pi_6 + \tau(Nb \pi_6 \pi_8 + Nt \pi_6^2) + \frac{Ec}{1 + \lambda_2} \{ \tau \pi_3^2 \\
&\quad + \beta \left[\tau(\pi_2 \pi_3^2 - \pi_1 \pi_3 \pi_4) + (1-\tau) \left(\tau \pi_3 \frac{\partial \pi_3}{\partial \tau} - 0.5\eta \pi_3 \pi_4 - 0.5\pi_3^2 \right) \right] \} - \tau^2(1-\tau) \frac{\partial \pi_5}{\partial \tau}, \\
\pi_7' &= \pi_8, \\
\frac{-1}{Sc} \tau \pi_8' &= 0.5\eta\tau(1-\tau)\pi_8 + \tau^2 \pi_1 \pi_8 + \frac{1}{Sc} \frac{Nt}{Nb} \tau \pi_6' - \tau^2(1-\tau) \frac{\partial \pi_7}{\partial \tau},
\end{aligned} \right\} \quad (14)$$

with boundary conditions (BCs)

$$\begin{aligned}
\pi_1(\tau, 0) &= 0, \pi_2(\tau, 0) = 1, \pi_5(\tau, 0) = 1, Nb \pi_8(\tau, 0) + Nt \pi_6(\tau, 0) = 0 \\
\pi_2(\tau, \eta_\infty) &= \pi_5(\tau, \eta_\infty) = \pi_7(\tau, \eta_\infty) = 0, \quad \eta_\infty \rightarrow \infty
\end{aligned} \quad (15)$$

The mesh grid-points (τ_j, η_κ) of the problem are defined by:

$$\tau_j = j\Delta_\tau, \quad \eta_\kappa = \kappa \Delta_\eta, \quad j = 0, 1, \dots, N_\tau, \quad \kappa = 0, 1, \dots, N_{\eta_\infty} \quad (16)$$

where Δ_τ , Δ_κ and N_τ , N_{η_∞} are the step-size and the number of subintervals in τ and η -orientations, respectively. Applying FFDCM on PDE system (14) results in the following system of algebraic equations.

$$\mathbf{F} = \left[\begin{aligned}
&\mathbf{DZ}_1 - \mathbf{Z}_2 \\
&\mathbf{DZ}_2 - \mathbf{Z}_3 \\
&\mathbf{DZ}_3 - \mathbf{Z}_4 \\
&\frac{\beta(\tau_j \mathbf{Z}_1 + 0.5\eta(1-\tau_j))}{\lambda_2 + 1} \cdot \mathbf{DZ}_4 - \left[\frac{\tau_j - (1-\tau_j)\beta}{\lambda_2 + 1} \mathbf{Z}_4 + \frac{\beta}{\lambda_2 + 1} \left(\tau_j(1-\tau_j) \frac{\mathbf{Z}_4 - \hat{\mathbf{Z}}_4}{\Delta_\tau} + \tau_j \mathbf{Z}_3 \cdot \mathbf{Z}_3 \right) \right. \\
&\quad \left. + \frac{1}{2} \tau_j \eta(1-\tau_j) \cdot \mathbf{Z}_3 + \tau_j^2 (\mathbf{Z}_1 \cdot \mathbf{Z}_3 - \mathbf{Z}_2 \cdot \mathbf{Z}_2 - Ha^2 \mathbf{Z}_2 + \lambda(\mathbf{Z}_5 - Nr \mathbf{Z}_7)) - \tau_j^2(1-\tau_j) \frac{\mathbf{Z}_2 - \hat{\mathbf{Z}}_2}{\Delta_\tau} \right] \\
&\mathbf{DZ}_5 - \mathbf{Z}_6 \\
&\frac{1}{Pr} \tau_j \mathbf{DZ}_6 + 0.5\eta \tau_j(1-\tau_j) \cdot \mathbf{Z}_6 + \tau_j^2 \mathbf{Z}_1 \cdot \mathbf{Z}_6 + \tau_j(Nb \mathbf{Z}_6 \cdot \mathbf{Z}_8 + Nt \mathbf{Z}_6 \cdot \mathbf{Z}_6) + \frac{Ec}{1 + \lambda_2} \{ \tau_j \mathbf{Z}_3 \cdot \mathbf{Z}_3 + \\
&\quad + \beta \left(\tau_j(\mathbf{Z}_3 \cdot \mathbf{Z}_3 \cdot \mathbf{Z}_2 - \mathbf{Z}_4 \cdot \mathbf{Z}_3 \cdot \mathbf{Z}_1) + (1-\tau_j) \left(\mathbf{Z}_3 \cdot \frac{\tau_j(\mathbf{Z}_3 - \hat{\mathbf{Z}}_3)}{\Delta_\tau} - 0.5\eta \cdot \mathbf{Z}_3 \cdot \mathbf{Z}_4 - 0.5\mathbf{Z}_3 \cdot \mathbf{Z}_3 \right) \right\} - \\
&\quad - \tau_j^2(1-\tau_j) \cdot \frac{\mathbf{Z}_5 - \hat{\mathbf{Z}}_5}{\Delta_\tau} \\
&\mathbf{DZ}_7 - \mathbf{Z}_8 \\
&\frac{1}{Sc} \tau_j \mathbf{DZ}_8 + 0.5\eta \tau_j(1-\tau_j) \cdot \mathbf{Z}_8 + \tau_j^2 \cdot \mathbf{Z}_1 \cdot \mathbf{Z}_8 + \frac{1}{Sc} \frac{Nt}{Nb} \tau_j \mathbf{DZ}_6 - \tau_j(1-\tau_j) \cdot \frac{\mathbf{Z}_7 - \hat{\mathbf{Z}}_7}{\Delta_\tau}
\end{aligned} \right] = \mathbf{0} \quad (17)$$

where $\mathbf{D}$ is the fourth-order finite-difference differentiation matrix defined in [27], $\mathbf{Z}_+$ and $\hat{\mathbf{Z}}_+$ are the solution vectors $[\pi_+(\tau_j, \eta_\kappa)]_{\kappa=0}^{N_{\eta_\infty}}$, $[\pi_+(\tau_{j-1}, \eta_\kappa)]_{\kappa=0}^{N_{\eta_\infty}}$, respectively, $\dagger = 1, 2, \dots, 8$, and η is the coordinate vector $[\eta_\kappa]_{\kappa=0}^{N_{\eta_\infty}}$.

BCs (15) are inserted into the system (17) as follows:

$$\left. \begin{aligned} \mathbf{F}(1) &= \mathbf{Z}_1(1) \\ \mathbf{F}(N_{\eta_\infty} + 1) &= \mathbf{Z}_2(1) - 1 \\ \mathbf{F}(2N_{\eta_\infty}) &= \mathbf{Z}_2(N_{\eta_\infty}) \\ \mathbf{F}(3N_{\eta_\infty}) &= \mathbf{Z}_3(N_{\eta_\infty}) \\ \mathbf{F}(4N_{\eta_\infty} + 1) &= \mathbf{Z}_5(1) - 1 \\ \mathbf{F}(5N_{\eta_\infty}) &= \mathbf{Z}_5(N_{\eta_\infty}) \\ \mathbf{F}(7N_{\eta_\infty}) &= \mathbf{Z}_7(N_{\eta_\infty}) \\ \mathbf{F}(7N_{\eta_\infty} + 1) &= Nb\mathbf{Z}_8(1) + Nt\mathbf{Z}_6(1) \end{aligned} \right\} \quad (18)$$

The nonlinear algebraic system (17)–(18) is solved at each line τ_j using the nonlinear built-in MATLAB solver ‘fsolve’ with an initial guess $\mathbf{Z}_+$ satisfying the BCs(10) such as:

$$\left. \begin{aligned} \bar{\mathbf{Z}}_1 &= 1 - e^{-\eta} \\ \bar{\mathbf{Z}}_5 &= e^{-\eta} \\ \bar{\mathbf{Z}}_7 &= -\frac{Nt}{Nb}e^{-\eta} \end{aligned} \right\} \quad (19)$$

The solution of (17)–(18) is obtained through a continuation technique in η -orientation taking η_∞ as our continuation parameter. Starting with an initial estimation of the parameter η_∞ and the initial guess $\mathbf{Z}_+$, an estimate solution $\bar{\mathbf{Z}}_+$ is obtained for system (17)–(18) which is taken as a new initial guess to get a new estimate solution $\hat{\mathbf{Z}}_+$ for system (17)–(18) after adding an increment $\Delta\eta_\infty$ in the continuation parameter and the process is repeated as shown in Figure 2. The continuation technique is stopped at:

$$E_j = \max\left(\left\|\bar{\mathbf{Z}}_2 - \hat{\mathbf{Z}}_2\right\|_\infty, \left\|\bar{\mathbf{Z}}_5 - \hat{\mathbf{Z}}_5\right\|_\infty\right) < 10^{-5} \quad (20)$$

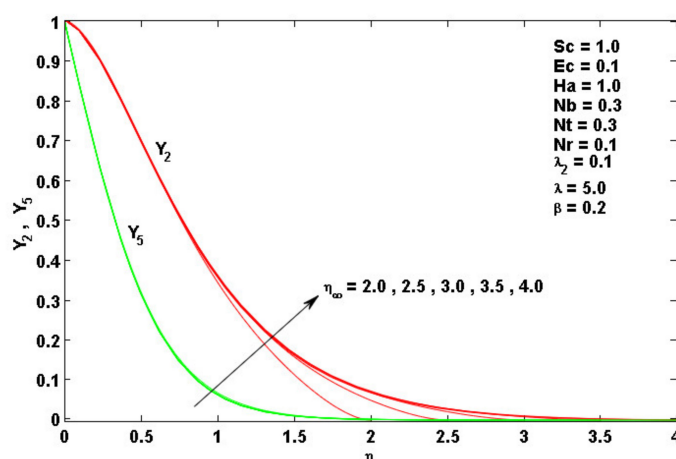


Figure 2. The continuation process on η_∞ for solving system (17)–(18) using $\Delta\eta_\infty = 0.5$.

The numerical results are obtained at $\Delta\tau = 0.05$, $\Delta\kappa = 0.005$ using absolute tolerance 10^{-6} and relative tolerance 10^{-3} for the nonlinear Matlab solver ‘fsolve’.

Table 1 shows the good agreement between the numerical results obtained by FFDCM and Ishak et al. [33] approach for different values of λ and Pr for ($Nr = Nt = Nb = \beta = Ha = \lambda_2 = 0$) at $\tau = 1.0$ (steady-state flow).

Table 1. Comparison of $f''(\tau, 0)$ and $-\theta'(\tau, 0)$ for different values of λ and Pr for ($Nr = Nt = Nb = \beta = Ha = \lambda_2 = 0$) at $\tau = 1.0$ (steady-state flow).

Pr	λ	Ishak et al. [33]		FFDCM Results	
		$f''(1,0)$	$-\theta'(1,0)$	$f''(1,0)$	$-\theta'(1,0)$
0.7	0	−1.0000	0.7937	−1.00000	0.79371
	1.0	−0.5076	0.8961	−0.50763	0.89613
	10	2.5777	1.1724	2.57772	1.17244
	100	21.8052	1.8075	21.80527	1.80756
1.0	0	−1.0000	1.0000	−1.00000	1.00000
	1.0	−0.5608	1.0873	−0.56081	1.08733
	10	2.3041	1.3716	2.30414	1.37164
	100	20.3786	2.0667	20.37862	2.06673
3.0	0	−1.0000	1.9237	−1.00000	1.92374
	1.0	−0.7092	1.9743	−0.70924	1.97433
	10	1.4567	2.2442	1.45673	2.24424
	100	15.9716	3.1042	15.97168	3.10427
7.0	0.0	−1.0000	3.0722	−1.00000	3.07221
	1.0	−0.7962	3.1055	−0.79621	3.10553
	10	0.8505	3.3318	0.85054	3.33185
	100	12.7216	4.2663	12.72167	4.26636

4. Discussion

This precise segment is dispensed to analyze the magneto-mixed convection and heat transport characteristics for convection Jeffrey-nanofluid flow towards an unsteady vertical moving surface. The graphical explanation is deliberated and the variation of velocity, temperature, skin friction and Nusselt number for various thermophysical parameters are characterized in Figures 3–8 and those parameters are fixed at $\beta = 0.2$, $Nb = 0.5$, $Nt = 0.3$, $Nr = 0.1$, $Sc = 1.0$, $Ec = 0.1$, $\lambda = 5.0$, $Ha = 1.0$, $\lambda_2 = 0.1$ and $Pr = 6.8$.

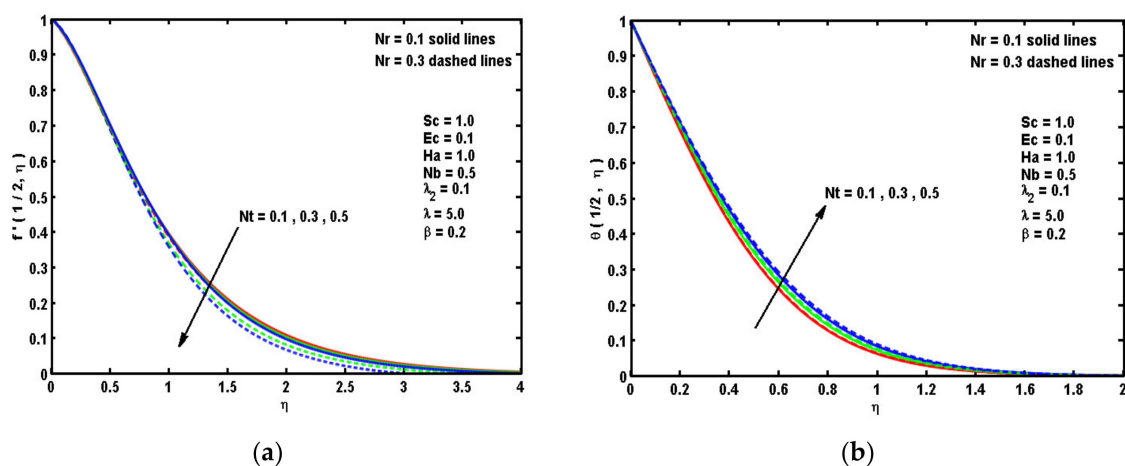


Figure 3. Impacts of Nr and Nt on (a) $f'(\tau, \eta)$ and (b) $\theta(\tau, \eta)$.

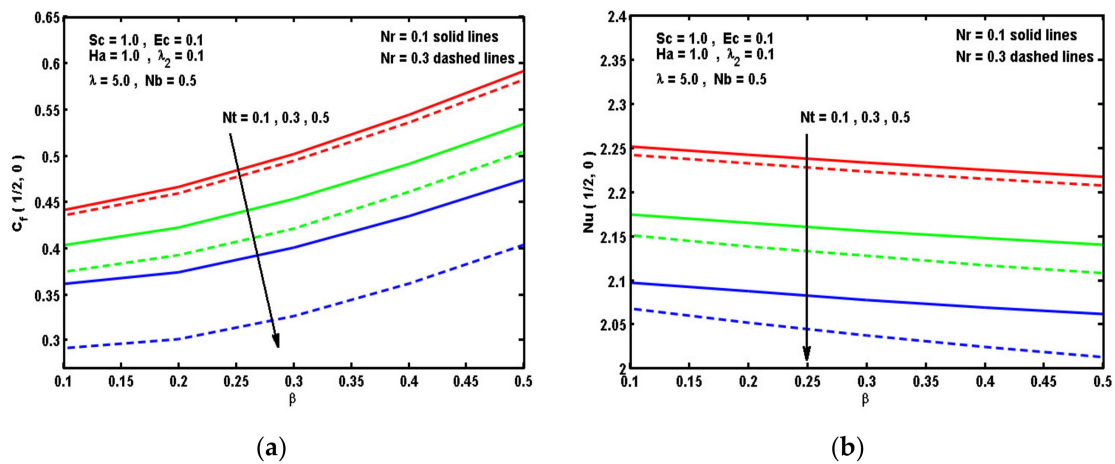


Figure 4. Impacts of Nr and Nt on (a) $C_f(\tau, 0)$ and (b) $Nu(\tau, 0)$.

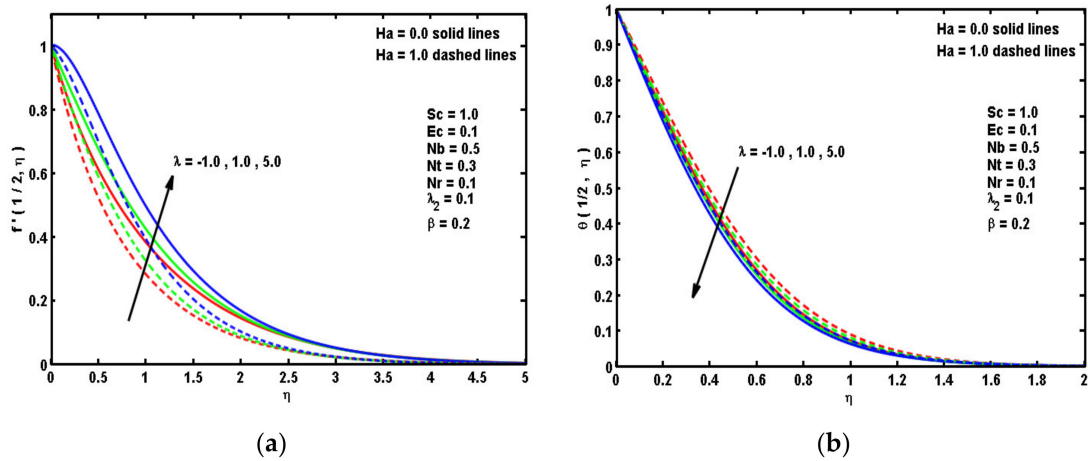


Figure 5. Impacts of λ and Ha on (a) $f'(\tau, \eta)$ and (b) $\theta(\tau, \eta)$.

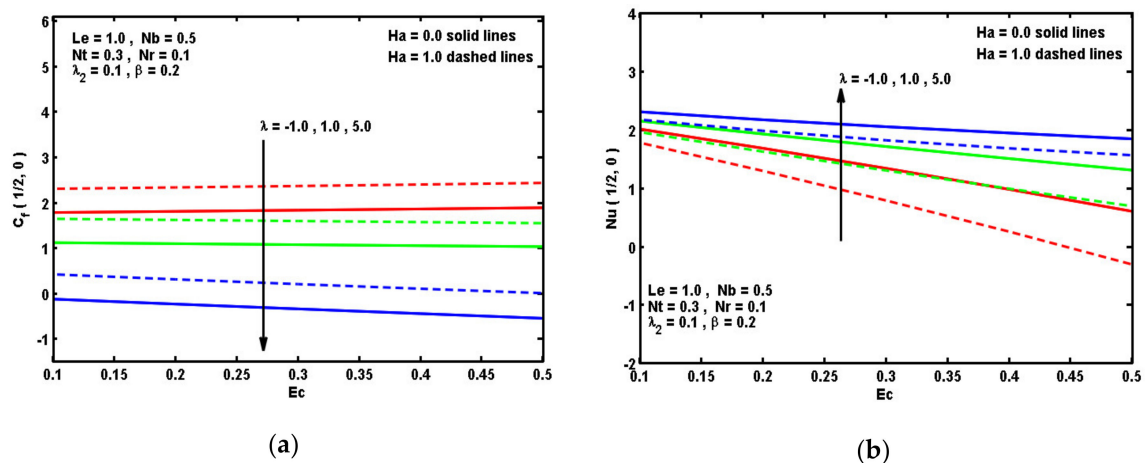


Figure 6. Impacts of λ and Ha on (a) $C_f(\tau, 0)$ and (b) $Nu(\tau, 0)$.

The nature of velocity $f'(\tau, \eta)$ and temperature $\theta(\tau, \eta)$ curves for magnifying the buoyancy ratio Nr and thermophoretic Nt is pointed out in Figures 2b and 3a. It is manifested that velocity diminishes with the augmentation of Nr and Nt , and this phenomenon yields an increase in temperature curves ability that boosts the thermal boundary. This is due to the reality that greater values of Nr boost the temperature fluid. Figure 4a,b uncovers the impacts of Nr and Nt on skin friction coefficient $C_f(\tau, 0)$ and the local Nusselt number $Nu(\tau, 0)$ with several values of β . It is observed that both of $C_f(\tau, 0)$ and $Nu(\tau, 0)$

diminish sufficiently for physical parameters Nr and Nt . This because the thermophoresis strength triggers the nanoparticles to transmit the warm sheet to cold sheet which creates the thermal related boundary layer thickness to magnify. In addition, an enlarge in β corresponds to an enhancement in the $Cf(\tau,0)$ and salient reduction in $Nu(\tau,0)$. This is because a weaker Deborah number, β , gives a viscous impact compared to the elastic impact, whilst a greater β exhibits in the elastically solid material in nature which leads to a reduction in Nusselt number.

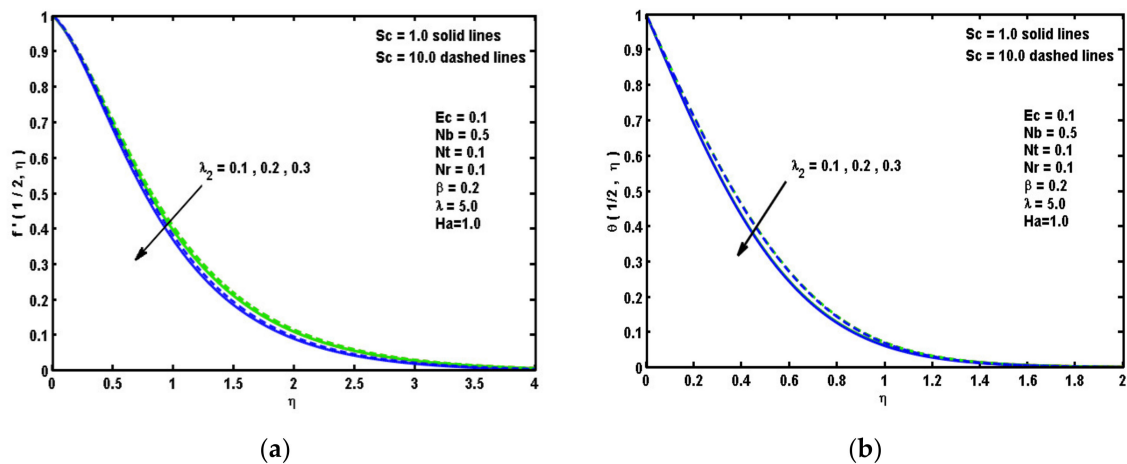


Figure 7. Impacts of Sc and λ_2 on (a) $f'(\tau, \eta)$ and (b) $\theta(\tau, \eta)$.

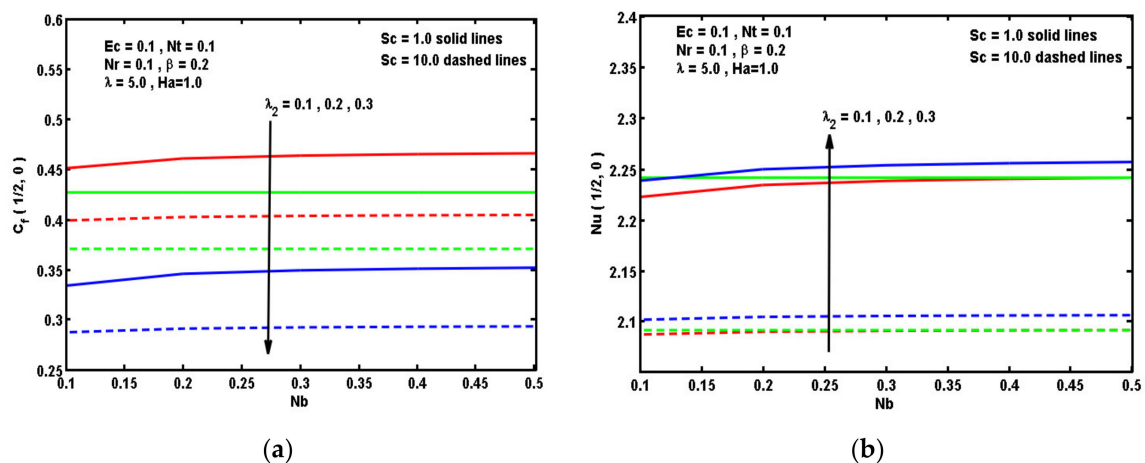


Figure 8. Impacts of Sc and λ_2 on (a) $Cf(\tau, 0)$ and (b) $Nu(\tau, 0)$.

The nature of velocity $f'(\tau, \eta)$ and temperature $\theta(\tau, \eta)$ curves against mixed convection parameter λ and Hartmann number Ha are sketched in Figure 5a,b. From these curves, it is concluded that intensifying λ improving the flow velocity and decaying the temperature curves. Because when growing λ , the impact of gravity is decreased, and then velocity within a boundary layer decreases. In addition, it is shown that the sway of magnetic strength reduces the fluid movement. This is realistic because magnetic strength is responsible to inspire Lorentz intensity which resists the fluid motion. Figure 6a,b determines the $Cf(\tau, 0)$ and $Nu(\tau, 0)$ against the λ and Ha with several values of Ec . It is noticed that $Nu(\tau, 0)$ enhances but $Cf(\tau, 0)$ reduces for the improved values of Ha , while an opposite reaction is found with λ . The reason, as mentioned above, is that Ha has a tendency to magnify the Lorentz intensity. This intensity resists the fluid movement and provides heat ability in the flowing. Furthermore, it is clear that an elevation in Ec results in an enhancement in the internal source of energy which boosts the thermal boundary layer, which results in a reduction in the Nusselt number for greater Ec .

Figure 7a,b are plotted to analyze the impacts of the ratio of relaxation to retardation times λ_2 and Schmidt number Sc on velocity $f'(\tau, \eta)$ and temperature $\theta(\tau, \eta)$ curves. It is apparent that both of velocity and temperature curves decline with the enhancement in λ_2 parameter while opposite results are obtained for the intensity of Sc . The consequences of Sc and λ_2 on $Cf(\tau, 0)$ and $Nu(\tau, 0)$ on along with Nb is visualized in Figure 8a,b. It is observed that $Cf(\tau, 0)$ reduces and $Nu(\tau, 0)$ enhances by augmenting λ_2 . Moreover, both $Cf(\tau, 0)$ and $Nu(\tau, 0)$ reduce sufficiently by enlarging Sc . This may occur owing to the cause that Schmidt number is reversely symmetrical to the Brownian diffusion coefficient. This Brownian diffusion coefficient becomes lower corresponding to the greater values of Sc and also Brownian movement happens in the system of the nanofluid due to contact of nanoparticles with the regular fluid.

5. Conclusions

The effectiveness of Brownian motion and viscous dissipation on magneto-mixed convection flow of Jeffrey nanofluid through an unsteady moving surface is examined with thermophoresis. An aiding transformation is adopted to renovate the governing equations into a set of PDEs which are sensitive to the initial conditions due to the singularity associated with the highest derivative-term and so the numerical solution is gotten with the aid of a new FFDCM and various graphical outcomes are discussed in detail with several employed parameters.

- A comparative investigation among the current outcomes and the former cited investigation are explored to trust our outcomes and a notable agreement is observed. The following observations are structured for the current investigation:
- Augmentation in buoyance ratio and thermophoretic parameters leads to diminish the velocity curves and increase the temperature curves ability that boosts the thermal boundary.
- A greater Deborah number exhibits increasing skin friction and salient decreasing heat transmission.
- The Nusselt number enhances and skin friction reduces for the improved magnetic strength, while an opposite reaction is found with mixed convection aspects.
- Both velocity and temperature curves decline with the enhancement in the ratio of relaxation to retardation times while opposite results are obtained for the intensity of Schmidt number.

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