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Certain Identities Associated with (p, q) -Binomial Coefficients and (p, q) -Stirling Polynomials of the Second Kind

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Received: 20 July 2020; Accepted: 27 August 2020; Published: 31 August 2020



Abstract: The q -Stirling numbers (polynomials) of the second kind have been investigated and applied in a variety of research subjects including, even, the q -analogue of Bernstein polynomials. The (p, q) -Stirling numbers (polynomials) of the second kind have been studied, particularly, in relation to combinatorics. In this paper, we aim to introduce new (p, q) -Stirling polynomials of the second kind which are shown to be fit for the (p, q) -analogue of Bernstein polynomials. We also present some interesting identities involving the (p, q) -binomial coefficients. We further discuss certain vanishing identities associated with the q -and (p, q) -Stirling polynomials of the second kind.

Keywords: (p, q) -analogue of the Chu-Vandermonde identity; Bernstein polynomials; q -analogue of Bernstein polynomials; (p, q) -analogue of Bernstein polynomials; q -Stirling polynomials of the second kind; (p, q) -Stirling polynomials of the second kind

1. Introduction and Preliminaries

Carlitz ([1], Equation (3.1)) used an explicit formula for q -differences to define the q -Stirling numbers of the second kind $S_q(\ell, \tau)$ by (see also ([2], Equation (4.7)), ([3], p. 505, Equation (35))

$$[\xi]_q^\ell = \sum_{\tau=0}^{\ell} q^{\frac{\tau(\tau-1)}{2}} \begin{bmatrix} \xi \\ \tau \end{bmatrix}_q [\tau]_q! S_q(\ell, \tau) \quad (\xi \in \mathbb{C}; \ell, \tau \in \mathbb{N}_0). \quad (1)$$

It is noted that the $S_q(\ell, \tau)$, being polynomials in q , are often called Stirling polynomials of the second kind (see [4]) or q -Stirling polynomials of the second kind (see, e.g., [5,6]). From (1) Carlitz ([1], Equation (3.3)) provided the following explicit expression (see also ([7], p. 1055), ([8], Equation (27)) ([4], Equation (2.5)), ([2], Equation (4.9)) ([9], Equation (2.2)), ([3], p. 505, Equation (37)))

$$S_q(\ell, \tau) = \frac{1}{[\tau]_q! q^{\frac{\tau(\tau-1)}{2}}} \sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q [\tau - \eta]_q^\ell \quad (\ell, \tau \in \mathbb{N}_0). \quad (2)$$

Here the involved notations are defined by (see, e.g., [10–15])

$$[\xi]_q := \frac{1 - q^\xi}{1 - q} = \frac{q^\xi - 1}{q - 1} \quad (\xi \in \mathbb{C}; q \in \mathbb{C} \setminus \{1\}; q^\xi \neq 1) \quad (3)$$

and

$$\left[\begin{matrix} \xi \\ \tau \end{matrix} \right]_q := \frac{[\xi]_{q;\tau}}{[\tau]_q!}, \quad (4)$$

where

$$[\xi]_{q;\tau} := [\xi]_q [\xi - 1]_q \cdots [\xi - \tau + 1]_q \quad (\xi \in \mathbb{C}; \tau \in \mathbb{N}_0). \quad (5)$$

Whenever a multiple-valued function appears, for example, q^ξ in (3), it is conveniently assumed to be chosen one of its appropriate branches, in particular, the principal branch. One also defines a q -binomial coefficient

$$\left[\begin{matrix} \mu \\ \nu \end{matrix} \right]_q = \frac{[\mu]_q!}{[\nu]_q! [\mu - \nu]_q!} \quad (\mu, \nu \in \mathbb{N}_0 \text{ with } \nu \leq \mu) \quad (6)$$

and otherwise is assumed to be zero, where $[\cdot]_q!$ is a q -factorial defined by

$$[\mu]_q! = \begin{cases} [\mu]_q \cdot [\mu - 1]_q \cdots [1]_q & (\mu \in \mathbb{N}) \\ 1 & (\mu = 0). \end{cases} \quad (7)$$

These q -binomial coefficients gratify the recursive relations (see, e.g., ([16], Section 8.2))

$$\left[\begin{matrix} \mu + 1 \\ \nu \end{matrix} \right]_q = q^{\mu - \nu + 1} \left[\begin{matrix} \mu \\ \nu - 1 \end{matrix} \right]_q + \left[\begin{matrix} \mu \\ \nu \end{matrix} \right]_q \quad (8)$$

and

$$\left[\begin{matrix} \mu + 1 \\ \nu \end{matrix} \right]_q = \left[\begin{matrix} \mu \\ \nu - 1 \end{matrix} \right]_q + q^\mu \left[\begin{matrix} \mu \\ \nu \end{matrix} \right]_q. \quad (9)$$

Both (8) and (9) when $q = 1$ give the Pascal identity for familiar binomial coefficients. It is noted that the q -binomial coefficient $\left[\begin{matrix} \mu \\ \nu \end{matrix} \right]_q$ ($0 \leq \nu \leq \mu$; $\nu, \mu \in \mathbb{N}_0$) is a polynomial of degree $\nu(\mu - \nu)$ in the variable q with all of its coefficients being positive, which is called Gaussian polynomial and may be shown by using either (8) or (9) (see, e.g., ([16], Section 8.2)). Here and elsewhere, let \mathbb{N} , \mathbb{R}^+ , \mathbb{R} , and \mathbb{C} denote the sets of positive integers, positive real numbers, real numbers, and complex numbers, respectively. Furthermore, put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The following recursive relation is satisfied

$$S_q(\ell + 1, \tau) = S_q(\ell, \tau - 1) + [\tau]_q S_q(\ell, \tau) \quad (\ell \in \mathbb{N}_0, \tau \in \mathbb{N}) \quad (10)$$

together with

$$S_q(\ell, \ell) = 1 \quad (\ell \in \mathbb{N}_0), \quad S_q(\ell, 0) = 0 \quad (\ell \in \mathbb{N}), \quad (11)$$

and

$$S_q(\ell, \tau) = 0 \quad (\tau > \ell). \quad (12)$$

Carlitz [1] employed the notation $a_{\ell, \tau}$ for $S_q(\ell, \tau)$ and commented that the numbers $S_q(\ell, \tau)$ emerged in relation to a problem in abelian groups (see ([17], p. 128)). Gould [18] exploited the notation $S_2(\ell, \tau, q)$ for $S_q(\ell, \tau)$ in order to distinguish q -Stirling numbers of the first kind $S_1(\ell, \tau, q)$. Since then, q -Stirling numbers of the second kind $S_q(\ell, \tau)$ in (2) have been involved in diverse research subjects, some of which are reviewed. In the course of introducing the q -Szász–Mirakyan operators, Aral ([7], Section 3) made use of q -Stirling polynomials of the second kind in (2). In investigation of a p -adic q -integral representation for q -Bernstein type polynomials which are different from those in (2), Kim et al. [19] explored q -Stirling numbers whose the first and second kinds were denoted, respectively, by $S_{1,q}(\ell, \tau)$ and $S_{2,q}(\ell, \tau)$. Alvarez [5] defined q -Stirling polynomials of the second kind

$S_q(\ell, \tau)$ by the recursive relation (10) together with $S_q(\ell, \tau) = 0$ ($\ell \in \mathbb{N}$, $\tau \leq 0$) and (12). Agratini and Radu ([20], Equation (5)) provided another q -Stirling numbers of the second kind denoted by $\sigma_q(\ell, \tau)$:

$$\sigma_q(\ell, \tau) = \frac{1}{[\tau]_q!} \sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{(\tau-\eta)(\tau-\eta-1)}{2}} \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q \frac{[\tau-\eta]_q^\ell}{q^{(\tau-\eta)\ell}} \quad (\ell, \tau \in \mathbb{N}_0). \quad (13)$$

Luo and Srivastava [2] expressed addition formulas of generalized q -Bernoulli and q -Euler polynomials in terms of certain finite double sums involving q -Stirling polynomials of the second kind $S_q(\ell, \tau)$ in (2). For further various usages and properties of q -Stirling numbers of the second kind, in particular, related to combinatorics, one may be referred to (for instance) [21–41].

The (p, q) -integers are presented by

$$[\mu]_{p,q} := \frac{p^\mu - q^\mu}{p - q} \quad (p, q \in \mathbb{C} \text{ with } p \neq q; \mu \in \mathbb{N}_0). \quad (14)$$

The (p, q) -factorial $[\mu]_{p,q}!$ of $\mu \in \mathbb{N}_0$ is given by

$$[\mu]_{p,q}! = \begin{cases} [\mu]_{p,q} \cdot [\mu-1]_{p,q} \cdots (p+q) \cdot 1 & (\mu \in \mathbb{N}) \\ 1 & (\mu = 0). \end{cases} \quad (15)$$

The (p, q) -binomial coefficients $\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q}$ are provided by

$$\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[\mu]_{p,q}! [n-\mu]_{p,q}!} \quad (n, \mu \in \mathbb{N}_0 \text{ with } \mu \leq n) \quad (16)$$

and otherwise is accepted to be zero.

The Pascal-type identity for (p, q) -binomial coefficient is given as (see, e.g., ([42], Equation (4.5)))

$$\begin{bmatrix} n+1 \\ \mu \end{bmatrix}_{p,q} = p^\mu \begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} + q^{n-\mu+1} \begin{bmatrix} n \\ \mu-1 \end{bmatrix}_{p,q} \quad (17)$$

$$(n, \mu \in \mathbb{N}_0 \text{ with } \mu \leq n),$$

which, upon using the following easily derivable identity from (16)

$$\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-\mu \end{bmatrix}_{p,q}, \quad (18)$$

gives another Pascal-type identity for (p, q) -binomial coefficient

$$\begin{bmatrix} n+1 \\ \mu \end{bmatrix}_{p,q} = p^{n+1-\mu} \begin{bmatrix} n \\ \mu-1 \end{bmatrix}_{p,q} + q^\mu \begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} \quad (19)$$

$$(n, \mu \in \mathbb{N}_0 \text{ with } \mu \leq n)$$

and vice versa. One observes from either (17) or (19) that the (p, q) -binomial coefficients $\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q}$ are polynomials in both p and q of degree $\mu(n-\mu)$.

When $p \neq q$ with $p, q \in \mathbb{C} \setminus \{0\}$, one gets (see, e.g., ([42], Equation (4.4)))

$$\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} = p^{\mu(n-\mu)} \begin{bmatrix} n \\ \mu \end{bmatrix}_{1, \frac{q}{p}} = p^{\mu(n-\mu)} \begin{bmatrix} n \\ \mu \end{bmatrix}_{\frac{q}{p}} \quad (20)$$

and

$$\begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} = q^{\mu(n-\mu)} \begin{bmatrix} n \\ \mu \end{bmatrix}_{1, \frac{p}{q}} = q^{\mu(n-\mu)} \begin{bmatrix} n \\ \mu \end{bmatrix}_{\frac{p}{q}}. \quad (21)$$

The q -Stirling polynomials of the second kind have been investigated in diverse research subjects including, even, the q -analogue of Bernstein polynomials, for example, [4]. The (p, q) -Stirling numbers (polynomials) of the second kind have been studied, particularly, in relation to combinatorics. In this paper, we aim to introduce new (p, q) -Stirling polynomials of the second kind (57) which are shown to be fit for the (p, q) -analogue of Bernstein polynomials (59). We also provide some interesting identities involving the (p, q) -binomial coefficients. We further discuss certain vanishing identities associated with the q -and (p, q) -Stirling polynomials of the second kind.

2. The q -Bernstein Polynomials Expressed in Terms of the q -Stirling Polynomials of the Second Kind

Sergei Natanovich Bernstein (1880–1968) made ingenious use of theory of probability to present afterwards Bernstein polynomials $B_n(h; x)$ associated with a function $h : [0, 1] \rightarrow \mathbb{R}$ (see [43–45]):

$$B_n(h; x) := \sum_{\mu=0}^n h\left(\frac{\mu}{n}\right) \binom{n}{\mu} x^\mu (1-x)^{n-\mu} \quad (n \in \mathbb{N}, x \in [0, 1]), \quad (22)$$

which were used to prove the Weierstrass approximation theorem constructively. Due to ensuing many findings of their remarkable and useful properties, Bernstein polynomials and their extensions have been fully and methodically explored (see, e.g., [4,46–49]).

Phillips [50,51] and ([16], Section 7.3) generalized (22) to give the following q -analogue of Bernstein polynomials (see also [8,52–59])

$$\mathcal{B}_n(h; x) = \sum_{\mu=0}^n h_\mu \begin{bmatrix} n \\ \mu \end{bmatrix}_q x^\mu \prod_{\nu=0}^{n-\mu-1} (1 - q^\nu x) \quad (0 \leq x \leq 1; n \in \mathbb{N}), \quad (23)$$

where $h : [0, 1] \rightarrow \mathbb{R}$, $h_\mu := h([\mu]_q / [n]_q)$. Here and elsewhere, the empty product is accepted to be 1.

Since $\mathcal{B}_n(h; 0) = h(0)$ and $\mathcal{B}_n(h; 1) = h(1)$, the q -analogue of Bernstein polynomials $\mathcal{B}_n(h; x)$ in (23) interpolate the function $h : [0, 1] \rightarrow \mathbb{R}$ at the endpoints, as the Bernstein polynomials $B_n(h; x)$ in (22) does.

Even though a generalization of Bernstein polynomials associated with q -integers was suggested in 1987 (see ([60], Section 1)), the q -analogue of Bernstein polynomials (23) have been received as a standard definition and investigated in such diverse ways as (an extension of several variables[61]; other q -polynomials and operators [7,20,62–65]; other types of Bernstein polynomials [19,66]; convergence and iterates [9,60,67]; monotonicity [48]; Cauchy kernel [68]; norm estimates [69]; unbounded function [70]; overview of the first decade [71]).

It is noticed that the product in (23) is connected with the following well known Euler's identity:

$$\prod_{\eta=0}^{\mu-1} (1 + q^\eta x) = (1+x) (1+qx) \cdots (1+q^{\mu-1}x) = \sum_{\nu=0}^{\mu} q^{\frac{\nu(\nu-1)}{2}} \begin{bmatrix} \mu \\ \nu \end{bmatrix}_q x^\nu. \quad (24)$$

One finds easily that the Euler's identity (24), which may be validated by induction on μ or another method (see, e.g., ([16], Chapter 7)), reduces to the ordinary binomial expansion when $q = 1$.

The q -difference was exploited to express the q -analogue of Bernstein polynomials (23) as follows (see [50], ([16], Theorem 7.3.1); see also ([72], Theorem 2)):

$$\mathcal{B}_n(h; x) = \sum_{\tau=0}^n \begin{bmatrix} n \\ \tau \end{bmatrix}_q \Delta_q^\tau h_0 x^\tau \quad (0 \leq x \leq 1), \quad (25)$$

where $h_\eta = h([\eta]_q / [n]_q)$ and

$$\Delta_q^\tau h_\eta := \begin{cases} \Delta_q^{\tau-1} h_{\eta+1} - q^{\tau-1} \Delta_q^{\tau-1} h_\eta & (\tau = 1, 2, \dots, n - \eta) \\ h_\eta & (\tau = 0). \end{cases} \quad (26)$$

It may be verified by induction that (cf., e.g., ([1], Equation (2.2)), ([18], Equation (2.13)), ([3], p. 504))

$$\Delta_q^\tau h_\mu = \sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q h_{\tau+\mu-\eta}. \quad (27)$$

The q -binomial coefficients were given in the following (see [4]):

$$\begin{bmatrix} n \\ \tau \end{bmatrix}_q = \frac{[n]_q^\tau}{[\tau]_q! q^{\frac{\tau(\tau-1)}{2}}} \psi_\tau^n \quad (0 \leq \tau \leq n), \quad (28)$$

where

$$\psi_\tau^n = \prod_{\eta=0}^{\tau-1} \left(1 - \frac{[\eta]_q}{[n]_q} \right).$$

They [4] used (25), (27) and (28) to obtain (cf., ([72], Corollary 4))

$$\mathcal{B}_n(x^\ell; x) = \sum_{\tau=0}^{\ell} \psi_\tau^n [n]_q^{\tau-\ell} S_q(\ell, \tau) x^\tau, \quad (29)$$

where $S_q(\ell, \tau)$ are q -Stirling polynomials of the second kind in (1) and (2).

From (2) and (12), the following vanishing identity is easily derived:

$$\sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q [\tau - \eta]_q^\ell = 0. \quad (30)$$

$$(\ell, \tau \in \mathbb{N}_0 \text{ with } \tau > \ell).$$

In particular,

$$\sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q = 0 \quad (\tau \in \mathbb{N}). \quad (31)$$

Goodman et al. [4] (see also ([9], p. 304)) assumed to define the vanishing identity (12), which is equivalent to (30). Indeed, (30) can be verified by induction on $\ell \in \mathbb{N}_0$. Carlitz ([1], p. 990) (see also Srivastava and Choi ([3], p. 505)) just noted that the left member of (30) under the condition vanishes. Mahmudov ([6], p. 1788) commented that (30) is obvious by setting a monomial in the q -difference formula ([6], Lemma 7) (see also ([7], p. 1054)).

Phillips [16] used $x_j = [j]_q$ ($j \in \mathbb{N}_0$) with $0 \leq j \leq n$ ($n \in \mathbb{N}$) and $q \in \mathbb{R}^+$. In this case, $x_j = [j]_q \in (1, \infty)$ if $j \geq 2$. Here we set $x_j = [j]_q / [n]_q$ so that $x_j \in [0, 1]$ ($j \in \mathbb{N}_0$) and summarize some slightly modified related formulas in the following theorem.

Theorem 1. Let $h : [0, 1] \rightarrow \mathbb{R}$. Furthermore, let $x_j = [j]_q / [n]_q$ for $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $q \in \mathbb{R}^+$ with $0 \leq j \leq n$. Then a relation between the Newton's divided difference $h[x_0, x_1, \dots, x_\tau]$ and the q -difference of order τ in (26) is given as follows:

$$h[x_j, x_{j+1}, \dots, x_{j+\tau}] = q^{-\frac{\tau(2j+\tau-1)}{2}} [n]_q^\tau \frac{\Delta_q^\tau h_j}{[\tau]_q!} \quad (32)$$

for $\tau \in \mathbb{N}_0$, where $h_j = h(x_j) = h([j]_q/[n]_q)$. Also

$$h[x_0, x_1, \dots, x_\tau] = \frac{h^{(\tau)}(\xi)}{\tau!} \quad (33)$$

for some $\xi \in (0, [\tau]_q/[n]_q)$, when $h \in C^\tau[0, 1]$. Hence

$$h[x_0, x_1, \dots, x_\tau] = \frac{[n]_q^\tau}{q^{\frac{\tau(\tau-1)}{2}} [\tau]_q!} \Delta_q^\tau h_0 = \frac{h^{(\tau)}(\xi)}{\tau!}, \quad (34)$$

for some $\xi \in (0, [\tau]_q/[n]_q)$, where $h \in C^\tau[0, 1]$.

It is noted that the explicit q -difference formula (27) remains the same under the conditions of Theorem 1. The proof is omitted. The interested reader may be referred (for example) to ([16], p. 268) and [73] (see also [74]).

If we choose a monomial $h(x)$, say, $h(x) = x^\ell$ ($\ell \in \mathbb{N}$) in (34) together with (2) and (27), we obtain

$$S_q(\ell, \tau) = \frac{[n]_q^{\ell-\tau}}{\tau!} h^{(\tau)}(\xi), \quad (35)$$

where the involved notations and conditions remain the same as above. From (35), obviously $S_q(\ell, \tau) = 0$ when $\tau > \ell$, which is equivalent to the vanishing identity (30).

It is noted that for a function h , the Newton's divided difference $h[x_j, x_{j+1}, \dots, x_{j+\tau}]$ with $\tau + 1$ distinct points and the q -difference $\Delta_q^\tau h$ of order τ , themselves, *vanish* when h is a monomial whose degree is less than τ . In this regard, we choose $h(x) = x^\ell$ with $\tau > \ell$ in (27) to obtain the following mild extension of the vanishing identity (30):

$$\sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \left[\begin{matrix} \tau \\ \eta \end{matrix} \right]_q [\tau + \mu - \eta]_q^\ell = 0 \quad (36)$$

$$(\ell, \tau, \mu \in \mathbb{N}_0 \text{ with } \tau > \ell).$$

We can prove (36) directly as follows: Let $\mathcal{L}(\tau + \mu, \ell)$ be the left member of (36). Noting $[\tau + \mu - \eta]_q = [\mu]_q + q[\tau - \eta]_q$ and using binomial theorem,

$$[\tau + \mu - \eta]_q^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} [\mu]_q^{\ell-k} q^{\mu k} [\tau - \eta]_q^k.$$

Then

$$\mathcal{L}(\tau + \mu, \ell) = \sum_{k=0}^{\ell} \binom{\ell}{k} [\mu]_q^{\ell-k} q^{\mu k} \sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{\eta(\eta-1)}{2}} \left[\begin{matrix} \tau \\ \eta \end{matrix} \right]_q [\tau - \eta]_q^k,$$

which, upon using (30), gives 0.

Agratini and Radu ([20], Equation (8)) proved that the numbers $\sigma_q(\ell, \tau)$ in (13) vanishes when $\tau > \ell$: Or, equivalently,

$$\sum_{\eta=0}^{\tau} (-1)^\eta q^{\frac{(\tau-\eta)(\tau-\eta-1)}{2}} \left[\begin{matrix} \tau \\ \eta \end{matrix} \right]_q \frac{[\tau - \eta]_q^\ell}{q^{(\tau-\eta)\ell}} = 0 \quad (37)$$

$$(\ell, \tau \in \mathbb{N}_0 \text{ with } \tau > \ell).$$

3. Certain Identities Involving the (p, q) -Binomial Coefficients

We recall the following (p, q) -analogue of the Euler's identity (24), which can be verified by induction (consult, e.g., ([73], Equation (1.5))) or the method in the proof of Theorem 2:

$$\prod_{v=0}^{n-1} (p^v + q^v x) = \sum_{s=0}^n p^{\frac{(n-s)(n-1)}{2}} q^{\frac{s(s-1)}{2}} \begin{bmatrix} n \\ s \end{bmatrix}_{p,q} x^s \quad (38)$$

$(p, q \in \mathbb{C} \text{ with } p \neq q; n \in \mathbb{N}).$

We may give the inverse of the (p, q) -binomial expansion (38), which is asserted in the following theorem.

Theorem 2. Let $p, q \in \mathbb{C}$ be such that $p \neq q$ and $|q/p| < 1$. Furthermore, let $n \in \mathbb{N}$. Then

$$\prod_{v=0}^{n-1} (p^v + q^v x)^{-1} = \sum_{s=0}^{\infty} (-1)^s p^{-\frac{(n-1)(n+2s)}{2}} \begin{bmatrix} n+s-1 \\ s \end{bmatrix}_{p,q} x^s. \quad (39)$$

Proof of Theorem 2. We write

$$\begin{aligned} \sum_{s=0}^{\infty} c_s x^s &= (1+x)^{-1} (p+qx)^{-1} \cdots (p^{n-1} + q^{n-1}x)^{-1} \\ &= p^{-\frac{n(n-1)}{2}} F_n(x), \end{aligned}$$

where

$$F_n(x) = (1+x)^{-1} \left(1 + \frac{q}{p}x\right)^{-1} \cdots \left(1 + \left(\frac{q}{p}\right)^{n-1}x\right)^{-1}.$$

Then we have

$$F_n(x) = p^{\frac{n(n-1)}{2}} \sum_{s=0}^{\infty} c_s x^s.$$

Replacing x by $\frac{q}{p}x$ in $F_n(x)$ gives

$$(1+x)F_n(x) = \left(1 + \left(\frac{q}{p}\right)^n x\right) F_n\left(\frac{q}{p}x\right).$$

We find from the last two identities that

$$\sum_{s=0}^{\infty} c_s x^s + \sum_{s=0}^{\infty} c_s x^{s+1} = \sum_{s=0}^{\infty} c_s \left(\frac{q}{p}\right)^s x^s + \sum_{s=0}^{\infty} c_s \left(\frac{q}{p}\right)^{n+s} x^{s+1},$$

which, upon equating the coefficients of x^s , yields

$$c_s + c_{s-1} = \left(\frac{q}{p}\right)^s c_s + \left(\frac{q}{p}\right)^{n+s-1} c_{s-1} \quad (s \in \mathbb{N}).$$

Or, equivalently,

$$c_s = -\frac{1 - \left(\frac{q}{p}\right)^{n+s-1}}{1 - \left(\frac{q}{p}\right)^s} c_{s-1} = -\frac{[n+s-1]_{q/p}}{[s]_{q/p}} c_{s-1} \quad (s \in \mathbb{N}).$$

From the last identity we derive

$$c_s = (-1)^s \begin{bmatrix} n+s-1 \\ s \end{bmatrix}_{q/p} c_0 \quad (s \in \mathbb{N}_0),$$

which, upon using (20) and $c_0 = p^{-\frac{n(n-1)}{2}}$, produces

$$c_s = (-1)^s p^{-\frac{(n-1)(n+2s)}{2}} \begin{bmatrix} n+s-1 \\ s \end{bmatrix}_{p,q} \quad (s \in \mathbb{N}_0).$$

This completes the proof. \square

If we multiply (38) and (39) side by side and equate the coefficients x^r of the resulting identity, we get a vanishing identity, which is given in the following corollary.

Corollary 1. Let $p, q \in \mathbb{C}$ be such that $p \neq q$ and $|q/p| < 1$. Furthermore, let $n, r \in \mathbb{N}$. Then

$$\sum_{s=0}^r (-1)^s p^{\frac{1}{2}\{(r-s)^2+r(1-2n)+s\}} q^{\frac{(r-s)(r-s-1)}{2}} \begin{bmatrix} n \\ r-s \end{bmatrix}_{p,q} \begin{bmatrix} n+s-1 \\ s \end{bmatrix}_{p,q} = 0. \quad (40)$$

We provide a (p, q) -analogue of the Chu-Vandermonde identity, which is asserted in the following theorem.

Theorem 3. Let $p, q \in \mathbb{C}$ be such that $p \neq q$. Furthermore, let $\ell, m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then

$$\begin{bmatrix} \ell+m \\ r \end{bmatrix}_{p,q} = \sum_{s=0}^r p^{s(s-r+m)} q^{(r-s)(\ell-s)} \begin{bmatrix} \ell \\ s \end{bmatrix}_{p,q} \begin{bmatrix} m \\ r-s \end{bmatrix}_{p,q}. \quad (41)$$

Proof of Theorem 3. We rewrite (38) to denote

$$\begin{aligned} E_n(x) &= (1+x) \left(1 + \frac{q}{p}x\right) \cdots \left(1 + \left(\frac{q}{p}\right)^{n-1}x\right) \\ &= p^{-\frac{n(n-1)}{2}} \sum_{v=0}^n p^{\frac{(n-v)(n-v-1)}{2}} q^{\frac{v(v-1)}{2}} \begin{bmatrix} n \\ v \end{bmatrix}_{p,q} x^v. \end{aligned} \quad (42)$$

Then we consider

$$\begin{aligned} E_{\ell+m}(x) &= (1+x) \left(1 + \frac{q}{p}x\right) \cdots \left(1 + \left(\frac{q}{p}\right)^{\ell-1}x\right) \\ &\quad \times \left(1 + \left(\frac{q}{p}\right)^{\ell}x\right) \cdots \left(1 + \left(\frac{q}{p}\right)^{\ell+m-1}x\right) \\ &= E_{\ell}(x) E_m \left(\left(\frac{q}{p}\right)^{\ell}x \right), \end{aligned}$$

which, upon using the right member of (42), yields

$$\begin{aligned} &p^{-\frac{(\ell+m)(\ell+m-1)}{2}} \sum_{r=0}^{\ell+m} p^{\frac{(\ell+m-r)(\ell+m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} \ell+m \\ r \end{bmatrix}_{p,q} x^r \\ &= p^{-\frac{\ell(\ell-1)}{2}} \sum_{s=0}^{\ell} p^{\frac{(\ell-s)(\ell-s-1)}{2}} q^{\frac{s(s-1)}{2}} \begin{bmatrix} \ell \\ s \end{bmatrix}_{p,q} x^s \\ &\quad \times p^{-\frac{m(m-1)}{2}} \sum_{t=0}^m p^{\frac{(m-t)(m-t-1)}{2}} q^{\frac{t(t-1)}{2}} \begin{bmatrix} m \\ t \end{bmatrix}_{p,q} \left(\left(\frac{q}{p}\right)^{\ell}x \right)^t. \end{aligned}$$

Equating the coefficients of x^r in the last identity yields (41). \square

Remark 1. Setting $p = 1$ in (41) gives a q -analogue of the Chu-Vandermonde identity (see, e.g., ([16], Equation (8.27))):

$$\begin{bmatrix} \ell + m \\ r \end{bmatrix}_q = \sum_{s=0}^r q^{(r-s)(\ell-s)} \begin{bmatrix} \ell \\ s \end{bmatrix}_q \begin{bmatrix} m \\ r-s \end{bmatrix}_q. \quad (43)$$

The q -identity (43) when $q = 1$ reduces to yield the Chu-Vandermonde identity:

$$\binom{\ell + m}{r} = \sum_{s=0}^r \binom{\ell}{s} \binom{m}{r-s}. \quad (44)$$

The Formula (41) is a generalization of the Pascal-type identity (19) because, when $r \geq 1$, choosing $\ell = 1$ and $m = n$ in (41) yields (19).

We may use the relation (20) to convert certain identities involving q -binomial coefficients into those associated with (p, q) -binomial coefficients. We illustrate two identities in the following theorem (cf., ([16], Problem 8.1.8)).

Theorem 4. Let $p, q \in \mathbb{C} \setminus \{0\}$ be such that $p \neq q$. Furthermore, put $n, s \in \mathbb{N}_0$ with $0 \leq s \leq n$. Then

$$\sum_{j=0}^s (-1)^j p^{\frac{j(j-2n-1)}{2}} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n+1 \\ j \end{bmatrix}_{p,q} = (-1)^s p^{\frac{s(s-2n-1)}{2}} q^{\frac{s(s+1)}{2}} \begin{bmatrix} n \\ s \end{bmatrix}_{p,q} \quad (45)$$

and

$$\sum_{j=0}^s (-1)^j p^{\frac{(j+1)(j-2n-2)}{2}} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n+1 \\ j \end{bmatrix}_{p,q} = (-1)^s p^{\frac{(s+1)(s-2n-2)}{2}} q^{\frac{s(s+1)}{2}} \begin{bmatrix} n \\ s \end{bmatrix}_{p,q}. \quad (46)$$

Proof of Theorem 4. One may use induction on n to verify these identities. The details are omitted. \square

4. (p, q) -Stirling Polynomials of the Second Kind Associated with (p, q) -Bernstein Polynomials

Wachs and White ([75], Equation (4)) introduced p, q -Stirling numbers of the second kind by using a recursive relation (see also ([76], Equation (1)), ([77], Theorem 2.1): For $\ell, \tau \in \mathbb{N}_0$,

$$S_{p,q}(\ell, \tau) = \begin{cases} p^{\tau-1} S_{p,q}(\ell-1, \tau-1) + [\tau]_{p,q} S_{p,q}(\ell-1, \tau) & (0 < \tau \leq \ell) \\ 1 & (\ell = \tau = 0) \\ 0 & (\text{otherwise}). \end{cases} \quad (47)$$

The particular case $p = 1$ of (47) gives (10). It is noted in [75] that the p, q -Stirling number in (47) is a generating function of two variables which produces the joint allocation of pairs of statistics. The definition (47) was recalled by Park ([78], p. 42) who noticed

$$S_q(\ell, \tau) = q^{-\binom{\tau}{2}} S_{q,q^2}(\ell, \tau).$$

Sagan ([37], Theorems 6.1 and 6.2) introduced another p, q -Stirling numbers of the second kind. Sagan ([39], Proposition 4.3) gave an inequality involving p, q -Stirling numbers of the second kind in (47). Remmel and Wachs [79] introduced generalized p, q -Stirling numbers $S_{\ell,\tau}^{i,j}(p, q)$ given by the recursive relation

$$S_{\ell+1,\tau}^{i,j}(p, q) = q^{i+(\tau-1)j} S_{\ell,\tau-1}^{i,j}(p, q) + p^{-j(\ell+1)} [i + \tau j]_{p,q} S_{\ell,\tau}^{i,j}(p, q), \quad (48)$$

with $S_{0,0}^{i,j}(p,q) = 1$ and $S_{\ell,\tau}^{i,j}(p,q) = 0$ when $\tau > \ell$ or $\tau < 0$ (see also [21,80]). For further properties and applications of (p,q) -Stirling numbers of the second kind, one may be referred (for instance) to [34,81–84].

Here and throughout, we assume that $0 < q < p \leq 1$. Mursaleen et al. [73] defined (p,q) -differences, recursively, as follows: For any function $h : [0, 1] \rightarrow \mathbb{R}$,

$$\Delta_{p,q}^0 h_j = h_j \quad (j \in \mathbb{N}_0), \quad (49)$$

$$\Delta_{p,q}^{v+1} h_j = p^v \Delta_{p,q}^v h_{j+1} - q^v \Delta_{p,q}^v h_j \quad (v = 0, 1, \dots, n-j-1), \quad (50)$$

where h_j equals $h\left(\frac{p^{n-j}[j]_{p,q}}{[n]_{p,q}}\right)$. They [73] derived the following explicit (p,q) -difference formula:

$$\Delta_{p,q}^v h_j = \sum_{\eta=0}^v (-1)^\eta p^{\frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} h_{j+v-\eta}. \quad (51)$$

They [73] established the following relation between the Newton's divided difference and the (p,q) -differences:

$$h[x_j, x_{j+1}, \dots, x_{j+v}] = \left(\frac{q}{p}\right)^{-\frac{v(2j+v-1)}{2}} [n]_{p,q}^v \frac{\Delta_{p,q}^v h_j}{[v]_{p,q}!}, \quad (52)$$

where $x_j = \frac{p^{n-j}[j]_{p,q}}{[n]_{p,q}}$.

Remark 2. We use the same reasoning in ([16], p. 268) together with (52) to get

$$\left(\frac{q}{p}\right)^{-\frac{v(v-1)}{2}} [n]_{p,q}^v \frac{\Delta_{p,q}^v h_0}{[v]_{p,q}!} = h[x_0, x_1, \dots, x_v] = \frac{h^{(v)}(\xi)}{v!} \quad (53)$$

for some $\xi \in (x_0, x_v)$, when $h \in C^v[0, 1]$. In particular, when $h(x)$ is a monomial x^ℓ whose degree ℓ is less than v , we find that

$$\Delta_{p,q}^v h_0 = h[x_0, x_1, \dots, x_v] = 0. \quad (54)$$

Here, in fact, when $v > \ell$ with $h(x) = x^\ell$, the (p,q) -difference $\Delta_{p,q}^v h_0$ and the Newton's divided difference $h[x_0, x_1, \dots, x_v]$, themselves, are seen to be zero. The identities and arguments here are easily found to reduce to yield those in Theorem 1.

Mursaleen et al. [85,86] (see also ([87], p. 2), ([73], p. 2), [88]) presented the (p,q) -analogue of Bernstein polynomials:

$$\mathcal{B}_{p,q}^n(h; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{\mu=0}^n h_\mu \begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} p^{\frac{\mu(\mu-1)}{2}} x^\mu \prod_{\eta=0}^{n-\mu-1} (p^\eta - q^\eta x). \quad (55)$$

They [73] expressed the generalized Bernstein polynomial (55) in terms of the (p,q) -differences in (50):

$$\mathcal{B}_{p,q}^n(h; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{\mu=0}^n \begin{bmatrix} n \\ \mu \end{bmatrix}_{p,q} p^{\frac{(n-\mu)(n-\mu-1)}{2}} \Delta_{p,q}^\mu h_0 x^\mu. \quad (56)$$

For more formulas and theories regarding (p,q) -calculus, one can be referred, for instance, to [42,73,85–93].

As in (29), in order to expand the (p,q) -analogue of Bernstein polynomials $\mathcal{B}_{p,q}^n(x^\ell; x)$ in powers of x , in Definition 1, we introduce new (p,q) -Stirling polynomials of the second kind, denoted by $S_{p,q}(\ell, v)$, which are different from those, including (47) and (48), in the reviewed literature.

Definition 1. Let $p, q \in \mathbb{R}$ be such that $1 \geq p > q > 0$. Furthermore, set $\ell, v \in \mathbb{N}_0$. Then we define (p, q) -Stirling polynomials of the second kind, denoted by $S_{p,q}(\ell, v)$, by

$$S_{p,q}(\ell, v) := \frac{1}{[v]_{p,q}!} \frac{p^{\frac{v(v+1)}{2}}}{q^{\frac{v(v-1)}{2}}} \times \sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} p^{\ell(\eta-v) + \frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} [v-\eta]_{p,q}^\ell. \quad (57)$$

Remark 3. Let $p, q \in \mathbb{R}$ be such that $1 \geq p > q > 0$. Furthermore, set $\ell, v \in \mathbb{N}_0$. Then

- (i) $S_{1,q}(\ell, v) = S_q(\ell, v)$ in (2).
- (ii) $S_{p,q}(0, 0) = 1, S_{p,q}(\ell, 0) = 0$ ($\ell \in \mathbb{N}$).
- (iii) $S_{p,q}(\ell, v) = 0$ ($v > \ell$).

We use (51) in (56) to express (p, q) -analogue of Bernstein polynomials (55) in the following double series like the form of the q -analogue of Bernstein polynomials which is obtained by substituting (27) for $\Delta^v h_0$ in (25) (see also ([72], Proposition 1)):

$$\mathcal{B}_{p,q}^n(h; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{v=0}^n \begin{bmatrix} n \\ v \end{bmatrix}_{p,q} p^{\frac{(n-v)(n-v-1)}{2}} x^v \times \sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} p^{\frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} h_{v-\eta}. \quad (58)$$

Like (29), we use (p, q) -Stirling polynomials of the second kind (57) to expand the (p, q) -analogue of Bernstein polynomials $\mathcal{B}_{p,q}^n(x^\ell; x)$ in powers of x , which is asserted in Theorem 5.

Theorem 5. Set $x \in [0, 1]$, $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Furthermore, let $p, q \in \mathbb{R}$ be such that $1 \geq p > q > 0$. Then

$$\mathcal{B}_{p,q}^n(x^\ell; x) = \sum_{v=0}^n \Phi_v^n \left(\frac{[n]_{p,q}}{p^n} \right)^{v-\ell} S_{p,q}(\ell, v) x^v, \quad (59)$$

where

$$\Phi_v^n := \prod_{k=0}^{v-1} \left(1 - \frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} \right).$$

Proof of Theorem 5. Choosing $h(x) = x^\ell$ ($\ell \in \mathbb{N}_0$) in (58), we get

$$\mathcal{B}_{p,q}^n(x^\ell; x) = \sum_{v=0}^n p^{\frac{v(v-2n+1)}{2}} [n]_{p,q}^{-\ell} x^v \begin{bmatrix} n \\ v \end{bmatrix}_{p,q} \times \sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} [v-\eta]_{p,q}^\ell p^{\ell(n-v+\eta) + \frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}}. \quad (60)$$

It is not hard to show that the (p, q) -binomial coefficients $\begin{bmatrix} n \\ v \end{bmatrix}_{p,q}$ can be expressed as follows:

$$\begin{bmatrix} n \\ v \end{bmatrix}_{p,q} = \frac{[n]_{p,q}^v}{[v]_{p,q}! q^{\frac{v(v-1)}{2}}} \Phi_v^n \quad (0 \leq v \leq n). \quad (61)$$

Finally, using (57) and (61) in (60), we obtain (59). \square

Like (30), we give some vanishing identities involving (p, q) -Stirling polynomials of the second kind (57), which are asserted in Theorems 6 and 7.

Theorem 6. Take $p, q \in \mathbb{R}$ be such that $1 \geq p > q > 0$. Furthermore, let $v, \ell \in \mathbb{N}_0$. Then

$$S_{p,q}(\ell, v) = 0 \quad (v > \ell). \quad (62)$$

Or, equivalently,

$$\sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} p^{\ell(\eta-v) + \frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} [v-\eta]_{p,q}^\ell = 0 \quad (63)$$

$$(v > \ell).$$

Proof of Theorem 6. Setting $h(x) = x^\ell$ and $j = 0$ in (51), whose resulting identity is compared with (57), gives

$$S_{p,q}(\ell, v) = \left(\frac{[n]_{p,q}}{p^n} \right)^{v-\ell} \frac{1}{[v]_{p,q}!} \frac{p^{\frac{v(v+1)}{2}}}{q^{\frac{v(v-1)}{2}}} \Delta_{p,q}^v h_0, \quad (64)$$

which, in view of (54), vanishes when $v > \ell$. Also, in view of (57), the statements (62) and (63) are immediately seen to be equivalent. \square

Theorem 7. Take $p, q \in \mathbb{R}$ be such that $1 \geq p > q > 0$. Furthermore, let $v, k \in \mathbb{N}_0$. Then

$$\sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} p^{\frac{(v-\eta-k)(v-\eta-k-1)}{2}} q^{\frac{(\eta-k)(\eta-k-1)}{2}} = 0 \quad (v \geq k+1). \quad (65)$$

In particular,

$$\sum_{\eta=0}^v (-1)^\eta \begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} p^{\frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} = 0 \quad (v \geq 1). \quad (66)$$

We also have

$$\begin{aligned} & \sum_{\eta=0}^{v-k} (-1)^\eta \begin{bmatrix} v-k \\ \eta \end{bmatrix}_{p,q} p^{k(\eta-v) + \frac{(v-\eta)(v-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} \\ &= p^{-\frac{k(k+1)}{2}} \sum_{\eta=0}^{v-k} (-1)^\eta \begin{bmatrix} v-k \\ \eta \end{bmatrix}_{p,q} p^{\frac{(v-\eta-k)(v-\eta-k-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} = 0 \quad (v \geq k+1). \end{aligned} \quad (67)$$

Proof of Theorem 7. We begin by noting that the vanishing identity (66) follows immediately from either (65) or (63) when, respectively, $k = 0$ and $\ell = 0$.

Let $\mathcal{L}_{p,q}(v; k)$ be the left-hand member of (65). We will use induction on k to prove (65). The case $\mathcal{L}_{p,q}(v; 0) = 0$ follows from (63) when $\ell = 0$.

We assume that $\mathcal{L}_{p,q}(v; k) = 0$ ($v \geq k+1$) for some $k \in \mathbb{N}_0$. Then, using the Pascal-type identity (19)

$$\begin{bmatrix} v \\ \eta \end{bmatrix}_{p,q} = p^{v-\eta} \begin{bmatrix} v-1 \\ \eta-1 \end{bmatrix}_{p,q} + q^\eta \begin{bmatrix} v-1 \\ \eta \end{bmatrix}_{p,q}, \quad (68)$$

we find that, for $\nu \geq k + 2$,

$$\begin{aligned}\mathcal{L}_{p,q}(\nu; k+1) &= \sum_{\eta=1}^{\nu} (-1)^{\eta} p^{\nu-\eta} \begin{bmatrix} \nu-1 \\ \eta-1 \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k-1)(\nu-\eta-k-2)}{2}} q^{\frac{(\eta-k-1)(\eta-k-2)}{2}} \\ &\quad + \sum_{\eta=0}^{\nu-1} (-1)^{\eta} q^{\eta} \begin{bmatrix} \nu-1 \\ \eta \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k-1)(\nu-\eta-k-2)}{2}} q^{\frac{(\eta-k-1)(\eta-k-2)}{2}} \\ &= p^{k+1} \sum_{\eta=1}^{\nu} (-1)^{\eta} \begin{bmatrix} \nu-1 \\ \eta-1 \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k-1)(\nu-\eta-k)}{2}} q^{\frac{(\eta-k-1)(\eta-k-2)}{2}} \\ &\quad + q^{k+1} \sum_{\eta=0}^{\nu-1} (-1)^{\eta} \begin{bmatrix} \nu-1 \\ \eta \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k-1)(\nu-\eta-k-2)}{2}} q^{\frac{(\eta-k-1)(\eta-k)}{2}}.\end{aligned}$$

Putting $\eta - 1 = \eta'$ in the first summation of the last equality, we obtain

$$\begin{aligned}\mathcal{L}_{p,q}(\nu; k+1) &= (q^{k+1} - p^{k+1}) \sum_{\eta=0}^{\nu-1} (-1)^{\eta} \begin{bmatrix} \nu-1 \\ \eta \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k-1)(\nu-\eta-k-2)}{2}} q^{\frac{(\eta-k)(\eta-k-1)}{2}},\end{aligned}$$

which, being $\nu - 1 \geq k + 1$ and using the induction hypothesis, is evaluated to be 0.

Let $\mathcal{D}_{p,q}(\nu, k)$ be the left-hand member of (67). Then

$$\begin{aligned}\mathcal{D}_{p,q}(\nu, k) &= \frac{1}{p} \sum_{\eta=0}^{\nu-k} (-1)^{\eta} \begin{bmatrix} \nu-k \\ \eta \end{bmatrix}_{p,q} p^{(k-1)(\eta-\nu)+\frac{(\nu-\eta-1)(\nu-\eta-2)}{2}} q^{\frac{\eta(\eta-1)}{2}} \\ &= \frac{1}{p \cdot p^2} \sum_{\eta=0}^{\nu-k} (-1)^{\eta} \begin{bmatrix} \nu-k \\ \eta \end{bmatrix}_{p,q} p^{(k-2)(\eta-\nu)+\frac{(\nu-\eta-2)(\nu-\eta-3)}{2}} q^{\frac{\eta(\eta-1)}{2}} \\ &\quad \vdots \\ &= \frac{1}{p \cdot p^2 \cdots p^k} \sum_{\eta=0}^{\nu-k} (-1)^{\eta} \begin{bmatrix} \nu-k \\ \eta \end{bmatrix}_{p,q} p^{\frac{(\nu-\eta-k)(\nu-\eta-k-1)}{2}} q^{\frac{\eta(\eta-1)}{2}},\end{aligned}$$

which, being $\nu - k \geq 1$ and using (66), is evaluated to be 0. \square

We find from (59) with the aid of the identities in Theorem 6 that (consult, e.g., ([87], Lemma 1.1); see also [86] and ([73], Equation (2.13)); cf., Equation (1.4) and ([93], Lemma 2))

$$\begin{aligned}\mathcal{B}_{p,q}^n(1; x) &= 1, \quad \mathcal{B}_{p,q}^n(x; x) = x, \\ \mathcal{B}_{p,q}^n(x^2; x) &= \frac{p^{n-1}}{[n]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2 \\ &= \frac{p^{n-1}}{[n]_{p,q}} x(1-x) + x^2,\end{aligned}$$

where the second equality can be given by the following identity (see ([73], Equation (2.12)))

$$[n]_{p,q} = p^{n-1} + q[n-1]_{p,q}. \quad (69)$$

Indeed, we also find from (59) and (62) that $\mathcal{B}_{p,q}^n(x^{\ell}; x)$ is a polynomial in the variable x whose degree is $\min\{\ell, n\}$.

5. Further Remark and an Open Question

Recall the recursive relation (10)

$$S_q(\ell + 1, \tau) = S_q(\ell, \tau - 1) + [\tau]_q S_q(\ell, \tau) \quad (\ell \in \mathbb{N}_0, \tau \in \mathbb{N}). \quad (70)$$

It is noted in ([4], p. 181, Equation (2.6)) that (10) or (70) may be verified by induction on ℓ . Here we prove (70) directly. Using the following identity (cf., ([20], Equation (9)))

$$[\tau - \eta]_q = [\tau]_q - q^{\tau-\eta} [\eta]_q, \quad (71)$$

it follows from (2) that

$$S_q(\ell + 1, \tau) = [\tau]_q S_q(\ell, \tau) + \mathcal{R}_q(\ell, \tau), \quad (72)$$

where

$$\mathcal{R}_q(\ell, \tau) = \frac{1}{[\tau]_q! q^{\frac{\tau(\tau-1)}{2}}} \sum_{\eta=1}^{\tau} (-1)^{\eta+1} q^{\frac{\eta(\eta-1)}{2}} q^{\tau-\eta} [\eta]_q \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q [\tau - \eta]_q^{\ell}.$$

Noting

$$[\eta]_q \begin{bmatrix} \tau \\ \eta \end{bmatrix}_q = [\tau]_q \begin{bmatrix} \tau - 1 \\ \eta - 1 \end{bmatrix}_q$$

and writing $q^{\tau-\eta} = q^{\tau-1} q^{1-\eta}$, we obtain

$$\mathcal{R}_q(\ell, \tau) = \frac{1}{[\tau - 1]_q! q^{\frac{(\tau-1)(\tau-2)}{2}}} \sum_{\eta=1}^{\tau} (-1)^{\eta+1} q^{\frac{(\eta-1)(\eta-2)}{2}} \begin{bmatrix} \tau - 1 \\ \eta - 1 \end{bmatrix}_q [\tau - \eta]_q^{\ell}.$$

Here, setting $\eta - 1 = \eta'$ and removing the prime on η , we find from (2) that

$$\mathcal{R}_q(\ell, \tau) = S_q(\ell, \tau - 1). \quad (73)$$

Finally, putting (73) in (72) proves (70).

Open Question

Find a recursive relation for the (p, q) -Stirling polynomials of the second kind $S_{p,q}(\ell, \nu)$ in (57) like the (47).

Here we use the method of proof of (70) above to try to give a recursive relation for the $S_{p,q}(\ell, \nu)$ in (57): We first give the following two identities:

$$[\nu - \eta]_{p,q} = \frac{[\nu]_{p,q}}{q^{\eta}} - \frac{p^{\nu-\eta}}{q^{\eta}} [\eta]_{p,q} \quad (74)$$

and

$$[\eta]_{p,q} \begin{bmatrix} \nu \\ \eta \end{bmatrix}_{p,q} = [\nu]_{p,q} \begin{bmatrix} \nu - 1 \\ \eta - 1 \end{bmatrix}_{p,q}. \quad (75)$$

Using first the relation (74) in (57) and then (75), we get

$$S_{p,q}(\ell + 1, \nu) = \mathcal{H}_{p,q}^{(1)}(\nu, \ell) + \mathcal{H}_{p,q}^{(2)}(\nu, \ell), \quad (76)$$

where

$$\begin{aligned} \mathcal{H}_{p,q}^{(1)}(\nu, \ell) &= \frac{1}{[\nu]_{p,q}!} p^{\frac{\nu(\nu+1)}{2}-1} q^{\frac{\nu(\nu-1)}{2}+1} \\ &\times [\nu]_{p,q} \sum_{\eta=0}^{\nu} (-1)^{\eta} \begin{bmatrix} \nu \\ \eta \end{bmatrix}_{p,q} p^{\ell(\eta-\nu)+\frac{(\nu-\eta-1)(\nu-\eta-2)}{2}} q^{\frac{(\eta-1)(\eta-2)}{2}} [\nu - \eta]_{p,q}^{\ell} \end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_{p,q}^{(2)}(v, \ell) &= \frac{1}{[v-1]_{p,q}!} \frac{p^{\frac{v(v+1)}{2}}}{q^{\frac{v(v-1)}{2}+1}} \\ &\quad \times \sum_{\eta=0}^{v-1} (-1)^{\eta} \begin{bmatrix} v-1 \\ \eta \end{bmatrix}_{p,q} p^{\ell(\eta+1-v) + \frac{(v-1-\eta)(v-1-\eta-1)}{2}} q^{\frac{\eta(\eta-1)}{2}} [v-1-\eta]_{p,q}^{\ell} \\ &= \left(\frac{p}{q}\right)^v s_{p,q}(\ell, v-1).\end{aligned}$$

6. Conclusions

In Section 3, some interesting and new identities involving the (p, q) -binomial coefficients including, in particular, the (p, q) -analogue of the Chu–Vandermonde identity (41), are presented. Using the same technique in Section 3 together with (20) and (21), some known identities associated with the q -binomial coefficients (if any) are believed to yield the corresponding identities involving the (p, q) -binomial coefficients.

In Section 4, the new (p, q) -Stirling polynomials of the second kind are introduced and shown to be fit for the (p, q) -analogue of Bernstein polynomials. A recursive relation for these new (p, q) -Stirling polynomials of the second kind remains to be an open question. More properties and applications of these new (p, q) -Stirling polynomials of the second kind are left to the authors and the interested researchers for future study.

Certain vanishing identities associated with the q - and (p, q) -Stirling polynomials of the second kind are also discussed.

Author Contributions: The authors have equally contributed to accomplish this research work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R111A1A01052440).

Acknowledgments: The authors are grateful to the anonymous referees for the favorable and valuable comments which improved this paper.

Conflicts of Interest: The authors have no conflict of interest.

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