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A Definition of Two-Dimensional Schoenberg Type Operators

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Abstract: In this paper, a way to build two-dimensional Schoenberg type operators with arbitrary knots or with equidistant knots, respectively, is presented. The order of approximation reached by these operators was studied by obtaining a Voronovskaja type asymptotic theorem and using estimates in terms of second-order moduli of continuity.

Keywords: two-dimensional splines; two-dimensional Schoenberg type operators; order of approximation; spline surfaces

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1. Introduction

The Schoenberg operators provide a concrete method for obtaining spline approximations of functions. These operators have very good approximation properties. However, they are not very present in the literature. The study of the positive linear Schoenberg operators was the subject of several recent papers, among which we mention here Beutel, Gonska, Kacso and Tachev [1], Tachev [2,3], and Tachev and Zapryanova [4]. In [1] variation-diminishing one-dimensional Schoenberg spline operators, especially with equidistant knots and inequalities in terms of moduli of continuity, were studied. An analysis of the second moment of one-dimensional Schoenberg spline operators moment was presented. A discussion about the degree of simultaneous approximation for multivariate case, more specifically for Boolean sums, was realized. Additionally, a similar discussion was presented for tensor products of one-dimensional Schoenberg spline operators. In [2] a lower bound for the second moment of one-dimensional Schoenberg spline operators is made. In [3] a Voronovskaja's type theorem for one-dimensional Schoenberg spline operators is presented. In [4] a generalized inverse theorem for one-dimensional Schoenberg spline operators is established.

In practice, the use of the one-dimensional Schoenberg operators offers many advantages. This fact was illustrated, for example, in the recent paper [5], in which these operators were applied for improving the clear sky models used to estimate the direct solar irradiance, with influences of the system design and financial benefits.

The subject of multivariate splines was approached by different methods and from various points of view, such as in the papers written by: Curry and Schoenberg [6], Goodman and Lee [7], de Boor and Hollig [8], Karlin et al. [9], Goodman [10], Chui [11], Schumaker [12], Conti and Morandi [13], Ugarte et al. [14], and Groselj and Knez [15]. Curry and Schoenberg indicated in [6] that the multivariate spline functions can be constructed from volumes of slices of polyhedra; therefore, papers can be found that were written toward that direction. For example, this idea led to the recurrence relations for multivariate splines presented by Karlin et al. in [9]. Goodman and Lee in [7] and Goodman in [10] approached the subject of multidimensional Bernstein–Schoenberg operators depending on m-dimensional volume. In [8] the subject of multidimensional B-splines is treated by de Boor and

Hollig as the m-shadow of the polyhedral convex body included in $\mathbb{R}^n$. In [11] the Box splines, multivariate truncated powers and many other aspects of multivariate splines are studied by Chui. Conti and Morandi used mixed splines to solve the interproximation problem for surfaces in the case of scattered data in [13]. The aim of Ugarte et al. in the paper [14] was to propose different possibilities of modeling the space–time interaction using one dimensional, two-dimensional and three dimensional B-splines. In [15] Groselj and Knez introduced the notion of a balanced 10-split for the construction of non-negative basis functions for the space of C^1 quadratic splines and showed that the considered split has potential to be used for the construction of C^2 splines.

The aim of our paper is to consider a new approach in spline approximation in the two-dimensional case, based on a two-dimensional version of Schoenberg operators. The two-dimensional Schoenberg type operators are constructed by generalization of the one-dimensional Schoenberg operators. As a result of this generalization we reach a particular form of tensor-product B-splines. The subject of the tensor-product B-splines is treated in several papers, for example in [12].

2. Two-Dimensional Schoenberg Type Operators on Arbitrary Nodes

We define two-dimensional Schoenberg type operators as follows.

Let us consider the knot sequence $\Delta_{n,h}$

$$0 = \alpha_{-h} = \alpha_{-h+1} = \dots = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n = \alpha_{n+1} = \dots = \alpha_{n+h} = 1, \tag{1}$$

where $n > 0,\ h > 0$ and the knot sequence $\Delta_{m,k}$

$$0 = \beta_{-k} = \beta_{-k+1} = \dots = \beta_0 < \beta_1 < \beta_2 < \dots < \beta_m = \beta_{m+1} = \dots = \beta_{m+k} = 1, \tag{2}$$

where $m > 0,\ k > 0$.

The Greville abscissas associated with knot sequence $\Delta_{n,h}$ are

$$\xi_{i,h} := \frac{\alpha_{i+1} + \alpha_{i+2} + \dots + \alpha_{i+h}}{h}, \quad \text{with } -h \le i \le n-1, \tag{3}$$

and the Greville abscissas associated with knot sequence $\Delta_{m,k}$ are

$$\zeta_{j,k} := \frac{\beta_{j+1} + \beta_{j+2} + \dots + \beta_{j+k}}{k}, \quad \text{with } -k \le j \le m-1. \tag{4}$$

The B-splines $N_{i,h}(\alpha)$ depending on $\Delta_{n,h}$ are defined in the following mode:

$$N_{i,h}(\alpha) = (\alpha_{i+h+1} - \alpha_i)[\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+h+1}](\cdot - \alpha)_+^h, \ -h \le i \le n-1, \tag{5}$$

and the B-splines $N_{j,k}(\beta)$ depending on $\Delta_{m,k}$, by:

$$N_{j,k}(\beta) = (\beta_{j+k+1} - \beta_j)[\beta_j, \beta_{j+1}, \dots, \beta_{j+k+1}](\cdot - \beta)_+^k, \ -k \le j \le m-1. \tag{6}$$

Remark 1. *If $\alpha \in [\alpha_q, \alpha_{q+1}]$ and $\beta \in [\beta_r, \beta_{r+1}]$ with $0 \le q \le n-1$, $0 \le r \le m-1$, then*

$$N_{i,h}(\alpha) = 0, \text{ for } i < q-h \text{ or } i \ge q+1, \text{ and } N_{i,h}(\alpha) \ge 0, \text{ for } q-h \le i \le q,$$

and

$$N_{j,k}(\beta) = 0, \text{ for } j < r-k \text{ or } j \ge r+1, \text{ and } N_{j,k}(\alpha) \ge 0, \text{ for } r-k \le j \le r.$$

The following relations are well known:

$$\sum_{i=-h}^{n-1} N_{i,h}(\alpha) = 1, \quad \sum_{i=-h}^{n-1} \xi_{i,h} N_{i,h}(\alpha) = \alpha, \text{ for } \alpha \in [0,1]. \tag{7}$$

Analogous relations are fulfilled for $N_{j,k}$.

This notation—$\Delta_1 = \Delta_{n,h}$, $\Delta_2 = \Delta_{m,k}$, $\tilde{\Delta} = \Delta_1 \times \Delta_2$—is used; i.e.,

$$\tilde{\Delta} := \tilde{\Delta}_{n,m}^{h,k} = \{(\alpha_i, \beta_j), -h \le i \le n+h, -k \le j \le m+k\}. \tag{8}$$

Definition 1. *Two-dimensional Schoenberg type operator associated with $\tilde{\Delta}$ has the form*

$$\left(S_{\tilde{\Delta}}\varphi\right)(\alpha,\beta) = \sum_{i=-h}^{n-1}\sum_{j=-k}^{m-1} N_{i,h}(\alpha) N_{j,k}(\beta) \varphi(\xi_{i,h}, \zeta_{j,k}), \tag{9}$$

where $\varphi : [0,1] \times [0,1] \to \mathbb{R}$, $(\alpha,\beta) \in [0,1]^2$.

Remark 2. *By taking into account Remark 1 it follows that if $\alpha \in [\alpha_q, \alpha_{q+1}]$, with $0 \le q \le n-1$ and $\beta \in [\beta_r, \beta_{r+1}]$ with $0 \le r \le m-1$, then*

$$\left(S_{\tilde{\Delta}}\varphi\right)(\alpha,\beta) = \sum_{i=q-h}^{q}\sum_{j=r-k}^{r} N_{i,h}(\alpha) N_{j,k}(\beta) \varphi(\xi_{i,h}, \zeta_{j,k}).$$

Remark 3. *Two-dimensional Schoenberg type operators are linear and positive like one-dimensional Schoenberg operators. This follows immediately from the linearity and from the positivity of one-dimensional Schoenberg operators. Additionally, obviously, $S_{\tilde{\Delta}}$ is a polynomial of degree at most h in the variable α and with degree at most k in variable β, on each rectangle $[\alpha_{i-1}, \alpha_i] \times [\beta_{j-1}, \beta_j]$, with $0 \le i \le n-1$ and $0 \le j \le m-1$. Moreover $S_{\tilde{\Delta}}$ is a B-spline in each variable.*

Two-dimensional Schoenberg type operators admit partial continuous derivatives on $[0,1] \times [0,1]$, since

$$\left(\frac{\partial^{i+j}}{\partial\alpha^i \partial\beta^j}(S_{\tilde{\Delta}}\varphi)\right)(\alpha,\beta) = \sum_{i=-h}^{n-1} \frac{\partial^i}{\partial\alpha^i} N_{i,h}(\alpha) \sum_{j=-k}^{m-1} \frac{\partial^j}{\partial\beta^j} N_{j,k}(\beta) \varphi(\xi_{i,h}, \zeta_{j,k}), \; i,j \ge 0.$$

Further, the functions $e_0, \pi_1, \pi_2 \in C([0,1]^2)$, defined by $e_0(\alpha,\beta) = 1$, $\pi_1(\alpha,\beta) = \alpha$, $\pi_2(\alpha,\beta) = \beta$, for $(\alpha,\beta) \in [0,1]^2$ are considered. The following propositions result directly from Definition 1 and relations (7).

Proposition 1. *For $(\alpha,\beta) \in [0,1]^2$ we have*

(i) $(S_{\tilde{\Delta}} e_0)(\alpha,\beta) = 1$;
(ii) $(S_{\tilde{\Delta}} \pi_1)(\alpha,\beta) = \alpha$; $(S_{\tilde{\Delta}} \pi_2)(\alpha,\beta) = \beta$;
(iii) $S_{\tilde{\Delta}}(\pi_1 \cdot \pi_2)(\alpha,\beta) = \alpha\beta$.

In the next proposition the notation $e_1(t) = t$, $t \in [0,1]$ is used, and e_0 denotes the constant function equal to 1, on the both sets $[0,1]$ and $[0,1]^2$.

Proposition 2. *For $(\alpha,\beta) \in [0,1]^2$ we have*

(i) $(S_{\tilde{\Delta}}(\pi_1 - \alpha e_0))(\alpha,\beta) = 0$ $(S_{\tilde{\Delta}}(\pi_2 - \beta e_0))(\alpha,\beta) = 0$;
(ii) $(S_{\tilde{\Delta}}(\pi_1 - \alpha e_0)(\pi_2 - \beta e_0))(\alpha,\beta) = 0$;
(iii) $(S_{\tilde{\Delta}}(\pi_1 - \alpha e_0)^2)(\alpha,\beta) = (S_{\Delta_1}(e_1 - \alpha e_0)^2)(\alpha)$;
(iv) $(S_{\tilde{\Delta}}(\pi_2 - \beta e_0)^2)(\alpha,\beta) = (S_{\Delta_2}(e_1 - \beta e_0)^2)(\beta)$.

Theorem 1. *For the two-dimensional Schoenberg type operators*

$$(S_{\tilde{\Delta}}\varphi)(\alpha,\beta) = \sum_{i=-h}^{n-1}\sum_{j=-k}^{m-1} N_{i,h}(\alpha)N_{j,k}(\beta)\varphi(\xi_{i,h},\zeta_{j,k})$$

to converge uniformly on $[0,1]\times[0,1]$ *to continuous function* φ *it is sufficient that for any* $\eta > 0$

$$\sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|<\eta} N_{i,h}(\alpha)N_{j,k}(\beta) \to 1, \tag{10}$$

uniformly for $0 \le \alpha \le 1$, $0 \le \beta \le 1$ *when* $n \to \infty$ *and* $m \to \infty$.

Proof. We assume that condition (10) is fulfilled. Since φ is continuous on $[0,1]\times[0,1]$, for $\forall\varepsilon > 0$, $\exists\eta_\varepsilon > 0$ such that for any $(\alpha_1,\beta_1) \in [0,1]\times[0,1]$ and $(\alpha_2,\beta_2) \in [0,1]\times[0,1]$ with $\|(\alpha_2,\beta_2) - (\alpha_1,\beta_1)\| < \eta_\varepsilon$ one has $|\varphi(\alpha_2,\beta_2) - \varphi(\alpha_1,\beta_1)| < \frac{\varepsilon}{2}$. Additionally $\exists M > 0$ such that $|\varphi(\alpha,\beta)| \le M$, $(\alpha,\beta) \in [0,1]^2$.

Let $n_\varepsilon,\ m_\varepsilon \in \mathbb{N}$ such that:

$$0 < 1 - \sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|<\eta_\varepsilon} N_{i,h}(\alpha)N_{j,k}(\beta) < \frac{\varepsilon}{4M},\ \text{for } n \ge n_\varepsilon,\ m \ge m_\varepsilon.$$

Then, for such n and m, we have

$$\begin{aligned}
& |(S_{\tilde{\Delta}}\varphi)(\alpha,\beta) - \varphi(\alpha,\beta)| \\
= \; & \Big|\sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|<\eta_\varepsilon} (\varphi(\xi_{i,h},\zeta_{j,k}) - \varphi(\alpha,\beta)) \\
& + \sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|\ge\eta_\varepsilon} (\varphi(\xi_{i,h},\zeta_{j,k}) - \varphi(\alpha,\beta))\Big| \\
\le \; & \frac{\varepsilon}{2}\sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|<\eta_\varepsilon} + 2M\sum\sum\nolimits_{\|(\xi_{i,h},\zeta_{j,k})-(\alpha,\beta)\|\ge\eta_\varepsilon} \\
< \; & \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon.
\end{aligned}$$

□

The norm of the knot sequence $\tilde{\Delta}$ is given by

$$\|\tilde{\Delta}\| := \|\Delta_1\| + \|\Delta_2\|, \tag{11}$$

where $\|\Delta_1\| = max_i(\alpha_{i+1} - \alpha_i)$ and $\|\Delta_2\| = max_i(\beta_{i+1} - \beta_i)$.

A quantitative version of the degree of approximation can be given using the first order modulus of continuity, defined as follows:

$$\begin{aligned}
\omega_1(\varphi,\rho) \;\; := \;\; & \sup\{|\varphi(u_1,u_2) - \varphi(v_1,v_2)|,\ (u_1,u_2),(v_1,v_2) \in [0,1]^2, \\
& \|(u_1 - v_1, u_2 - v_2)\| \le \rho\},
\end{aligned} \tag{12}$$

where $\varphi \in C([0,1]^2)$, $\rho > 0$.

Theorem 2. *For any* $\varphi \in C([0,1]^2)$, *operators* $S_{\tilde{\Delta}}$ *given in (9) satisfy inequality*

$$\|(S_{\tilde{\Delta}}\varphi) - \varphi\| \le \ell\omega_1(\varphi,\|\tilde{\Delta}\|), \tag{13}$$

where $\ell = \frac{1}{2}\max\{h+1,k+1\}$.

Proof. Let the continuous function φ and let $(\alpha,\beta) \in [0,1]^2$. There exist $q \in \{0,1,...,n-1\}$ and $r \in \{0,1,...,m-1\}$ so that $(\alpha,\beta) \in [\alpha_q,\alpha_{q+1}] \times [\beta_r,\beta_{r+1}]$.

Let $q-h \leq i \leq q$ and $r-k \leq j \leq r$. Then

$$\begin{aligned} \alpha - \xi_{i,h} &\leq \alpha_{q+1} - \frac{\alpha_{i+1} + \ldots + \alpha_{i+h}}{h} \leq \alpha_{q+1} - \frac{\alpha_{q-h+1} + \ldots + \alpha_q}{h} \\ &\leq \frac{1}{h}(h + (h-1) + \ldots + 1)\|\Delta_1\| = \frac{h+1}{2}\|\Delta_1\|, \end{aligned}$$

and

$$\begin{aligned} \alpha - \xi_{i,h} &\geq \alpha_q - \frac{\alpha_{i+1} + \ldots + \alpha_{i+h}}{h} \geq \alpha_q - \frac{\alpha_{q+1} + \ldots + \alpha_{q+h}}{h} \\ &\geq -\frac{1}{h}(1 + 2 + \ldots + h)\|\Delta_1\| = -\frac{h+1}{2}\|\Delta_1\|. \end{aligned}$$

Therefore $|\alpha - \xi_{i,h}| \leq \frac{h+1}{2}\|\Delta_1\|$. Similarly $|\beta - \zeta_{j,k}| \leq \frac{k+1}{2}\|\Delta_2\|$. Then

$$\|(\alpha,\beta) - (\xi_{i,h}, \zeta_{j,k})\| \leq |\alpha - \xi_{i,h}| + |\beta - \zeta_{j,k}| \leq \frac{h+1}{2}\|\Delta_1\| + \frac{k+1}{2}\|\Delta_2\| \leq \ell\|\tilde{\Delta}\|.$$

From Remark 2 it results

$$\begin{aligned} |(S_{\tilde{\Delta}}\varphi)(\alpha,\beta) - \varphi(\alpha,\beta)| &= \left| \sum_{i=q-h}^{q} \sum_{j=r-k}^{r} N_{i,h}(\alpha) N_{j,k}(\beta) \varphi(\xi_{i,h}, \zeta_{j,k}) - \varphi(\alpha,\beta) \right| \\ &\leq \sum_{i=q-h}^{q} \sum_{j=r-k}^{r} N_{i,h}(\alpha) N_{j,k}(\beta) |\varphi(\xi_{i,h}, \zeta_{j,k}) - \varphi(\alpha,\beta)| \\ &\leq \sum_{i=q-h}^{q} \sum_{j=r-k}^{r} N_{i,h}(\alpha) N_{j,k}(\beta) \omega_1(\varphi, \|(\alpha,\beta) - (\xi_{i,h}, \zeta_{j,k})\|) \\ &\leq \omega_1(\varphi, \ell\|\tilde{\Delta}\|) \\ &\leq \ell\omega_1(\varphi, \|\tilde{\Delta}\|). \end{aligned}$$

□

Corollary 1. *If*

$$\|\tilde{\Delta}\| \to 0,$$

then two-dimensional Schoenberg type operators

$$(S_{\tilde{\Delta}}\varphi)(\alpha,\beta) = \sum_{i=-h}^{n-1} \sum_{j=-k}^{m-1} N_{i,h}(\alpha) N_{j,k}(\beta) \varphi(\xi_{i,h}, \zeta_{j,k})$$

converge uniformly on $[0,1] \times [0,1]$ *to* φ*, for any continuous function* φ*.*

In [16], the subject of the second moment of variation-diminishing splines is approached. The second moment of the second degree Schoenberg one-dimensional operators was established in [17] and of the third degree Schoenberg one-dimensional operators in [18]. Further on, the form of second moment of two-dimensional Schoenberg type operators with $h = k = 3$ is presented.

Theorem 3. *The second moment of the two-dimensional Schoenberg type operators $S_{\tilde{\Delta}}$ for $h = k = 3$, is*

$$\begin{aligned}
&\left(S_{\tilde{\Delta}}\left((\pi_1 - \alpha e_0)^2 + (\pi_2 - \beta e_0)^2\right)\right)(\alpha, \beta) \\
= \; & \frac{1}{9}\Big[\frac{\alpha_{q+3} - \alpha_{q-1}}{\alpha_{q+2} - \alpha_q} \cdot \frac{(\alpha - \alpha_q)^3}{\alpha_{q+1} - \alpha_q} - \frac{\alpha_{q+2} - \alpha_{q-2}}{\alpha_{q+1} - \alpha_{q-1}} \cdot \frac{(\alpha - \alpha_{q+1})^3}{\alpha_{q+1} - \alpha_q} \\
- \; & \sum_{q-1 \le i < j \le q+2} (\alpha - \alpha_i)(\alpha - \alpha_j) + \frac{\beta_{r+3} - \beta_{r-1}}{\beta_{r+2} - \beta_r} \cdot \frac{(\beta - \beta_r)^3}{\beta_{r+1} - \beta_r} \\
- \; & \frac{\beta_{r+2} - \beta_{r-2}}{\beta_{r+1} - \beta_{r-1}} \cdot \frac{(\beta - \beta_{r+1})^3}{\beta_{r+1} - \beta_r} - \sum_{r-1 \le i < j \le r+2} (\beta - \beta_i)(\beta - \beta_j)\Big]
\end{aligned} \tag{14}$$

where $(\alpha, \beta) \in [\alpha_q, \alpha_{q+1}] \times [\beta_r, \beta_{r+1}]$ with $0 \le q \le n-1$ and $0 \le r \le m-1$.

Proof. By applying the linearity, Proposition 2(iii) and (iv) follow and the result given in [18]. □

3. Two-Dimensional Schoenberg Type Operators with Equidistant Knots

Now the case $h = k = 3$, $m = n$ and equidistant knots is analyzed. More precisely, the equidistant knots are $\alpha_i = \frac{i}{n}$, $0 \le i \le n$, and the extra-knots are $\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0$ and $\alpha_{n+1} = \alpha_{n+2} = \alpha_{n+3} = 1$, respectively $\beta_j = \frac{j}{n}$, $0 \le j \le n$, with extra-knots $\beta_{-3} = \beta_{-2} = \beta_{-1} = 0$ and $\beta_{n+1} = \beta_{n+2} = \beta_{n+3} = 1$.

The Greville abscissas are in this case

$$\xi_{i,3} := \frac{\alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3}}{3} = \begin{cases} \alpha_{i+2}, & i \in \{-3, \dots, n-1\} \setminus \{-2, n-2\} \\ \frac{1}{3n}, & i = -2, \\ \frac{3n-1}{3n} & i = n-2 \end{cases} \tag{15}$$

respectively

$$\zeta_{j,3} := \frac{\beta_{j+1} + \beta_{j+2} + \beta_{j+3}}{3} = \begin{cases} \beta_{j+2}, & j \in \{-3, \dots, n-1\} \setminus \{-2, n-2\} \\ \frac{1}{3n}, & j = -2, \\ \frac{3n-1}{3n} & j = n-2. \end{cases} \tag{16}$$

The B-splines are

$$N_{i,3}(\alpha) = (\alpha_{i+4} - \alpha_i)[\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}](\cdot - \alpha)^3_+, \tag{17}$$

respectively

$$N_{j,3}(\beta) = (\beta_{j+4} - \beta_j)[\beta_j, \beta_{j+1}, \beta_{j+2}, \beta_{j+3}, \beta_{j+4}](\cdot - \beta)^3_+. \tag{18}$$

Two-dimensional Schoenberg type operators, with $h = k = 3$ and $m = n$, with equidistant knots is denoted by $\tilde{S}_{n,3}$:

$$(\tilde{S}_{n,3}\varphi)(\alpha, \beta) = \sum_{i=-3}^{n-1} \sum_{j=-3}^{n-1} N_{i,3}(\alpha) N_{j,3}(\beta) \varphi(\xi_{i,3}, \zeta_{j,3}), \tag{19}$$

and the one-dimensional k degree Schoenberg operators with equidistant knots are denoted by $S_{n,k}$.

Lemma 1. *The second moment of the two-dimensional Schoenberg type operators $\tilde{S}_{n,3}$, with $n \ge 5$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, verifies the relation*

$$\left(\tilde{S}_{n,3}\left((\pi_1 - \alpha e_0)^2\right)(\alpha, \beta) \le \frac{1}{3n^2}. \tag{20}$$

Moreover,

$$\left(\tilde{S}_{n,3}\left((\pi_1-\alpha e_0)^2\right)(\alpha,\beta)=\frac{1}{3n^2},\tag{21}$$

for $\alpha\in\left[\frac{2}{n},\frac{n-2}{n}\right]$ *and* $\beta\in[0,1]$.

Similar relations are true for $\left(\tilde{S}_{n,3}\left((\pi_2-\beta e_0)^2\right)(\alpha,\beta)$.

Proof. The exact form of the second moment of the one-dimensional Schoenberg operators was established in [18]. We have the next cases:

(i) For $\chi\in[0,\frac{1}{n}]$ we have $(S_{n,3}(e_1-\chi e_0)^2)(\chi)=-\frac{\chi^3 n}{18}+\frac{\chi}{3n}$ with the maximum $\frac{2\sqrt{2}}{9n^2}\le\frac{1}{3n^2}$;

(ii) For $\chi\in[\frac{1}{n},\frac{2}{n}]$ we have $(S_{n,3}(e_1-\chi e_0)^2)(\chi)=\frac{\chi^3 n}{18}-\frac{\chi^2}{3}+\frac{2\chi}{3n}-\frac{1}{9n^2}$, which is an increasing function on $[\frac{1}{n},\frac{2}{n}]$ with the maximum $\frac{1}{3n^2}$;

(iii) For $\chi\in[\frac{2}{n},\frac{n-2}{n}]$ we have $(S_{n,3}(e_1-\chi e_0)^2)(\chi)=\frac{1}{3n^2}$.

By symmetry, the inequality $(S_{n,3}(e_1-\chi e_0)^2)(\chi)\le\frac{1}{3n^2}$ is also obtained for $\alpha\in[\frac{n-2}{n},1]$.

Finally Proposition 2-(iii) can be applied. In the case of $\left(\tilde{S}_{n,3}\left((\pi_2-\beta e_0)^2\right)(\alpha,\beta)$ Proposition 2-(iv) can be applied . □

Lemma 2. *For* $n\ge 5$ *one has*

$$\left(\tilde{S}_{n,3}\left((\pi_1-\alpha e_0)^4+(\pi_2-\beta e_0)^4\right)\right)(\alpha,\beta)\le\frac{8}{3n^4}.\tag{22}$$

Proof. From [3] we have

$$(S_{n,k}(e_1-\alpha e_0)^4)(\alpha)\le\left(\frac{k+1}{2n}\right)^2(S_{n,k}(e_1-\alpha e_0)^2)(\alpha).$$

Therefore,

$$\begin{aligned}&\left(\tilde{S}_{n,3}\left((\pi_1-\alpha e_0)^4+(\pi_2-\beta e_0)^4\right)\right)(\alpha,\beta)\\ \le\;&\left(\frac{2}{n}\right)^2\left(\tilde{S}_{n,3}\left((\pi_1-\alpha e_0)^2+(\pi_2-\beta e_0)^2\right)\right)(\alpha,\beta)\le 2\cdot\frac{4}{n^2}\cdot\frac{1}{3n^2}=\frac{8}{3n^4}.\end{aligned}$$

□

The Voronovskaja type theorems are a main topic in studying the convergence properties of the sequences of linear operators. We mention only [3,19–22]. For two-dimensional Schoenberg type operators $\tilde{S}_{n,3}$, we obtain the following Voronovskaja type theorem.

Theorem 4. *The following limit is true:*

$$\lim_{n\to\infty}n^2\left((\tilde{S}_{n,3}\varphi)(\alpha,\beta)-\varphi(\alpha,\beta)\right)=\frac{1}{6}\left[\frac{\partial^2\varphi}{\partial\alpha^2}(\alpha,\beta)+\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta)\right].\tag{23}$$

for any $\varphi\in C^2([0,1]^2)$, $(\alpha,\beta)\in(0,1)^2$.

Proof. From Taylor's formula, for any $(\gamma,\theta) \in [0,1]\times[0,1]$ it follows that

$$\begin{aligned} \varphi(\theta,\gamma) &= \varphi(\alpha,\beta) + \frac{\partial\varphi}{\partial\alpha}(\alpha,\beta)(\theta-\alpha) + \frac{\partial\varphi}{\partial\beta}(\alpha,\beta)(\gamma-\beta) \\ &+ \frac{1}{2}\frac{\partial^2\varphi}{\partial\alpha^2}(\alpha,\beta)(\theta-\alpha)^2 + \frac{\partial^2\varphi}{\partial\alpha\partial\beta}(\alpha,\beta)(\theta-\alpha)(\gamma-\beta) \\ &+ \frac{1}{2}\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta)(\gamma-\beta)^2 + R((\theta,\gamma),(\alpha,\beta)), \end{aligned} \tag{24}$$

with the remainder

$$R((\theta,\gamma),(\alpha,\beta)) = \|(\theta-\alpha,\gamma-\beta)\|^2 g(\theta,\gamma), \tag{25}$$

where $g(\theta,\gamma) \to 0$ when $(\theta,\gamma) \to (\alpha,\beta)$.

Applying operator $\tilde{S}_{n,3}$ in relation (24) and taking into account Proposition 2 results in

$$\begin{aligned} &(\tilde{S}_{n,3}\varphi)(\theta,\gamma) = \varphi(\alpha,\beta) \\ &+\frac{1}{2}\frac{\partial^2\varphi}{\partial\alpha^2}(\alpha,\beta)(\tilde{S}_{n,3}((\pi_1-\alpha e_0)^2)(\alpha,\beta) + \frac{1}{2}\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta)(\tilde{S}_{n,3}(\pi_2-\beta e_0)^2)(\alpha,\beta) \\ &+\big(\tilde{S}_{n,3}(R((\theta,\gamma),(\alpha,\beta)))\big)(\alpha,\beta). \end{aligned} \tag{26}$$

Let $\varepsilon > 0$. Then there is $\eta > 0$ so that $|g(\theta,\gamma)| < \varepsilon$ if $\|(\theta-\alpha,\gamma-\beta)\| < \eta$.

It takes place that

$$|g(\theta,\gamma)| \le \varepsilon + \frac{M}{\eta^2}\|(\theta-\alpha,\gamma-\beta)\|^2, \tag{27}$$

where $(\theta,\gamma) \in [0,1]\times[0,1]$, when $M = \|g\|$.

We have $\|(\theta-\alpha,\gamma-\beta)\|^2 = (\theta-\alpha)^2 + (\gamma-\beta)^2$ and $\|(\theta-\alpha,\gamma-\beta)\|^4 \le 2(\theta-\alpha)^4 + 2(\gamma-\beta)^4$. Then,

$$|R((\theta,\gamma),(\alpha,\beta))| \le \varepsilon\big((\theta-\alpha)^2 + (\gamma-\beta)^2\big) + \frac{2M}{\eta^2}\big((\theta-\alpha)^4 + (\gamma-\beta)^4\big). \tag{28}$$

From the relation (28) it follows that

$$\begin{aligned} &\big|\big(\tilde{S}_{n,3}(R((\theta,\gamma),(\alpha,\beta)))\big)(\alpha,\beta)\big| \le \big(\tilde{S}_{n,3}(|R((\theta,\gamma),(\alpha,\beta))|)\big)(\alpha,\beta) \\ <\; &\varepsilon\Big(\big(\tilde{S}_{n,3}(\pi_1-\alpha e_0)^2\big)(\alpha,\beta) + \big(\tilde{S}_{n,3}(\pi_2-\beta e_0)^2\big)(\alpha,\beta)\Big) \\ +\; &\frac{2M}{\eta^2}\Big(\big(\tilde{S}_{n,3}(\pi_1-\alpha e_0)^4\big)(\alpha,\beta) + \big(\tilde{S}_{n,3}(\pi_2-\beta e_0)^4\big)(\alpha,\beta)\Big). \end{aligned} \tag{29}$$

From Lemmas 1 and 2 applied in (29), one has

$$|\big(\tilde{S}_{n,3}(R((\theta,\gamma),(\alpha,\beta)))\big)(\alpha,\beta)| \le \varepsilon\frac{2}{3n^2} + \frac{2M}{\eta^2}\cdot\frac{8}{3n^4}.$$

Therefore,

$$\lim_{n\to\infty} n^2\big(\tilde{S}_{n,3}(R((\theta,\gamma),(\alpha,\beta)))\big)(\alpha,\beta) = 0. \tag{30}$$

Since $(\alpha,\beta) \in (0,1)^2$, for sufficiently great n we have $\alpha,\beta \in \left[\frac{2}{n}, \frac{n-2}{n}\right]$. Then, for such n from Lemma 1 it follows that $(\tilde{S}_{n,3}((\pi_1 - \alpha e_0)^2)(\alpha,\beta) = (\tilde{S}_{n,3}((\pi_2 - \beta e_0)^2)(\alpha,\beta) = \frac{1}{3n^2}$. Replacing these in (26) and taking into account relation (30), Equation (23) is immediate. □

Moduli of continuity are a powerful tool in evaluating the approximation order. To evaluate the approximation order through the operators $\tilde{S}_{n,3}$ we use general evaluations, expressed with second-order moduli of continuity, demonstrated in [23]. These give a finer evaluation than the evaluations with the first order modulus. For this we introduce the following notation. Let $(X, \|\cdot\|_X)$ be a normed space and $D \subset X$ be a compact and convex set. Let $e_0 : X \to \mathbb{R}$, $e_0(t) = 1,\ t \in D$.

If $\varphi \in C(X,\mathbb{R})$ and $h > 0$, then the usual second-order modulus of a function $\varphi \in C(X,\mathbb{R})$ is defined by

$$\tilde{\omega}_2(\varphi,h) := \sup\left\{\left|\varphi(u) - 2f\left(\frac{u+v}{2}\right) + \varphi(v)\right|, u,v \in D, \|u-v\|_X \le 2h\right\}. \tag{31}$$

With these elements, a particular version of a more general result given in [23] can be expressed in the form:

Theorem 5. *Let $L : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ be a positive linear operator. Suppose that X is finite dimensional space with* $\dim X = p$. *Let $x \in X$. Suppose also that*

$$(L\psi)(x) = \psi(x), \text{ for all affine functions } \psi : X \to \mathbb{R}.$$

then

$$|(L\varphi)(x) - \varphi(x)| \le \left(p(Le_0)(x) + \frac{1}{2h^2}(L\|\cdot - x\|_X^2)(x)\right)\tilde{\omega}_2(\varphi,h), \tag{32}$$

for any $\varphi \in C(D,\mathbb{R}),\ h > 0$.

In the case of operators $\tilde{S}_{n,3}$ we get:

Theorem 6.

$$|(\tilde{S}_{n,3}\varphi)(\alpha,\beta) - \varphi(\alpha,\beta)| \quad \le \quad \left(2 + \frac{1}{3h^2n^2}\right)\tilde{\omega}_2(\varphi,h), \tag{33}$$

where $\varphi \in C([0,1]^2),\ h > 0,\ (\alpha,\beta) \in [0,1]^2,\ n \in \mathbb{N},\ n \ge 5$.

Consequently:

$$\|(\tilde{S}_{n,3}\varphi) - \varphi\| \le \frac{7}{3}\tilde{\omega}_2\left(\varphi, \frac{1}{n}\right),\ \varphi \in C([0,1]^2),\ n \in \mathbb{N},\ n \ge 5. \tag{34}$$

Proof. We take $X = \mathbb{R}^2$, $p = 2$, $D = [0,1] \times [0,1]$. Therefore

$$\begin{aligned}
&|(\tilde{S}_{n,3}\varphi)(\alpha,\beta) - \varphi(\alpha,\beta)| \\
\le\ & \left(2(S_{n,3}e_0)(\alpha,\beta) + \frac{1}{2h^2}(S_{n,3}((\pi_1 - \alpha e_0)^2 + (\pi_2 - \beta e_0)^2))(\alpha,\beta)\right)\tilde{\omega}_2(\varphi,h) \\
=\ & \left(2 + \frac{1}{3h^2n^2}\right)\tilde{\omega}_2(\varphi,h),
\end{aligned}$$

where $h > 0,\ (\alpha,\beta) \in [0,1] \times [0,1]$.

The particular case is obtained if $h = \frac{1}{n}$ is chosen. □

In [23] an other second-order global continuity modulus is defined:

$$\tilde{\omega}_2^{\star}(\varphi,h) = sup\Big\{\Big|\sum_{i=1}^{n}\lambda_i f(\alpha_i)-\varphi(\alpha)\Big|,\ \alpha\in D,\ \alpha_i\in D,\ \|\alpha_i-\alpha\|_X\le h,$$
$$\lambda_i\in(0,1),\ 0\le i\le n,\ \lambda_1+...+\lambda_n=1\Big\}, \tag{35}$$

where $\varphi\in C(D,\mathbb{R})$, $D\subset X$ is a compact and convex set in the normed space X and $h>0$.

For a more general result given in [23], in a particular case we have the next estimate, which does not depend on the dimension of the space X.

Theorem 7. *Let $L: C(X,\mathbb{R})\to C(X,\mathbb{R})$ be a positive linear operator. Suppose that X is finite dimensional space. Let $x\in X$. Suppose also that*

$$(L\psi)(x)=\psi(x),\ \textit{for all affine functions}\ \psi: X\to\mathbb{R},$$

then

$$|(L\varphi)(x)-\varphi(x)|\le\Big((Le_0)(x)+\frac{1}{h^2}(L\|\cdot-x\|_X^2)(x)\Big)\tilde{\omega}_2^{\star}(\varphi,h), \tag{36}$$

for any $\varphi\in C(D,\mathbb{R}),\ h>0$.

Applying this theorem to operators $\tilde{S}_{n,3}$ we get:

Theorem 8. *For a function φ continue on $[0,1]\times[0,1]$ and $h>0$ we have*

$$|(\tilde{S}_{n,3}\varphi)(\alpha,\beta)-\varphi(\alpha,\beta)|\le\Big(1+\frac{2}{3n^2h^2}\Big)\tilde{\omega}_2^{\star}(\varphi,h).$$

Consequently

$$\|(\tilde{S}_{n,3}\varphi)-\varphi\|\le\frac{5}{3}\tilde{\omega}_2^{\star}\left(\varphi,\frac{1}{n}\right),\ \varphi\in C([0,1]\times[0,1]),\ n\in\mathbb{N}. \tag{37}$$

Proof. The proof is similar to the proof of Theorem 6. □

Remark 4. *We can make a comparison between the order of approximation reached by the two-dimensional Schoenberg type operators $\tilde{S}_{n,3}$ and that obtained by the two-dimensional Bernstein operators of degree n, which are given by*

$$(B_n\varphi)(\alpha,\beta)=\sum_{i=0}^{n}\sum_{j=0}^{n}\varphi\left(\frac{i}{n},\frac{j}{n}\right)p_{n,i}(\alpha)p_{n,j}(\beta),\ \varphi:[0,1]^2\to\mathbb{R},\ (\alpha,\beta)\in[0,1]^2,\ n\in\mathbb{N}, \tag{38}$$

where $p_{n,i}(t)=\binom{n}{i}t^i(1-t)^{n-i}$, for $t=\alpha,\beta$. These operators are the most common approximation polynomial operators.

For these, the following relation is well known:

$$(B_n((\pi_1-\alpha)^2+(\pi_2-\beta)^2))(\alpha,\beta)=\frac{\alpha(1-\alpha)}{n}+\frac{\beta(1-\beta)}{n}\le\frac{1}{2n},\ (\alpha,\beta)\in[0,1]^2,\ n\in\mathbb{N}.$$

Thus, if for a function $\varphi : [0,1]^2 \to \mathbb{R}$, its values at $(n+1)^2$ knots $\left(\frac{i}{n}, \frac{j}{n}\right)$, $0 \le i,j \le n$ are known, then applying two-dimensional Bernstein operators we get:

$$\|(B_n\varphi) - \varphi\| \le \frac{5}{2}\tilde{\omega}_2\left(\varphi, \frac{1}{\sqrt{2n}}\right),\ \varphi \in C([0,1]^2),\ n \in \mathbb{N}, \tag{39}$$

$$\|(B_n\varphi) - \varphi\| \le \frac{3}{2}\tilde{\omega}_2^{\star}\left(\varphi, \frac{1}{\sqrt{2n}}\right),\ \varphi \in C([0,1]^2),\ n \in \mathbb{N}. \tag{40}$$

On the other hand, if the values of function φ are known at $(n+3)^2$ knots (ξ_i, ζ_j), $-3 \le i,j \le n-1$, where $\xi_i, \zeta_j \in \left\{0, \frac{1}{3n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{3n-1}{3n}, 1\right\}$ (see (15) and (16)), then applying two-dimensional Schoenberg operators $\tilde{S}_{n,3}$ given in (19) it results relations (34) and (37). Thus, when using the same tools of measuring for the order of approximations, the advantage is clearly in the favor of Schoenberg operators. Additionally, the volume of computations and the rounding errors are higher in the case of Bernstein operators. Only the smoothness of the image $(B_n\varphi)$ is better than the smoothness of $(\tilde{S}_{n,3}\varphi)$. However, for practical applications, the fact that $(\tilde{S}_{n,3}\varphi)$ has continuous partial derivatives of degree 2 offers a sufficient order of smoothness.

4. Conclusions

In this study, a definition of two-dimensional Schoenberg type operators and their properties has been established. The definition was obtained by generalizing the one-dimensional Schoenberg operators' formula. The exact forms of the second moment of two-dimensional Schoenberg type operators on arbitrary knots, and on equidistant knots, respectively, alongside a Voronovskaja type theorem, are given. The study presented here also contains estimates with moduli of continuity.

The extension of the Schoenberg operators to the two-dimensional case increases significantly the applicability area of the Schoenberg approximation method. The two-dimensional Schoenberg type operators $\tilde{S}_{n,3}$ generate sufficient smooth surfaces for practical applications and also offer a very good order of approximation of functions. An important advantage of the definition of two-dimensional Schoenberg type operators established in this study consists of the fact that they have a simple form, and this can lead to their easy application in practice.

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