Article

Complex T-Spherical Fuzzy Aggregation Operators with Application to Multi-Attribute Decision Making

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Abstract: In this paper, the novel approach of complex T-spherical fuzzy sets (CTSFSs) and their operational laws are explored and also verified with the help of examples. CTSFS composes the grade of truth, abstinence, and falsity with a condition that the sum of q-power of the real part (also for imaginary part) of the truth, abstinence, and falsity grades cannot be exceeded from a unit interval. Additionally, to examine the interrelationships among the complex T-spherical fuzzy numbers (CTSFNs), we propose two aggregation operators, called complex T-spherical fuzzy weighted averaging (CTSFWA) and complex T-spherical fuzzy weighted geometric (CTSFWG) operators. A multi-attribute decision making (MADM) problem is resolved based on CTSFNs by using the proposed CTSFWA and CTSFWG operators. To examine the proficiency and reliability of the explored works, we use an example to make comparisons between the proposed operators and some existing operators. Based on the comparison results, the proposed CTSFWA and CTSFWG operators are well suited in the fuzzy environment with legitimacy and prevalence by contrasting other existing operators.

Keywords: fuzzy sets; T-spherical fuzzy sets; complex T-spherical fuzzy sets; aggregation operators; multi-attribute decision making

1. Introduction

Multi-attribute decision making (MADM) can be delineated as a method for choosing the most ideal alternative(s) among accessible choices. Settling on choices and decisions is a part of our normal life because we always face decisions. In reality, with the unpredictability of decision-making (DM) issues and the fuzziness of DM conditions, MADM accepts a significant part as a result of which we cannot generally have complete exact data [1]. To manage such issues, Zadeh [2] coordinated the creative thought of fuzzy sets (FSs) which relate each part of the universal set to a one of a kind genuine number, called a participation degree (PD), in [0,1]. Additionally, Atanassove [3] explored the intuitionistic fuzzy set (IFS) which contains the grades ‘satisfactory’ and ‘dissatisfactory’, whose sum is not exceeded from a unit interval. The conditions of IFS are very restricted for choosing the grade of ‘satisfactory’ or ‘dissatisfactory’. To improve the retractions of IFS, the theory of Pythagorean FS (PFS) was proposed by Yager [4], with a new condition that the sum of the squares of both is not exceeded from a unit interval. Because of the expansion in volumes and multifaceted nature of ongoing data, the theory of q-rung orthopair FS (QROFS) was considered by Yager [5], with a modified condition that the sum of q-powers of the satisfactory and the dissatisfactory grades is limited to a unit interval. Both IFS and PFS are exceptional instances of QROFS. We can contend that QROFS is progressively broad in light of the fact that as the bar builds, the satisfactory space of the orthopair increments and more orthopairs meet the limited conditions.

In real decision making, DMs may offer their feelings with more responses for a decision index, such as positive, negative, neutral, and abstinence. In order to express this information, Cuong [6] investigated the picture FS (PiFS)—which contains positive, abstinence, and negative grades—whose sum is bounded to $[0,1]$. However, in some cases, for a MD, it is very difficult to face some limitations, and so the spherical FS (SFS), established by Ullah et al. [7], is more powerful compared to IFS and PiFS. The constraint of SFS is that the sum of squares of positive, abstinence, and negative grades belongs to $[0,1]$. When a DM gives 0.9 for positive grade, 0.85 for abstinence grade, and 0.8 for negative grade, the PiFS and SFS are not able to parse it, and so the idea of T-spherical fuzzy set (TSFS) was established by Mahmood et al. [8], in which the sum of q-powers of positive, abstinence, and negative grades belong to $[0,1]$. The TSFS is more powerful compared to PiFS and SFS, and had been applied on evaluation of investment policy [9].

Many researchers had also raised a question: what will be the result if we change the range of the FS into unit disc in a complex plane? For coping with such kind of issue, the theory of complex FS (CFS), explored by Ramot et al. [10], which contains the grade of complex-valued supporting belonging to unit disc in a complex plane in the form of polar co-ordinates. Additionally, Alkouri and Salleh [11] improved CFS to explore complex intuitionistic FS (CIFS) that contains the grade of supporting and the grade of supporting against in the form of complex number in a unit disc with the condition that the sum of real part (also for the imaginary part) of the supporting grade and supporting against grade is not exceeded from a unit interval. Additionally, Ullah et al. [12] modified CIFS to explore the complex PFS (CPFS) with the condition that the sum of the squares of the real parts (also for imaginary parts) of the supporting and supporting against grades cannot be exceeded from a unit interval. After CPFS was introduced, the complex QROFS (CQROFS) [13] was demonstrated as a compelling device for delineating the vulnerability of MADM issues. The CQROFS is additionally described by the participation degree and the non-membership degree, whose aggregate of q-powers of the real part (also for imaginary part) is not exactly or equivalent to 1. Thus, CQROFS is broader than CIFS and CPFS.

Until now, CQROFS seems to be a broader extension of fuzzy sets, but in some circumstances, CQROFS cannot handle well. In this paper, we propose a novel approach of complex T-spherical fuzzy sets (CTSFSs). The proposed CTSFS can deal some circumstances better than CFS, CIFS, CPFS, or even CQROFS. We will give an example to explain it in the next section. Thus, the contributions of the paper are described as follows: (1) the novel approaches of CSFS and CTSFS with their operational laws are explored; (2) to examine the interrelationships among CTSFNs, we propose the aggregation operators, called CTSFWA and CTSFWG operators; (3) an MADM problem is resolved based on CTSFNs by using the CTSFWA and CTSFWG operators. To examine the proficiency and reliability of our works, we resolve an example and solve it by using the proposed operators and some existing operators; (4) the proposed operators are compared with existing operators with an MADM example.

The rest of the paper is organized as follows. In Section 2, we give a literature review and some preliminary definitions of FS, CFS, and TSFS with operational laws. In Section 3, the novel approach of CTSFS and its operational laws are explored. CTSFS composes the grade of truth, abstinence, and falsity with a conditions that the sum of q-power of the real part (also for imaginary part) of the truth, abstinence, and falsity grades is not exceeded from a unit interval. In Section 4, to examine the interrelationships among CTSFNs, we give the aggregation operators, called CTSFWA and CTSFWG. We also consider their special cases. In Section 5, a novel MADM based on the CTSFWA and CTSFWG operators is proposed. To examine the proficiency and reliability of the explored work, we resolve an MADM example and solve it by using the explored operators and some existing operators. These operators are also compared with some other operators. In Section 6, we give our conclusions.

2. Literature Review and Preliminary Definitions

In this section, we first have a literature review and then give some preliminary definitions.

2.1. Literature Review

Since Zadeh [2] first introduced fuzzy sets (FSs) in 1965 which give an approach to treating uncertainty different from that of probability, there were various extensions of FSs in the literature. Atanassov [3] considered intuitionistic FSs (IFSs) which contain the grades of membership and non-membership. Yager [4] extended IFS to Pythagorean FS (PFS) and then Yager [5] generalize IFS and PFS to the q-rung orthopair FS (QROFS). As a result, QROFS gives a more noteworthy range to select producers to communicate their questionable data with a key gadget to be adjusted to awkward and inconvenient fuzzy information. Since it was set up, it has gotten the thought of various investigators and it is utilized in various fields [14–22] in which Peng et al. [14] and Riaz et al. [15] considered aggregation operators; Du [16] proposed Makowski-type distance measures; Bai et al. [17], Liu et al. [18], and Liu and Wang [19] gave the Maclaurin symmetric mean; Wei et al. [20] considered the Heronian mean operator; Yin et al. [21] proposed product operations on QROFS graph; Shu et al. [22] studied integrations of QROFS.

Based on a similar idea of the picture FS (PiFS) proposed by Cuong [6], Ullah et al. [7] gave the spherical FS (SFS), and Mahmood et al. [8] recently established the idea of T-spherical fuzzy set (TSFS) that presents more powerfully than IFS, PiFS, and SFS. For extending FSs to unit disc in a complex plane, Ramot et al. [10] first proposed complex FSs (CFSs) in 2002 which contain the grade of complex-valued supporting belonging to unit disc in a complex plane. Recently, Liu et al. [13] extended QROFS to the complex QROFS (CQROFS). The CQROFS additionally presents that the aggregate of q-powers of the real part (also for imaginary part) is not exactly equivalent to 1. In general, CQROFS can take care of more issues that CIFS and CPFS cannot, for instance, if a DM problem gives the enrollment degree and the non-membership degree as $0.9e^{i2\pi(0.9)}$ and $0.8e^{i2\pi(0.8)}$, separately, at that point it is just substantial for the CQROFS. All the CIFS and CPFS degrees are a piece of CQROFS degrees, which demonstrates that CQROFS is all the more impressive for dealing with unsure issues.

In view of sign quality for most parts in a wi-fi or versatile system, a gadget could either be associated or not associated with a fundamental switch. However, there is another situation regarding its availability in a flash change. Leave t alone be a fixed limit estimation of a wi-fi signal at which a gadget may get associated to a switch. At that point, if signal quality is over this limit regarding t , the gadget could be associated. In the event that signal quality is underneath t , the gadget could not be associated. Be that as it may, there are few situations when signal quality changes about this limit regarding t . In such circumstances, the status of gadget changes from associated with detached and the other way around in a short time frame. Along these lines, the portable or wi-fi gadgets can have the accompanying three cases: connected appropriately, disconnected, and fluctuates between the over two states. The above-discussed issue has multiple circumstances that cannot be well demonstrated with CFS, CIFS, CPFS, or even the broader CQROFS. In this sense, we propose a novel approach of complex T-spherical fuzzy sets (CTSFSs) in this paper where CTSFS composes the grade of truth, abstinence, and falsity with a condition that the sum of q-power of the real part (also for imaginary part) of the truth, abstinence, and falsity grades cannot be exceeded from a unit interval. Based on the complex T-spherical fuzzy numbers (CTSFNs), we propose two aggregation operators, complex T-spherical fuzzy weighted averaging (CTSFWA) and complex T-spherical fuzzy weighted geometric (CTSFWG). According to the CTSFWA and CTSFWG operators, we give a MADM solution.

2.2. Preliminary Definitions

The aim of this subsection is to review some fundamental definitions of FS, CFS, TSFS and their operational laws.

Definition 1 [2]. A FS $\mathfrak{F}$ is defined as:

$$\mathfrak{F} = \{(x, m_{\mathfrak{F}}(x)) / x \in X\} \quad (1)$$

where $m_{\mathbf{T}}(x)$ denotes the grade of truth with a condition: $0 \leq m_{\mathbf{T}}(x) \leq 1$.

Definition 2 [10]. A CFS $\mathbf{T}$ is defined as:

$$\mathbf{T} = \{(x, m_{\mathbf{T}}(x)) / x \in X\} \quad (2)$$

where $m_{\mathbf{T}}(x) = m_{\mathbf{T}}(x) \cdot e^{i.2\pi \omega_{m_{\mathbf{T}}}(x)}$ denotes the grade of complex-valued truth with a condition: $0 \leq m_{\mathbf{T}}(x), \omega_{m_{\mathbf{T}}}(x) \leq 1$.

Definition 3 [8]. A TSFS $\mathbf{T}$ is defined as:

$$\mathbf{T} = \{(x, m_{\mathbf{T}}(x), A_{\mathbf{T}}(x), F_{\mathbf{T}}(x)) / x \in X\} \quad (3)$$

where $m_{\mathbf{T}}(x), A_{\mathbf{T}}(x)$ and $F_{\mathbf{T}}(x)$ denote the grade of truth, abstinence, and falsity with a condition: $0 \leq m_{\mathbf{T}}^q(x) + A_{\mathbf{T}}^q(x) + F_{\mathbf{T}}^q(x) \leq 1$. Additionally, the term $H_{\mathbf{T}}(x) = \left(1 - \left(m_{\mathbf{T}}^q(x) + A_{\mathbf{T}}^q(x) + F_{\mathbf{T}}^q(x)\right)\right)^{1/q}$ expresses the hesitancy grade of x . Moreover $\mathbf{T} = (m_{\mathbf{T}}, A_{\mathbf{T}}, F_{\mathbf{T}})$ is called a TSFN.

Definition 4 [8]. For any two TSFNs $\mathbf{T}_j = (m_j, A_j, F_j)$, $j = 1, 2$ with $\gamma > 0$, then

1. $\mathbf{T}_j^c = (F_j, A_j, m_j)$, $j = 1, 2$, where $\mathbf{T}_j^c$ is the complement of $\mathbf{T}_j$.
2. $\mathbf{T}_1 \oplus \mathbf{T}_2 = \left(\left(m_1^q + m_2^q - m_1^q m_2^q \right)^{\frac{1}{q}}, (A_1 A_2), (F_1 F_2) \right);$
3. $\mathbf{T}_1 \otimes \mathbf{T}_2 = \left((m_1 m_2), \left(F_1^q + F_2^q - F_1^q F_2^q \right)^{\frac{1}{q}}, \left(F_1^q + F_2^q - F_1^q F_2^q \right)^{\frac{1}{q}} \right);$
4. $\gamma \mathbf{T}_j = \left(\left(1 - \left(1 - m_j^q \right)^{\gamma} \right)^{\frac{1}{q}}, A_j^{\gamma}, F_j^{\gamma} \right)$, $j = 1, 2$;
5. $\mathbf{T}_j^{\gamma} = \left(m_j^{\gamma}, \left(1 - \left(1 - A_j^q \right)^{\gamma} \right)^{\frac{1}{q}}, \left(1 - \left(1 - F_j^q \right)^{\gamma} \right)^{\frac{1}{q}} \right)$, $j = 1, 2$.

Definition 5. [8] For any TSFN $\mathbf{T}_j = (m_j, A_j, F_j)$, the score and accuracy function are defined as:

$$S(\mathbf{T}_j) = (m_j^q - A_j^q - F_j^q) \quad (4)$$

$$H(\mathbf{T}_j) = (m_j^q + A_j^q + F_j^q) \quad (5)$$

where $(\mathbf{T}_j) \in [-1, 1]$, $H(\mathbf{T}_j) \in [0, 1]$. The relation between two TSFNs $\mathbf{T}_j$ and $\mathbf{T}_j'$ can be defined as

1. If $S(\mathbf{T}_j) > S(\mathbf{T}_j')$ then $\mathbf{T}_j > \mathbf{T}_j'$.
2. If $S(\mathbf{T}_j) = S(\mathbf{T}_j')$ and
 - (1) If $H(\mathbf{T}_j) > H(\mathbf{T}_j')$ then $\mathbf{T}_j > \mathbf{T}_j'$.
 - (2) If $H(\mathbf{T}_j) = H(\mathbf{T}_j')$ then $\mathbf{T}_j = \mathbf{T}_j'$.

3. Proposed Complex T-Spherical Fuzzy Sets

One aim of this study is to explore the novel approach of CTSFSs and their operational laws. These operational laws are also verify with the help of a numerical example.

Definition 6. A CTSFS $\mathfrak{T}$ is defined as:

$$\mathfrak{T} = \{(x, m_{\mathfrak{T}}(x), A_{\mathfrak{T}}(x), F_{\mathfrak{T}}(x)) / x \in X\} \quad (6)$$

where $m_{\mathfrak{T}}(x) = m_{\mathfrak{T}}(x) \cdot e^{i.2\pi(\omega_{m_{\mathfrak{T}}}(x))}$, $A_{\mathfrak{T}}(x) = A_{\mathfrak{T}}(x) \cdot e^{i.2\pi(\omega_{A_{\mathfrak{T}}}(x))}$ and $F_{\mathfrak{T}}(x) = F_{\mathfrak{T}}(x) \cdot e^{i.2\pi(\omega_{F_{\mathfrak{T}}}(x))}$ denote the grade of truth, abstinence, and falsity with the conditions: $0 \leq m_{\mathfrak{T}}^q(x) + A_{\mathfrak{T}}^q(x) + F_{\mathfrak{T}}^q(x) \leq 1$ and $0 \leq \omega_{m_{\mathfrak{T}}}^q(x) + \omega_{A_{\mathfrak{T}}}^q(x) + \omega_{F_{\mathfrak{T}}}^q(x) \leq 1$. Additionally, the term $H_{\mathfrak{T}}(x) = R \cdot e^{i.2\pi(\omega_{Rc}(x))}$ such that $R = \left(1 - \left(m_{\mathfrak{T}}^q(x) + A_{\mathfrak{T}}^q(x) + F_{\mathfrak{T}}^q(x)\right)\right)^{1/q}$ and $\omega_R(x) = \left(1 - \left(\omega_{m_{\mathfrak{T}}}^q(x) + \omega_{A_{\mathfrak{T}}}^q(x) + \omega_{F_{\mathfrak{T}}}^q(x)\right)\right)^{1/q}$ expresses the complex hesitancy grade of x . Moreover, $\mathfrak{T} = (m \cdot e^{i.2\pi(\omega_m)}, A \cdot e^{i.2\pi(\omega_A)}, F \cdot e^{i.2\pi(\omega_F)})$ is called a CTSFN.

Definition 7. For any two CTSFNs $\mathfrak{T}_j = (m_j \cdot e^{i.2\pi(\omega_{m_j})}, A_j \cdot e^{i.2\pi(\omega_{A_j})}, F_j \cdot e^{i.2\pi(\omega_{F_j})})$, $j = 1, 2$ with $\gamma > 0$, then

1. $\mathfrak{T}_j^c = (F_j \cdot e^{i.2\pi(\omega_{F_j})}, A_j \cdot e^{i.2\pi(\omega_{A_j})}, m_j \cdot e^{i.2\pi(\omega_{m_j})})$, $j = 1, 2$, where $\mathfrak{T}_j^c$ is the complement of $\mathfrak{T}_j$.
2. $\mathfrak{T}_1 \oplus \mathfrak{T}_2 = \left((m_1^q + m_2^q - m_1^q m_2^q)^{\frac{1}{q}} \cdot e^{i.2\pi(\omega_{m_1}^q + \omega_{m_2}^q - \omega_{m_1}^q \omega_{m_2}^q)^{\frac{1}{q}}}, (A_1 A_2) \cdot e^{i.2\pi(\omega_{A_1} \omega_{A_2})}, (F_1 F_2) \cdot e^{i.2\pi(\omega_{F_1} \omega_{F_2})} \right);$
3. $\mathfrak{T}_1 \oplus \mathfrak{T}_2 = \left((m_1 m_2) \cdot e^{i.2\pi(\omega_{m_1} \omega_{m_2})}, (A_1^q + A_2^q - A_1^q A_2^q)^{\frac{1}{q}} \cdot e^{i.2\pi(\omega_{A_1}^q + \omega_{A_2}^q - \omega_{A_1}^q \omega_{A_2}^q)^{\frac{1}{q}}}, (F_1^q + F_2^q - F_1^q F_2^q)^{\frac{1}{q}} \cdot e^{i.2\pi(\omega_{F_1}^q + \omega_{F_2}^q - \omega_{F_1}^q \omega_{F_2}^q)^{\frac{1}{q}}} \right);$
4. $\gamma \mathfrak{T}_j = \left(\left(1 - (1 - m_j^q)^\gamma\right)^{\frac{1}{q}} \cdot e^{i.2\pi(1 - (1 - \omega_{m_j}^q)^\gamma)^{\frac{1}{q}}}, A_j^\gamma \cdot e^{i.2\pi(\omega_{A_j}^\gamma)}, F_j^\gamma \cdot e^{i.2\pi(\omega_{F_j}^\gamma)} \right), j = 1, 2;$
5. $\mathfrak{T}_j^\gamma = \left(m_j^\gamma \cdot e^{i.2\pi(\omega_{m_j}^\gamma)}, \left(1 - (1 - A_j^q)^\gamma\right)^{\frac{1}{q}} \cdot e^{i.2\pi(1 - (1 - \omega_{A_j}^q)^\gamma)^{\frac{1}{q}}}, \left(1 - (1 - F_j^q)^\gamma\right)^{\frac{1}{q}} \cdot e^{i.2\pi(1 - (1 - \omega_{F_j}^q)^\gamma)^{\frac{1}{q}}} \right), j = 1, 2.$

Theorem 1. For any two CTSFNs $\mathfrak{T}_j = (m_j \cdot e^{i.2\pi(\omega_{m_j})}, A_j \cdot e^{i.2\pi(\omega_{A_j})}, F_j \cdot e^{i.2\pi(\omega_{F_j})})$, $j = 1, 2$ with $n_1, n_2 > 0$, then

1. $\mathfrak{T}_1 \oplus \mathfrak{T}_2 = \mathfrak{T}_2 \oplus \mathfrak{T}_1;$
2. $\mathfrak{T}_1 \otimes \mathfrak{T}_2 = \mathfrak{T}_2 \otimes \mathfrak{T}_1;$
3. $n_1(\mathfrak{T}_1 \oplus \mathfrak{T}_2) = n_1 \mathfrak{T}_1 \oplus n_1 \mathfrak{T}_2;$
4. $n_1 \mathfrak{T}_1 \oplus n_2 \mathfrak{T}_1 = (n_1 + n_2) \mathfrak{T}_1;$
5. $\mathfrak{T}_1^{n_1} \otimes \mathfrak{T}_2^{n_1} = \mathfrak{T}_1^{n_1 + n_2};$
6. $\mathfrak{T}_1^{n_1} \otimes \mathfrak{T}_2^{n_1} = (\mathfrak{T}_1 \otimes \mathfrak{T}_2)^{n_1}.$

Proof.

1. By Definition 7, we have

$$\begin{aligned} & \mathfrak{T}_1 \oplus \mathfrak{T}_2 \\ &= \left((m_1^q + m_2^q - m_1^q m_2^q)^{\frac{1}{q}} \cdot e^{i.2\pi \cdot (\omega_{m_1}^q + \omega_{m_2}^q - \omega_{m_1}^q \omega_{m_2}^q)^{\frac{1}{q}}}, (A_1 A_2) \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})}, (F_1 F_2) \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})} \right) \\ &= \left((m_2^q + m_1^q - m_2^q m_1^q)^{\frac{1}{q}} \cdot e^{i.2\pi \cdot (\omega_{m_2}^q + \omega_{m_1}^q - \omega_{m_2}^q \omega_{m_1}^q)^{\frac{1}{q}}}, (A_2 A_1) \cdot e^{i.2\pi (\omega_{A_2} \omega_{A_1})}, (F_2 F_1) \cdot e^{i.2\pi (\omega_{F_2} \omega_{F_1})} \right) \\ &= \mathfrak{T}_2 \oplus \mathfrak{T}_1 \end{aligned}$$

2. Obviously.

3. By Definition 7, we have

$$\begin{aligned} & n_1(\mathfrak{T}_1 \oplus \mathfrak{T}_2) \\ &= n_1 \left((m_1^q + m_2^q - m_1^q m_2^q)^{\frac{1}{q}} \cdot e^{i.2\pi \cdot (\omega_{m_1}^q + \omega_{m_2}^q - \omega_{m_1}^q \omega_{m_2}^q)^{\frac{1}{q}}}, (A_1 A_2) \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})}, (F_1 F_2) \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})} \right) \\ &= \left(\left((1 - (1 - (m_1^q + m_2^q - m_1^q m_2^q))^{n_1})^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \cdot \left((1 - (\omega_{m_1}^q + \omega_{m_2}^q - \omega_{m_1}^q \omega_{m_2}^q))^{n_1} \right)^{\frac{1}{q}}}, \right. \\ & \quad \left. (A_1 A_2)^{n_1} \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})^{n_1}}, (F_1 F_2)^{n_1} \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})^{n_1}} \right) \end{aligned}$$

Similarly, we have

$$n_1 \mathfrak{T}_1 = \left((1 - (1 - m_1^q)^{n_1})^{\frac{1}{q}} \cdot e^{i.2\pi (1 - (\omega_{m_1}^q)^{n_1})^{\frac{1}{q}}}, A_1^{n_1} \cdot e^{i.2\pi \omega_{A_1}^{n_1}}, F_1^{n_1} \cdot e^{i.2\pi \omega_{F_1}^{n_1}} \right)$$

$$\text{and } n_1 \mathfrak{T}_2 = \left((1 - (1 - m_2^q)^{n_1})^{\frac{1}{q}} \cdot e^{i.2\pi (1 - (\omega_{m_2}^q)^{n_1})^{\frac{1}{q}}}, A_2^{n_1} \cdot e^{i.2\pi \omega_{A_2}^{n_1}}, F_2^{n_1} \cdot e^{i.2\pi \omega_{F_2}^{n_1}} \right).$$

Thus, we have

$$\begin{aligned} n_1 \mathfrak{T}_1 \oplus n_1 \mathfrak{T}_2 &= \left((1 - (1 - m_1^q)^{n_1})^{\frac{1}{q}} \cdot e^{i.2\pi (1 - (\omega_{m_1}^q)^{n_1})^{\frac{1}{q}}}, \right. \\ & \quad \left. A_1^{n_1} \cdot e^{i.2\pi \omega_{A_1}^{n_1}}, F_1^{n_1} \cdot e^{i.2\pi \omega_{F_1}^{n_1}} \right) \\ & \oplus \left((1 - (1 - m_2^q)^{n_1})^{\frac{1}{q}} \cdot e^{i.2\pi (1 - (\omega_{m_2}^q)^{n_1})^{\frac{1}{q}}}, \right. \\ & \quad \left. A_2^{n_1} \cdot e^{i.2\pi \omega_{A_2}^{n_1}}, F_2^{n_1} \cdot e^{i.2\pi \omega_{F_2}^{n_1}} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - (1 - m_1^q)^{n_1}\right)^{\frac{1}{q}} + \\ \left(1 - (1 - m_2^q)^{n_1}\right)^{\frac{1}{q}} - \\ \left(1 - (1 - m_1^q)^{n_1}\right)^{\frac{1}{q}} \left(1 - (1 - m_2^q)^{n_1}\right)^{\frac{1}{q}} \end{array} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(\begin{array}{c} \left(1 - (1 - \omega_{m_1}^q)^{n_1}\right)^{\frac{1}{q}} + \\ \left(1 - (1 - \omega_{m_2}^q)^{n_1}\right)^{\frac{1}{q}} - \\ \left(1 - (1 - \omega_{m_1}^q)^{n_1}\right)^{\frac{1}{q}} \left(1 - (1 - \omega_{m_2}^q)^{n_1}\right)^{\frac{1}{q}} \end{array} \right)^{\frac{1}{q}}} \\ (A_1 A_2)^{n_1} \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})^{n_1}}, (F_1 F_2)^{n_1} \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})^{n_1}} \end{array} \right) \\
&= \left(\begin{array}{c} \left(1 - (1 - m_1^q - m_2^q + m_1^q m_2^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{m_1}^q - \omega_{m_2}^q + \omega_{m_1}^q \omega_{m_2}^q)^{n_1}\right)^{\frac{1}{q}}} \\ (A_1 A_2)^{n_1} \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})^{n_1}}, (F_1 F_2)^{n_1} \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})^{n_1}} \end{array} \right) \\
&= \left(\begin{array}{c} \left(1 - (1 - m_1^q - m_2^q + m_1^q m_2^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{m_1}^q - \omega_{m_2}^q + \omega_{m_1}^q \omega_{m_2}^q)^{n_1}\right)^{\frac{1}{q}}} \\ (A_1 A_2)^{n_1} \cdot e^{i.2\pi (\omega_{A_1} \omega_{A_2})^{n_1}}, (F_1 F_2)^{n_1} \cdot e^{i.2\pi (\omega_{F_1} \omega_{F_2})^{n_1}} \end{array} \right)
\end{aligned}$$

Hence, it is proven.

4. Obviously.
5. By Definition 7, we have

$$\begin{aligned}
\mathfrak{T}_1^{n_1} &= \left(m_1^{n_1} \cdot e^{i.2\pi \omega_{m_1}^{n_1}}, \left(1 - (1 - A_1^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{A_1}^q)^{n_1}\right)^{\frac{1}{q}}}, \left(1 - (1 - F_1^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{F_1}^q)^{n_1}\right)^{\frac{1}{q}}} \right) \\
\text{and } \mathfrak{T}_1^{n_2} &= \left(m_1^{n_2} \cdot e^{i.2\pi \omega_{m_1}^{n_2}}, \left(1 - (1 - A_1^q)^{n_2}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{A_1}^q)^{n_2}\right)^{\frac{1}{q}}}, \left(1 - (1 - F_1^q)^{n_2}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{F_1}^q)^{n_2}\right)^{\frac{1}{q}}} \right).
\end{aligned}$$

Then

$$\mathfrak{T}_1^{n_1} \otimes \mathfrak{T}_1^{n_2} = \left(\begin{array}{c} m_1^{n_1} \cdot e^{i.2\pi \omega_{m_1}^{n_1}}, \\ \left(1 - (1 - A_1^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{A_1}^q)^{n_1}\right)^{\frac{1}{q}}}, \\ \left(1 - (1 - F_1^q)^{n_1}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{F_1}^q)^{n_1}\right)^{\frac{1}{q}}} \end{array} \right) \oplus \left(\begin{array}{c} m_1^{n_2} \cdot e^{i.2\pi \omega_{m_1}^{n_2}}, \\ \left(1 - (1 - A_1^q)^{n_2}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{A_1}^q)^{n_2}\right)^{\frac{1}{q}}}, \\ \left(1 - (1 - F_1^q)^{n_2}\right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_{F_1}^q)^{n_2}\right)^{\frac{1}{q}}} \end{array} \right)$$

$$\begin{aligned}
& \left(m_1^{n_1} m_1^{n_2} \right) \cdot e^{i.2\pi \left(\omega_{m_1}^{n_1} \omega_{m_1}^{n_2} \right)}, \\
& \left[\left(\left(1 - \left(1 - A_1^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q + \left(\left(1 - \left(1 - A_1^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \left(1 - A_1^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q \left(\left(1 - \left(1 - A_1^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \cdot e^{i.2\pi \left[\left(\left(1 - \left(1 - \omega_{A_1}^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q + \left(\left(1 - \left(1 - \omega_{A_1}^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \left(1 - \omega_{A_1}^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q \left(\left(1 - \left(1 - \omega_{A_1}^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q \right]} \\
& \left[\left(\left(1 - \left(1 - F_1^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q + \left(\left(1 - \left(1 - F_1^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \left(1 - F_1^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q \left(\left(1 - \left(1 - F_1^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \cdot e^{i.2\pi \left[\left(\left(1 - \left(1 - \omega_{F_1}^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q + \left(\left(1 - \left(1 - \omega_{F_1}^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q - \left(\left(1 - \left(1 - \omega_{F_1}^q \right)^{n_1} \right)^{\frac{1}{q}} \right)^q \left(\left(1 - \left(1 - \omega_{F_1}^q \right)^{n_2} \right)^{\frac{1}{q}} \right)^q \right]} \\
& = \left[\left(m_1^{n_1} m_1^{n_2} \right) \cdot e^{i.2\pi \left(\omega_{m_1}^{n_1} \omega_{m_1}^{n_2} \right)}, \left(1 - \left(1 - A_1^q \right)^{n_1} \left(1 - A_1^q \right)^{n_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \left(1 - \omega_{A_1}^q \right)^{n_1} \left(1 - \omega_{A_1}^q \right)^{n_2} \right)^{\frac{1}{q}}}, \right. \\
& \quad \left. \left(1 - \left(1 - F_1^q \right)^{n_1} \left(1 - F_1^q \right)^{n_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \left(1 - \omega_{F_1}^q \right)^{n_1} \left(1 - \omega_{F_1}^q \right)^{n_2} \right)^{\frac{1}{q}}} \right] \\
& = \left[m_1^{n_1+n_2} \cdot e^{i.2\pi \omega_{m_1}^{n_1+n_2}}, \left(1 - \left(1 - A_1^q \right)^{n_1+n_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \left(1 - \omega_{A_1}^q \right)^{n_1+n_2} \right)^{\frac{1}{q}}}, \right. \\
& \quad \left. \left(1 - \left(1 - F_1^q \right)^{n_1+n_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \left(1 - \omega_{F_1}^q \right)^{n_1+n_2} \right)^{\frac{1}{q}}} \right] = \mathfrak{T}_1^{n_1+n_2}
\end{aligned}$$

Hence, $\mathfrak{T}_1^{n_1} \otimes \mathfrak{T}_1^{n_2} = \mathfrak{T}_1^{n_1+n_2}$.

6. Obviously. $\square$

Definition 8. For any CTSFN $\mathfrak{T} = \left(m \cdot e^{i.2\pi \omega_m}, A \cdot e^{i.2\pi \omega_A}, F \cdot e^{i.2\pi \omega_F} \right)$, the score and accuracy functions are defined as:

$$S(\mathfrak{T}) = \frac{1}{2} \left| (m^q - A^q - F^q) + (\omega_{m^q} - \omega_{A^q} - \omega_{F^q}) \right| \quad (7)$$

$$H(\mathfrak{T}) = \frac{1}{2} \left| (m^q + A^q + F^q) + (\omega_{m^q} + \omega_{A^q} + \omega_{F^q}) \right| \quad (8)$$

where $(\mathfrak{T}) \in [-1, 1]$, $H(\mathfrak{T}) \in [0, 1]$. The relation between two CTSFNs $\mathfrak{T}$ and $\dot{\mathfrak{T}}$ can be defined as

1. If $S(\mathfrak{T}) > S(\dot{\mathfrak{T}})$, then $\mathfrak{T} > \dot{\mathfrak{T}}$,

2. If $S(\mathfrak{T}) = S(\dot{\mathfrak{T}})$ and

(1) If $H(\mathfrak{T}) > H(\dot{\mathfrak{T}})$, then $\mathfrak{T} > \dot{\mathfrak{T}}$.

(2) If $H(\mathfrak{T}) = H(\dot{\mathfrak{T}})$, then $\mathfrak{T} = \dot{\mathfrak{T}}$.

4. Complex T-Spherical Fuzzy Aggregation Operators and Theorems

The aims of this section is to explore aggregation operators, that are the complex T-spherical fuzzy weighted averaging (CTSFWA) and the complex T-spherical fuzzy weighted geometric (CTSFWG) operators. The weight vector is given as $\omega = (\omega_1, \omega_2, \dots, \omega_k)^m$ with $\sum_{j=1}^k \omega_j = 1, \omega_j \in [0, 1], j = 1, 2, \dots, k$. Additionally, $\mathfrak{T}_j = \left(m_j \cdot e^{i.2\pi \mathfrak{U}_{m_j}}, A_j \cdot e^{i.2\pi \mathfrak{U}_{A_j}}, F_j \cdot e^{i.2\pi \mathfrak{U}_{F_j}} \right), j = 1, 2, \dots, k$ are used to express the family of CTSFNs.

Definition 9. The CTSFWA is defined as $CmSF(\mathfrak{U})A : m^k \rightarrow m$, and

$$CmSF(\mathfrak{U})A(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_k) = \omega_1 \mathfrak{T}_1 \oplus \omega_2 \mathfrak{T}_2 \oplus \dots \oplus \omega_k \mathfrak{T}_k = \bigoplus_{j=1}^k \omega_j \mathfrak{T}_j \quad (9)$$

Based on the operational laws of Cq-ROFNs, we give the following Theorem.

Theorem 2. We have the following form of $CmSF(\mathfrak{U})A(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_k)$:

$$CmSF(\mathfrak{U})A(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_k) = \left(\left(\left(1 - \prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \mathfrak{U}_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}}, \right. \right. \quad (10)$$

$$\left. \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \mathfrak{U}_{A_j}^{\omega_j} \right)}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \mathfrak{U}_{F_j}^{\omega_j} \right)} \right)$$

Proof. By using the mathematical indication, we prove Equation (10) as follows:

Case 1. For $k = 2$, since

$$\omega_1 \mathfrak{T}_1 = \left(\left(1 - (1 - m_1^q)^{\omega_1} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \mathfrak{U}_{m_1}^q)^{\omega_1} \right)^{\frac{1}{q}}}, A_1^{\omega_1} \cdot e^{i.2\pi \mathfrak{U}_{A_1}^{\omega_1}}, F_1^{\omega_1} \cdot e^{i.2\pi \mathfrak{U}_{F_1}^{\omega_1}} \right),$$

$$\text{and } \omega_2 \mathfrak{T}_2 = \left(\left(1 - (1 - m_2^q)^{\omega_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \mathfrak{U}_{m_2}^q)^{\omega_2} \right)^{\frac{1}{q}}}, A_2^{\omega_2} \cdot e^{i.2\pi \mathfrak{U}_{A_2}^{\omega_2}}, F_2^{\omega_2} \cdot e^{i.2\pi \mathfrak{U}_{F_2}^{\omega_2}} \right), \text{ we have}$$

$$\begin{aligned} CmSF(\mathfrak{U})A(\mathfrak{T}_1, \mathfrak{T}_2) &= \omega_1 \mathfrak{T}_1 \oplus \omega_2 \mathfrak{T}_2 \\ &= \left(\left(1 - (1 - m_1^q)^{\omega_1} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \mathfrak{U}_{m_1}^q)^{\omega_1} \right)^{\frac{1}{q}}}, \right. \\ &\quad \left. A_1^{\omega_1} \cdot e^{i.2\pi \mathfrak{U}_{A_1}^{\omega_1}}, F_1^{\omega_1} \cdot e^{i.2\pi \mathfrak{U}_{F_1}^{\omega_1}} \right) \\ &\oplus \left(\left(1 - (1 - m_2^q)^{\omega_2} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \mathfrak{U}_{m_2}^q)^{\omega_2} \right)^{\frac{1}{q}}}, \right. \\ &\quad \left. A_2^{\omega_2} \cdot e^{i.2\pi \mathfrak{U}_{A_2}^{\omega_2}}, F_2^{\omega_2} \cdot e^{i.2\pi \mathfrak{U}_{F_2}^{\omega_2}} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\left(\begin{aligned} &\left((1 - (1 - m_1^q)^{\omega_1})^{\frac{1}{q}} \right)^q + \\ &\left((1 - (1 - m_2^q)^{\omega_2})^{\frac{1}{q}} \right)^q - \\ &\left((1 - (1 - m_1^q)^{\omega_1})^{\frac{1}{q}} \right)^q \left((1 - (1 - m_2^q)^{\omega_2})^{\frac{1}{q}} \right)^q \end{aligned} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(\begin{aligned} &\left((1 - (1 - \omega_{m_1}^q)^{\omega_1})^{\frac{1}{q}} \right)^q + \\ &\left((1 - (1 - \omega_{m_2}^q)^{\omega_2})^{\frac{1}{q}} \right)^q - \\ &\left((1 - (1 - \omega_{m_1}^q)^{\omega_1})^{\frac{1}{q}} \right)^q \left((1 - (1 - \omega_{m_2}^q)^{\omega_2})^{\frac{1}{q}} \right)^q \end{aligned} \right)^{\frac{1}{q}}} \right)^{\frac{1}{q}} \\
&\quad (A_1^{\omega_1} A_2^{\omega_2}) \cdot e^{i.2\pi \left(\omega_{A_1}^{\omega_1} \omega_{A_2}^{\omega_2} \right)}, (F_1^{\omega_1} F_2^{\omega_2}) \cdot e^{i.2\pi \left(\omega_{F_1}^{\omega_1} \omega_{F_2}^{\omega_2} \right)} \\
&= \left(\begin{aligned} &(1 - (1 - m_1^q)^{\omega_1} (1 - m_2^q)^{\omega_2})^{\frac{1}{q}} \cdot e^{i.2\pi (1 - (1 - \omega_{m_1}^q)^{\omega_1} (1 - \omega_{m_2}^q)^{\omega_2})^{\frac{1}{q}}} \\ &(A_1^{\omega_1} A_2^{\omega_2}) \cdot e^{i.2\pi \left(\omega_{A_1}^{\omega_1} \omega_{A_2}^{\omega_2} \right)}, (F_1^{\omega_1} F_2^{\omega_2}) \cdot e^{i.2\pi \left(\omega_{F_1}^{\omega_1} \omega_{F_2}^{\omega_2} \right)} \end{aligned} \right) \\
&= \left(\left(\left(1 - \prod_{j=1}^2 (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^2 (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right)^{\frac{1}{q}}, \prod_{j=1}^2 A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^2 \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^2 F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^2 \omega_{F_j}^{\omega_j} \right)} \right)
\end{aligned}$$

Obviously, Equation (10) is kept for $k = 2$.

Case 2. For $k = m$, let Equation (10) be held with

$$CmSF\omega A(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_m) = \left(\left(\left(1 - \prod_{j=1}^m (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^m (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right)^{\frac{1}{q}}, \prod_{j=1}^m A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^m F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{F_j}^{\omega_j} \right)} \right)$$

We next check that, for $k = m + 1$, Equation (10) is also held:

$$\begin{aligned}
CmSF\omega A(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_{m+1}) &= \left(\left(\left(1 - \prod_{j=1}^m (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^m (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right)^{\frac{1}{q}}, \prod_{j=1}^m A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^m F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{F_j}^{\omega_j} \right)} \right) \oplus \omega_{m+1} \mathfrak{t}_{m+1} \\
&= \left(\left(\left(\begin{aligned} &\left(\left(1 - \prod_{j=1}^m (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \\ &\left((1 - (1 - m_{m+1}^q)^{\omega_{m+1}})^{\frac{1}{q}} \right)^q - \\ &\left(\left(1 - \prod_{j=1}^m (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \left((1 - (1 - m_{m+1}^q)^{\omega_{m+1}})^{\frac{1}{q}} \right)^q \end{aligned} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(\begin{aligned} &\left((1 - \prod_{j=1}^m (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \\ &\left((1 - (1 - \omega_{m_{m+1}}^q)^{\omega_{m+1}})^{\frac{1}{q}} \right)^q - \\ &\left(\left(1 - \prod_{j=1}^m (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \left((1 - (1 - \omega_{m_{m+1}}^q)^{\omega_{m+1}})^{\frac{1}{q}} \right)^q \end{aligned} \right)^{\frac{1}{q}}} \right)^{\frac{1}{q}} \right) \\
&\quad \prod_{j=1}^m A_j^{\omega_j} A_{m+1}^{\omega_{m+1}} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{A_j}^{\omega_j} \omega_{A_{m+1}}^{\omega_{m+1}} \right)}, \prod_{j=1}^m F_j^{\omega_j} F_{m+1}^{\omega_{m+1}} \cdot e^{i.2\pi \left(\prod_{j=1}^m \omega_{F_j}^{\omega_j} \omega_{F_{m+1}}^{\omega_{m+1}} \right)}
\end{aligned}$$

$$= \left(\left(\left(1 - \prod_{j=1}^{m+1} (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^{m+1} (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right), \prod_{j=1}^{m+1} A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^{m+1} \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^{m+1} F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^{m+1} \omega_{F_j}^{\omega_j} \right)} \right)$$

So, Equation (10) is kept for $k = m + 1$. Thus, we prove it. $\square$

Theorem 3. (Idempotent) Let CTSFNs $\mathfrak{T}_j = \mathfrak{T} = (m \cdot e^{i.2\pi \omega_m}, A \cdot e^{i.2\pi \omega_A}, F \cdot e^{i.2\pi \omega_F})$, $j = 1, 2, \dots, k$. Then

$$CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k) = \mathfrak{T} \quad (11)$$

Proof. Because

$$\begin{aligned} CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k) &= \left(\left(\left(1 - \prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right), \right. \\ &\quad \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{F_j}^{\omega_j} \right)} \right) \\ &= \left(\left(\left(1 - \prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \omega_m^q)^{\omega_j} \right)^{\frac{1}{q}}} \right), \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_A^{\omega_j} \right)}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_F^{\omega_j} \right)} \right) \\ &= \left(\left(\left(1 - (1 - m^q)^{\sum_{j=1}^k \omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - (1 - \omega_m^q)^{\sum_{j=1}^k \omega_j} \right)^{\frac{1}{q}}} \right), A^{\sum_{j=1}^k \omega_j} \cdot e^{i.2\pi \left(\omega_A^{\sum_{j=1}^k \omega_j} \right)}, F^{\sum_{j=1}^k \omega_j} \cdot e^{i.2\pi \left(\omega_F^{\sum_{j=1}^k \omega_j} \right)} \right) \\ &= (m \cdot e^{i.2\pi \omega_m}, A \cdot e^{i.2\pi \omega_A}, F \cdot e^{i.2\pi \omega_F}) = \mathfrak{T}. \end{aligned}$$

$\square$

Theorem 4. (Monotonicity) For any two families of the CTSFNs $\mathfrak{T}_j = (m_j \cdot e^{i.2\pi \omega_{m_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, F_j \cdot e^{i.2\pi \omega_{F_j}})$ and $\mathfrak{T}'_j = (m'_j \cdot e^{i.2\pi \omega'_{m_j}}, A'_j \cdot e^{i.2\pi \omega'_{A_j}}, F'_j \cdot e^{i.2\pi \omega'_{F_j}})$, $j = 1, 2, \dots, k$, if $m_j \geq m'_j, A_j \leq A'_j, F_j \leq F'_j, \omega_{m_j} \geq \omega'_{m_j}, \omega_{A_j} \leq \omega'_{A_j}$ and $\omega_{F_j} \leq \omega'_{F_j}$, then

$$CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k) \geq CmSF\omega A(\mathfrak{T}'_1, \mathfrak{T}'_2, \mathfrak{T}'_3, \dots, \mathfrak{T}'_k) \quad (12)$$

Proof. Because $m_j \geq m'_j, A_j \leq A'_j, F_j \leq F'_j, \omega_{m_j} \geq \omega'_{m_j}, \omega_{A_j} \leq \omega'_{A_j}$ and $\omega_{F_j} \leq \omega'_{F_j}$, we have

$$\begin{aligned} 1 - m_j^q \leq 1 - m'^q_j &\Rightarrow \left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right) \right)^{\frac{1}{q}} \right)^q \geq \left(\left(1 - \left(\prod_{j=1}^k (1 - m'^q_j)^{\omega_j} \right) \right)^{\frac{1}{q}} \right)^q \\ \prod_{j=1}^k A_j^{\omega_j} &\leq \prod_{j=1}^k A'^{\omega_j}_j \text{ and } \prod_{j=1}^k F_j^{\omega_j} \leq \prod_{j=1}^k F'^{\omega_j}_j. \end{aligned}$$

Similarly, for imaginary parts, we have

$$\begin{aligned}
 1 - \omega_{m_j}^q &\leq 1 - \omega'_{m_j} \Rightarrow \left(\left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \right. \\
 &\geq \left. \left(\left(1 - \left(\prod_{j=1}^k (1 - \omega'_{m_j}{}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \right) \\
 \prod_{j=1}^k \omega_{A_j}^{\omega_j} &\leq \prod_{j=1}^k \omega'_{A_j}{}^{\omega_j} \text{ and } \prod_{j=1}^k \omega_{F_j}^{\omega_j} \leq \prod_{j=1}^k \omega'_{F_j}{}^{\omega_j}.
 \end{aligned}$$

Combining both real and imaginary parts, we have

$$\begin{aligned}
 &\left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q} - \right. \right. \\
 &\quad \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}^{\omega_j}}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}^{\omega_j}} \right) \\
 &\geq \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega'_{F_j}{}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q} - \right. \right. \\
 &\quad \left. \prod_{j=1}^k A_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}'^{\omega_j}}, \prod_{j=1}^k F_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}'^{\omega_j}} \right)
 \end{aligned}$$

We assume that $CmSF\omega A(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \dots, \mathfrak{t}_k) = \mathfrak{t}$ and $CmSF\omega A(\mathfrak{t}'_1, \mathfrak{t}'_2, \mathfrak{t}'_3, \dots, \mathfrak{t}'_k) = \mathfrak{t}'$. By using Definition (8) and Equation (2), we have $S(\mathfrak{t}) \geq S(\mathfrak{t}')$.

Here there are two possibility which are discussed one by one.

1. When $S(\mathfrak{t}) > S(\mathfrak{t}')$, by Definition 8, we have

$$CmSF\omega A(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \dots, \mathfrak{t}_k) > CmSF\omega A(\mathfrak{t}'_1, \mathfrak{t}'_2, \mathfrak{t}'_3, \dots, \mathfrak{t}'_k);$$

2. When $S(\mathfrak{t}) = S(\mathfrak{t}')$, we have

$$\begin{aligned}
 &\left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q} - \right. \right. \\
 &\quad \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}^{\omega_j}} - \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}^{\omega_j}} \right) \\
 &= \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega'_{m_j}{}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q} - \right. \right. \\
 &\quad \left. \prod_{j=1}^k A_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}'^{\omega_j}} - \prod_{j=1}^k F_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}'^{\omega_j}} \right)
 \end{aligned}$$

Because $m_j \geq m'_j, A_j \leq A'_j, F_j \leq F'_j, \omega_{m_j} \geq \omega'_{m_j}, \omega_{A_j} \leq \omega'_{A_j}$ and $\omega_{F_j} \leq \omega'_{F_j}$, then

$$\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q = \left(\left(1 - \left(\prod_{j=1}^k (1 - m_j'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \right. \\ \left. \prod_{j=1}^k A_j^{\omega_j} = \prod_{j=1}^k A_j'^{\omega_j} \text{ and } \prod_{j=1}^k F_j^{\omega_j} = \prod_{j=1}^k F_j'^{\omega_j} \right.$$

Similarly for imaginary parts, we have

$$\left(\left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q = \left(\left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \right. \\ \left. \prod_{j=1}^k \omega_{A_j}^{\omega_j} = \prod_{j=1}^k \omega_{A_j}'^{\omega_j} \text{ and } \prod_{j=1}^k \omega_{F_j}^{\omega_j} = \prod_{j=1}^k \omega_{F_j}'^{\omega_j} \right.$$

Because the score functions are equal, we use the accuracy function such that

$$H(\mathfrak{T}) = H \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)} \right. \right. \\ \left. \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}^{\omega_j}}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}^{\omega_j}} \right) \right) \\ = \frac{1}{2} \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \prod_{j=1}^k A_j^{\omega_j} + \prod_{j=1}^k F_j^{\omega_j} + \right. \right. \\ \left. \left(\left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \prod_{j=1}^k \omega_{A_j}^{\omega_j} + \prod_{j=1}^k \omega_{F_j}^{\omega_j} \right) \right) \\ = \frac{1}{2} \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \prod_{j=1}^k A_j'^{\omega_j} + \prod_{j=1}^k F_j'^{\omega_j} + \right. \right. \\ \left. \left(\left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q + \prod_{j=1}^k \omega_{A_j}'^{\omega_j} + \prod_{j=1}^k \omega_{F_j}'^{\omega_j} \right) \right) \\ = H \left(\left(\left(1 - \left(\prod_{j=1}^k (1 - m_j'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)^q \cdot e^{i.2\pi \left(1 - \left(\prod_{j=1}^k (1 - \omega_{m_j}'^q)^{\omega_j} \right)^{\frac{1}{q}} \right)} \right. \right. \\ \left. \left. \prod_{j=1}^k A_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{A_j}'^{\omega_j}}, \prod_{j=1}^k F_j'^{\omega_j} \cdot e^{i.2\pi \prod_{j=1}^k \omega_{F_j}'^{\omega_j}} \right) \right) = H(\mathfrak{T}').$$

From cases (1) and (2), it is proven. $\square$

Theorem 5. (Boundedness) For any CTSFNs $\mathfrak{T}_j = \left(m_j \cdot e^{i.2\pi \omega_{m_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, F_j \cdot e^{i.2\pi \omega_{F_j}} \right), j = 1, 2, \dots, k$, if $\mathfrak{T}_j^+ = \left(\max_{1 \leq j \leq k} m_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{m_j}}, \min_{1 \leq j \leq k} A_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{A_j}}, \min_{1 \leq j \leq k} F_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{F_j}} \right)$ and $\mathfrak{T}_j^- = \left(\min_{1 \leq j \leq k} m_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{m_j}}, \max_{1 \leq j \leq k} A_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{A_j}}, \max_{1 \leq j \leq k} F_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{F_j}} \right)$, then

$$\mathfrak{T}_j^- \leq Cmsf\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k) \leq \mathfrak{T}_j^+ \quad (13)$$

Proof. We discuss two case separately for membership and non-membership grades (for real and imaginary parts) as follows:

1. For membership grade of $CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k)$, we get

$$\begin{aligned} & \left(1 - \left(\prod_{j=1}^k \left(1 - \min_{1 \leq j \leq k} m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} \leq \left(1 - \left(\prod_{j=1}^k \left(1 - m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} \\ & \leq \left(1 - \left(\prod_{j=1}^k \left(1 - \max_{1 \leq j \leq k} m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} \\ & \Rightarrow \left(1 - \left(\left(1 - \min_{1 \leq j \leq k} m_j^q \right)^{\sum_{j=1}^k \omega_j} \right) \right)^{\frac{1}{q}} \leq \left(1 - \left(\prod_{j=1}^k \left(1 - m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} \leq \left(1 - \left(\left(1 - \max_{1 \leq j \leq k} m_j^q \right)^{\sum_{j=1}^k \omega_j} \right) \right)^{\frac{1}{q}} \end{aligned}$$

Because $\sum_{j=1}^k \omega_j = 1$, we have $\min_{1 \leq j \leq k} m_j \leq \left(1 - \left(\prod_{j=1}^k \left(1 - m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} \leq \max_{1 \leq j \leq k} m_j$.

2. For abstinence and non-membership grades of $CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k)$, we obtain

$$\prod_{j=1}^k \min_{1 \leq j \leq k} A_j^{\omega_j} \leq \prod_{j=1}^k A_j^{\omega_j} \leq \prod_{j=1}^k \max_{1 \leq j \leq k} A_j^{\omega_j} \Rightarrow \min_{1 \leq j \leq k} A_j^{\sum_{j=1}^k \omega_j} \leq \prod_{j=1}^k A_j^{\omega_j} \leq \max_{1 \leq j \leq k} A_j^{\sum_{j=1}^k \omega_j}$$

Because $\sum_{j=1}^k \omega_j = 1$, we have $\min_{1 \leq j \leq k} A_j \leq \prod_{j=1}^k A_j^{\omega_j} \leq \max_{1 \leq j \leq k} A_j$ and then $\min_{1 \leq j \leq k} F_j \leq \prod_{j=1}^k F_j^{\omega_j} \leq \max_{1 \leq j \leq k} F_j$

By combining the above two cases, we have

$$\begin{aligned} \min_{1 \leq j \leq k} m_j - \max_{1 \leq j \leq k} A_j - \max_{1 \leq j \leq k} F_j & \leq \left(\left(1 - \left(\prod_{j=1}^k \left(1 - m_j^q \right)^{\omega_j} \right) \right)^{\frac{1}{q}} - \prod_{j=1}^k A_j^{\omega_j} - \prod_{j=1}^k F_j^{\omega_j} \right) \\ & \leq \max_{1 \leq j \leq k} m_j - \min_{1 \leq j \leq k} A_j - \min_{1 \leq j \leq k} F_j \end{aligned}$$

By the score function, we get

$$s(\mathfrak{T}^-) \leq s(\mathfrak{T}) \leq s(\mathfrak{T}^+).$$

According to cases (1) and (2) and the definition of the score function, we obtain

$$\mathfrak{T}_j^- \leq CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \dots, \mathfrak{T}_k) \leq \mathfrak{T}_j^+.$$

Next, we discuss the special cases of the proposed operator.

1. If $\mathfrak{Q} = 1$, then CTSFWA (Equation (10)) is reduced to CPFWA, i.e.,

$$\begin{aligned} CmSF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_k) & = \left(\left(\left(1 - \prod_{j=1}^k \left(1 - m_j^1 \right)^{\omega_j} \right)^1 \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k \left(1 - \omega_{m_j}^1 \right)^{\omega_j} \right)^1} \right), \right. \\ & \quad \left. \prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{A_j}^{\omega_j} \right)}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{F_j}^{\omega_j} \right)} \right) \\ & = CPF\omega A(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_k) \end{aligned}$$

2. If $q = 2$, then CTSFWA (Equation (10)) is reduced to CSFWA, i.e.,

$$\begin{aligned} C m S F \omega A(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k) &= \left(\left(\left(1 - \prod_{j=1}^k (1 - m_j^2)^{\omega_j} \right)^{\frac{1}{2}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \omega_{m_j}^2)^{\omega_j} \right)^{\frac{1}{2}}} \right), \right. \\ &\quad \left. \left(\prod_{j=1}^k A_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{A_j} \right)^{\omega_j}}, \prod_{j=1}^k F_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{F_j} \right)^{\omega_j}} \right) \right) \\ &= C S F \omega A(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k). \quad \square \end{aligned}$$

Next, we propose the notion of the CTSFWG operator.

Definition 10. The CTSFWG is expressed as $C m S F \omega G : m^k \rightarrow m$ with

$$C m S F \omega G(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k) = \mathfrak{t}_1^{\omega_1} \otimes \mathfrak{t}_2^{\omega_2} \otimes \dots \otimes \mathfrak{t}_k^{\omega_k} = \otimes_{j=1}^k \mathfrak{t}_j^{\omega_j} \quad (14)$$

Based on the operational laws of the CTSFNs, we can obtain the following Theorem.

Theorem 6. By using Definition 10, we get

$$\begin{aligned} C m S F \omega G(\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k) &= \left(\prod_{j=1}^k m_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{m_j} \right)^{\omega_j}}, \left(\left(1 - \prod_{j=1}^k (1 - A_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \omega_{A_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right), \right. \\ &\quad \left. \left(\left(1 - \prod_{j=1}^k (1 - F_j^q)^{\omega_j} \right)^{\frac{1}{q}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k (1 - \omega_{F_j}^q)^{\omega_j} \right)^{\frac{1}{q}}} \right) \right) \end{aligned} \quad (15)$$

Proof. Straightforward that is similar to Theorem 2. $\square$

Theorem 7. (Idempotent) For any family of CTSFNs $\mathfrak{t}_j = \mathfrak{t} = (m \cdot e^{i.2\pi \omega_m}, A \cdot e^{i.2\pi \omega_A}, F \cdot e^{i.2\pi \omega_F})$, $j = 1, 2, \dots, k$, then

$$C m S F \omega G(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \dots, \mathfrak{t}_k) = \mathfrak{t} \quad (16)$$

Proof. Straightforward that is similar to Theorem 3. $\square$

Theorem 8. (Monotonicity) For any family of Cq-ROFNs $\mathfrak{t}_j = (m_j \cdot e^{i.2\pi \omega_{m_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, F_j \cdot e^{i.2\pi \omega_{F_j}})$ and $\mathfrak{t}'_j = (m'_j \cdot e^{i.2\pi \omega'_{m_j}}, A'_j \cdot e^{i.2\pi \omega'_{A_j}}, F'_j \cdot e^{i.2\pi \omega'_{F_j}})$, $j = 1, 2, \dots, k$, if $m_j \geq m'_j, A_j \leq A'_j, F_j \leq F'_j, \omega_{m_j} \geq \omega'_{m_j}, \omega_{A_j} \leq \omega'_{A_j}$ and $\omega_{F_j} \leq \omega'_{F_j}$, then

$$C m S F \omega G(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \dots, \mathfrak{t}_k) \geq C m S F \omega G(\mathfrak{t}'_1, \mathfrak{t}'_2, \mathfrak{t}'_3, \dots, \mathfrak{t}'_k) \quad (17)$$

Proof. Straightforward that is similar to Theorem 4. $\square$

Theorem 9. (Boundedness) For any family of CTSFNs $\hat{\tau}_j = \left(m_j \cdot e^{i.2\pi \omega_{m_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, F_j \cdot e^{i.2\pi \omega_{F_j}} \right), j = 1, 2, \dots, k$, if $\hat{\tau}_j^+ = \left(\max_{1 \leq j \leq k} m_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{m_j}}, \min_{1 \leq j \leq k} A_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{A_j}}, \min_{1 \leq j \leq k} F_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{F_j}} \right)$ and $\hat{\tau}_j^- = \left(\min_{1 \leq j \leq k} m_j \cdot e^{i.2\pi \min_{1 \leq j \leq k} \omega_{m_j}}, \max_{1 \leq j \leq k} A_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{A_j}}, \max_{1 \leq j \leq k} F_j \cdot e^{i.2\pi \max_{1 \leq j \leq k} \omega_{F_j}} \right)$, then

$$\hat{\tau}_j^- \leq CmsFWG(\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \dots, \hat{\tau}_k) \leq \hat{\tau}_j^+ \quad (18)$$

Proof. Straightforward that is similar to Theorem 5. $\square$

Next, we discuss the special cases of the proposed operators.

1. If $q = 1$, then CTSFWG (Equation (15)) is reduced to CPFWG, i.e.,

$$\begin{aligned} & CmsFWG(\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_k) \\ &= \left(\prod_{j=1}^k m_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{m_j} \right)}, \left(1 - \prod_{j=1}^k \left(1 - A_j^1 \right)^{\omega_j} \right)^1 \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k \left(1 - \omega_{A_j}^1 \right)^{\omega_j} \right)^1}, \right. \\ & \quad \left. \left(1 - \prod_{j=1}^k \left(1 - F_j^1 \right)^{\omega_j} \right)^1 \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k \left(1 - \omega_{F_j}^1 \right)^{\omega_j} \right)^1} \right) \\ &= CPFWG(\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_k). \end{aligned}$$

2. If $q = 2$, then CTSFWG (Equation (15)) is reduced to CSFWG, i.e.,

$$\begin{aligned} & CmsFWG(\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_k) \\ &= \left(\prod_{j=1}^k m_j^{\omega_j} \cdot e^{i.2\pi \left(\prod_{j=1}^k \omega_{m_j} \right)}, \left(1 - \prod_{j=1}^k \left(1 - A_j^2 \right)^{\omega_j} \right)^{\frac{1}{2}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k \left(1 - \omega_{A_j}^2 \right)^{\omega_j} \right)^{\frac{1}{2}}}, \right. \\ & \quad \left. \left(1 - \prod_{j=1}^k \left(1 - F_j^2 \right)^{\omega_j} \right)^{\frac{1}{2}} \cdot e^{i.2\pi \left(1 - \prod_{j=1}^k \left(1 - \omega_{F_j}^2 \right)^{\omega_j} \right)^{\frac{1}{2}}} \right) \\ &= CSFWG(\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_k). \end{aligned}$$

5. Application to MADM Using the CTSFWA and CTSFGA Operators

To proficiently address the reliability and effectiveness of the explored works, we construct a multi-attribute decision making (MADM) problem based on CTSFNs by using the CTSFWA and CTSFWG operators. For this, we choose the following things, such as the family of alternatives, the family of attributes and their weight vectors. Let $X = \{X_1, X_2, \dots, X_m\}$, $\hat{\tau} = \{c_1, c_2, \dots, c_n\}$, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^m$ where $\sum_{i=1}^n \omega_i = 1$ and each $\omega_i \in [0, 1]$. Thus, the constructed MADM procedures are stated as follows:

Step 1: Based on Equation (19), we normalize the decision matrix $D = (\hat{\tau}_{ij})_{m \times n}$, such that

$$R_{ij} = \begin{cases} \left(m_j \cdot e^{i.2\pi \omega_{m_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, F_j \cdot e^{i.2\pi \omega_{F_j}} \right) & \text{for benefit} \\ \left(F_j \cdot e^{i.2\pi \omega_{F_j}}, A_j \cdot e^{i.2\pi \omega_{A_j}}, m_j \cdot e^{i.2\pi \omega_{m_j}} \right) & \text{for cost} \end{cases} \quad (19)$$

Step 2: Based on Equation (10) of Theorem 2 and Equation (15) of Theorem 6, we aggregate the decision matrix which is given in Step 1.

Step 3: Based on Equation (7) of Definition 8, we examine the score values of the aggregated values, which are given in Step 2.

Step 4: Rank to all alternatives

Step 5: End.

Example 1. A company wants to invest with another company to increase income, where there are four political companies called alternatives with four attributes as follows:

c_1 : Risk analysis.

c_2 : Growth conditions.

c_3 : Social political impact.

c_4 : Environmental impact.

To resolve this MADM problem, we may choose the weight vector with $(0.25, 0.45, 0.20, 0.1)^m$ that is a better fitted weight vector for this example. We mention that the users can assign their preferred weight vector. The information for the alternatives $x_i (i = 1, 2, 3, 4)$ and for the criteria $\hat{T} = \{c_1, c_2, c_3, c_4\}$ is expressed by the CTSFNs. The complex T-spherical fuzzy information is shown in Table 1.

Table 1. Original decision matrix.

	x_1	x_2	x_3	x_4
c_1	$\left(\begin{matrix} 0.3e^{i.2\pi(0.3)}, 0.2e^{i.2\pi(0.2)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.31e^{i.2\pi(0.31)}, 0.21e^{i.2\pi(0.21)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.32e^{i.2\pi(0.32)}, 0.22e^{i.2\pi(0.22)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.33e^{i.2\pi(0.33)}, 0.23e^{i.2\pi(0.23)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$
c_2	$\left(\begin{matrix} 0.4e^{i.2\pi(0.4)}, 0.3e^{i.2\pi(0.3)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.41e^{i.2\pi(0.41)}, 0.31e^{i.2\pi(0.31)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.42e^{i.2\pi(0.42)}, 0.32e^{i.2\pi(0.32)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.43e^{i.2\pi(0.43)}, 0.33e^{i.2\pi(0.33)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$
c_3	$\left(\begin{matrix} 0.5e^{i.2\pi(0.5)}, 0.1e^{i.2\pi(0.1)} \\ 0.2e^{i.2\pi(0.2)} \end{matrix} \right)$	$\left(\begin{matrix} 0.51e^{i.2\pi(0.51)}, 0.11e^{i.2\pi(0.11)} \\ 0.21e^{i.2\pi(0.21)} \end{matrix} \right)$	$\left(\begin{matrix} 0.52e^{i.2\pi(0.52)}, 0.12e^{i.2\pi(0.12)} \\ 0.22e^{i.2\pi(0.22)} \end{matrix} \right)$	$\left(\begin{matrix} 0.53e^{i.2\pi(0.53)}, 0.13e^{i.2\pi(0.13)} \\ 0.23e^{i.2\pi(0.23)} \end{matrix} \right)$
c_4	$\left(\begin{matrix} 0.6e^{i.2\pi(0.6)}, 0.1e^{i.2\pi(0.1)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.61e^{i.2\pi(0.61)}, 0.11e^{i.2\pi(0.11)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.62e^{i.2\pi(0.62)}, 0.12e^{i.2\pi(0.12)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.63e^{i.2\pi(0.63)}, 0.13e^{i.2\pi(0.13)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$

Step 1: Based on Equation (19), we normalize the decision matrix $D = (\hat{T}_{ij})_{m \times n}$, but there are not needed to normalize the information of Table 1.

Step 2: Based on Equations (10) and (15), we aggregate the decision matrix of Table 1 by using the weight vector $\omega = (0.25, 0.45, 0.20, 0.1)^m$, $q = 1$. The results are shown in Table 2.

Table 2. Aggregated values by using the proposed operators.

	CTSFWA	CTSFWG
c_1	$\left(\begin{matrix} 0.3116e^{i.2\pi(0.3116)}, 0.2113e^{i.2\pi(0.2113)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.3114e^{i.2\pi(0.3114)}, 0.2116e^{i.2\pi(0.2116)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$
c_2	$\left(\begin{matrix} 0.4116e^{i.2\pi(0.4116)}, 0.3114e^{i.2\pi(0.3114)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.4114e^{i.2\pi(0.4114)}, 0.3116e^{i.2\pi(0.3116)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$
c_3	$\left(\begin{matrix} 0.5116e^{i.2\pi(0.5116)}, 0.1111e^{i.2\pi(0.1111)} \\ 0.2113e^{i.2\pi(0.2113)} \end{matrix} \right)$	$\left(\begin{matrix} 0.5114e^{i.2\pi(0.5114)}, 0.1115e^{i.2\pi(0.1115)} \\ 0.2116e^{i.2\pi(0.2116)} \end{matrix} \right)$
c_4	$\left(\begin{matrix} 0.6116e^{i.2\pi(0.6116)}, 0.1111e^{i.2\pi(0.1111)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.6114e^{i.2\pi(0.6114)}, 0.1115e^{i.2\pi(0.1115)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$

Step 3: Based on Equation (7), we examine the score values of the aggregated values, which are given in Step 2. The results are shown in Table 3.

Table 3. Score values of CTSFWA and CTSFWG.

	Scores of CTSFWA	Scores of CTSFWG
c_1	0.0109	0.0117
c_2	0.0109	0.0117
c_3	0.1891	0.1883
c_4	0.3893	0.3883

Step 4: Rank to all c_1, c_2, c_3, c_4 with

$$c_4 \geq c_3 \geq c_2 \geq c_1 \quad c_4 \geq c_3 \geq c_2 \geq c_1$$

From the above analysis, we get c_4 as the best one. We find that the result well matches the goal of the company.

Step 5: End.

5.1. Advantages of the Proposed Operators

To evaluate the proficiency and improve the quality of the research work, in this section, we examine the effectiveness of the proposed work. We choose the complex spherical fuzzy (CSF) kind of information and complex T-spherical fuzzy (CTS) kind of information and solve it by using the proposed operators. In Example 1, we had chosen the complex picture fuzzy (CPF) kind of information to resolve it by using the proposed operators to express their reliability and effectiveness. We next choose the CSF type of information, as shown in Table 4.

Table 4. Original decision matrix.

	x_1	x_2	x_3	x_4
c_1	$\left(\begin{matrix} 0.9e^{i.2\pi(0.9)}, 0.2e^{i.2\pi(0.2)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.91e^{i.2\pi(0.91)}, 0.21e^{i.2\pi(0.21)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.92e^{i.2\pi(0.92)}, 0.22e^{i.2\pi(0.22)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.93e^{i.2\pi(0.93)}, 0.23e^{i.2\pi(0.23)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$
c_2	$\left(\begin{matrix} 0.8e^{i.2\pi(0.8)}, 0.3e^{i.2\pi(0.3)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.81e^{i.2\pi(0.81)}, 0.31e^{i.2\pi(0.31)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.82e^{i.2\pi(0.82)}, 0.32e^{i.2\pi(0.32)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.83e^{i.2\pi(0.83)}, 0.33e^{i.2\pi(0.33)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$
c_3	$\left(\begin{matrix} 0.9e^{i.2\pi(0.9)}, 0.1e^{i.2\pi(0.1)} \\ 0.2e^{i.2\pi(0.2)} \end{matrix} \right)$	$\left(\begin{matrix} 0.91e^{i.2\pi(0.91)}, 0.11e^{i.2\pi(0.11)} \\ 0.21e^{i.2\pi(0.21)} \end{matrix} \right)$	$\left(\begin{matrix} 0.92e^{i.2\pi(0.92)}, 0.12e^{i.2\pi(0.12)} \\ 0.22e^{i.2\pi(0.22)} \end{matrix} \right)$	$\left(\begin{matrix} 0.93e^{i.2\pi(0.93)}, 0.13e^{i.2\pi(0.13)} \\ 0.23e^{i.2\pi(0.23)} \end{matrix} \right)$
c_4	$\left(\begin{matrix} 0.8e^{i.2\pi(0.8)}, 0.1e^{i.2\pi(0.1)} \\ 0.1e^{i.2\pi(0.1)} \end{matrix} \right)$	$\left(\begin{matrix} 0.81e^{i.2\pi(0.81)}, 0.11e^{i.2\pi(0.11)} \\ 0.11e^{i.2\pi(0.11)} \end{matrix} \right)$	$\left(\begin{matrix} 0.82e^{i.2\pi(0.82)}, 0.12e^{i.2\pi(0.12)} \\ 0.12e^{i.2\pi(0.12)} \end{matrix} \right)$	$\left(\begin{matrix} 0.83e^{i.2\pi(0.83)}, 0.13e^{i.2\pi(0.13)} \\ 0.13e^{i.2\pi(0.13)} \end{matrix} \right)$

Based on Equation (19), we normalize the decision matrix $D = (\hat{f}_{ij})_{m \times n}$, but there are not needed to normalize the information of Table 1. Based on Equations (10) and (15), we aggregate the decision matrix of Table 4 by using the weight vector $\omega = (0.25, 0.45, 0.20, 0.1)^m$, $q = 2$. The results are shown in Table 5.

Table 5. Aggregated values by using the proposed operators.

	CTSFWA	CTSFWG
c_1	$\left(\begin{matrix} 0.9116e^{i.2\pi(0.9116)}, 0.2113e^{i.2\pi(0.2113)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.9114e^{i.2\pi(0.9114)}, 0.2116e^{i.2\pi(0.2116)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$
c_2	$\left(\begin{matrix} 0.8116e^{i.2\pi(0.8116)}, 0.3114e^{i.2\pi(0.3114)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.8114e^{i.2\pi(0.8114)}, 0.3116e^{i.2\pi(0.3116)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$
c_3	$\left(\begin{matrix} 0.9116e^{i.2\pi(0.9116)}, 0.1111e^{i.2\pi(0.1111)} \\ 0.2113e^{i.2\pi(0.2113)} \end{matrix} \right)$	$\left(\begin{matrix} 0.9114e^{i.2\pi(0.9114)}, 0.1115e^{i.2\pi(0.1115)} \\ 0.2116e^{i.2\pi(0.2116)} \end{matrix} \right)$
c_4	$\left(\begin{matrix} 0.8116e^{i.2\pi(0.8116)}, 0.1111e^{i.2\pi(0.1111)} \\ 0.1111e^{i.2\pi(0.1111)} \end{matrix} \right)$	$\left(\begin{matrix} 0.8114e^{i.2\pi(0.8114)}, 0.1115e^{i.2\pi(0.1115)} \\ 0.1115e^{i.2\pi(0.1115)} \end{matrix} \right)$

Based on Equation (7), we examine the score values of the aggregated values of Table 5, and the results are shown in Table 6.

Table 6. Score values of CTSFWA and CTSFWG.

	Scores of CTSFWA	Scores of CTSFWG
c_1	0.774	0.7734
c_2	0.45	0.5488
c_3	0.774	0.7734
c_4	0.634	0.6327

Step 4: Rank to all c_1, c_2, c_3, c_4 with

$$c_1 \geq c_3 \geq c_4 \geq c_2$$

From the above analysis, we get c_1 to be the best one.

Next, we choose the CTSF type of information, which is shown in Table 7.

Table 7. Original decision matrix.

	x_1	x_2	x_3	x_4
c_1	$\begin{pmatrix} 0.9e^{i.2\pi(0.9)}, 0.7e^{i.2\pi(0.7)} \\ 0.8e^{i.2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.91e^{i.2\pi(0.91)}, 0.71e^{i.2\pi(0.71)} \\ 0.81e^{i.2\pi(0.81)} \end{pmatrix}$	$\begin{pmatrix} 0.92e^{i.2\pi(0.92)}, 0.72e^{i.2\pi(0.72)} \\ 0.82e^{i.2\pi(0.82)} \end{pmatrix}$	$\begin{pmatrix} 0.93e^{i.2\pi(0.93)}, 0.73e^{i.2\pi(0.73)} \\ 0.83e^{i.2\pi(0.83)} \end{pmatrix}$
c_2	$\begin{pmatrix} 0.8e^{i.2\pi(0.8)}, 0.6e^{i.2\pi(0.6)} \\ 0.7e^{i.2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.81e^{i.2\pi(0.81)}, 0.61e^{i.2\pi(0.61)} \\ 0.71e^{i.2\pi(0.71)} \end{pmatrix}$	$\begin{pmatrix} 0.82e^{i.2\pi(0.82)}, 0.62e^{i.2\pi(0.62)} \\ 0.72e^{i.2\pi(0.72)} \end{pmatrix}$	$\begin{pmatrix} 0.83e^{i.2\pi(0.83)}, 0.63e^{i.2\pi(0.63)} \\ 0.73e^{i.2\pi(0.73)} \end{pmatrix}$
c_3	$\begin{pmatrix} 0.9e^{i.2\pi(0.9)}, 0.6e^{i.2\pi(0.6)} \\ 0.5e^{i.2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.91e^{i.2\pi(0.91)}, 0.61e^{i.2\pi(0.61)} \\ 0.51e^{i.2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.92e^{i.2\pi(0.92)}, 0.62e^{i.2\pi(0.62)} \\ 0.52e^{i.2\pi(0.52)} \end{pmatrix}$	$\begin{pmatrix} 0.93e^{i.2\pi(0.93)}, 0.63e^{i.2\pi(0.63)} \\ 0.53e^{i.2\pi(0.53)} \end{pmatrix}$
c_4	$\begin{pmatrix} 0.8e^{i.2\pi(0.8)}, 0.7e^{i.2\pi(0.7)} \\ 0.7e^{i.2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.81e^{i.2\pi(0.81)}, 0.71e^{i.2\pi(0.71)} \\ 0.71e^{i.2\pi(0.71)} \end{pmatrix}$	$\begin{pmatrix} 0.82e^{i.2\pi(0.82)}, 0.72e^{i.2\pi(0.72)} \\ 0.72e^{i.2\pi(0.72)} \end{pmatrix}$	$\begin{pmatrix} 0.83e^{i.2\pi(0.83)}, 0.73e^{i.2\pi(0.73)} \\ 0.73e^{i.2\pi(0.73)} \end{pmatrix}$

Based on Equations (10) and (15), we aggregate the decision matrix of Table 7 by using the weight vector $\omega = (0.25, 0.45, 0.20, 0.1)^m$, $q = 7$. The results are shown in Table 8.

Table 8. Aggregated values by using the proposed operators.

	CTSFWA	CTSFWG
c_1	$\begin{pmatrix} 0.9116e^{i.2\pi(0.9116)}, 0.7113e^{i.2\pi(0.7113)} \\ 0.8113e^{i.2\pi(0.8113)} \end{pmatrix}$	$\begin{pmatrix} 0.9114e^{i.2\pi(0.9114)}, 0.7116e^{i.2\pi(0.7116)} \\ 0.8115e^{i.2\pi(0.8115)} \end{pmatrix}$
c_2	$\begin{pmatrix} 0.8116e^{i.2\pi(0.8116)}, 0.6114e^{i.2\pi(0.6114)} \\ 0.7111e^{i.2\pi(0.7111)} \end{pmatrix}$	$\begin{pmatrix} 0.8114e^{i.2\pi(0.8114)}, 0.6116e^{i.2\pi(0.6116)} \\ 0.7115e^{i.2\pi(0.7115)} \end{pmatrix}$
c_3	$\begin{pmatrix} 0.9116e^{i.2\pi(0.9116)}, 0.6111e^{i.2\pi(0.6111)} \\ 0.5113e^{i.2\pi(0.5113)} \end{pmatrix}$	$\begin{pmatrix} 0.9114e^{i.2\pi(0.9114)}, 0.6115e^{i.2\pi(0.6115)} \\ 0.5116e^{i.2\pi(0.5116)} \end{pmatrix}$
c_4	$\begin{pmatrix} 0.8116e^{i.2\pi(0.8116)}, 0.7111e^{i.2\pi(0.7111)} \\ 0.7111e^{i.2\pi(0.7111)} \end{pmatrix}$	$\begin{pmatrix} 0.8114e^{i.2\pi(0.8114)}, 0.7115e^{i.2\pi(0.7115)} \\ 0.7115e^{i.2\pi(0.7115)} \end{pmatrix}$

Based on Equation (7), we examine the score values of the aggregated values of Table 8, and the results are shown in Table 9.

Table 9. Score values of CTSTFWA and CTSTFWG.

	Scores of CTSTFWA	Scores of CTSTFWG
c_1	0.1973	0.1982
c_2	0.1081	0.1072
c_3	0.4822	0.4812
c_4	0.0481	0.0469

Step 4: Rank to all c_1, c_2, c_3, c_4 with

$$c_3 \geq c_1 \geq c_2 \geq c_4, c_3 \geq c_1 \geq c_2 \geq c_4$$

From the above analysis, we get c_3 as the best one.

5.2. Comparisons of the Proposed Operators with Other Existing Operators

To express the reliability and effectiveness of the presented work in this study, we consider the following ideas for resolving the information of Table 1, Table 4, and Table 7 to show the proficiency of the proposed approach. To make the comparisons of the proposed approach, we consider the following existing operators. Garg et al. [23] explored the attractive aggregation operators based on T-SFS and their application in MAGDM problems. Liu et al. [24] established power muirhead mean operators based on T-SFS and their application in MAGDM problems. Ullah et al. [25] explored geometric aggregation operators based on T-SFS and their application in decision making. Munir et al. [26] evaluated the Einstein hybrid aggregation operator based on T-SFS and their application in decision making. Ashraf and Abdullah [27] examined the spherical fuzzy aggregation operator and their application. Ashraf et al. [28] examined the spherical fuzzy Dombi aggregation operator and their application in decision making. Jin et al. [29] explored the spherical fuzzy logarithmic aggregation operator and their application in decision making. Wang et al. [30] explored the geometric aggregation for PFS and their application in decision making. Garg [31] established the picture fuzzy aggregation operators and their application. Wei [32] explored the picture fuzzy aggregation operators and their application, and picture fuzzy hamacher aggregation operators was explored by Wei [33]. The comparative analysis of the proposed work with these existing operators for Table 1 is shown in Table 10.

Table 10. Comparative analysis of the proposed work with existing operators for Table 1.

Methods	Operators	Score Values	Ranking Results
Garg et al. [23]	WA	Failed	—
	WG	Failed	—
Liu et al. [24]	WA	Failed	—
	WG	Failed	—
Ullah et al. [25]	WA	Failed	—
	WG	Failed	—
Munir et al. [26]	WA	Failed	—
	WG	Failed	—
Ashraf and Abdullah [27]	WA	Failed	—
	WG	Failed	—
Ashraf et al. [28]	WA	Failed	—
	WG	Failed	—
Jin et al. [29]	WA	Failed	—
	WG	Failed	—
Wang et al. [30]	WA	Failed	—
	WG	Failed	—
Garg [31]	WA	Failed	—
	WG	Failed	—

Table 10. Cont.

Methods	Operators	Score Values	Ranking Results
Wei [32]	WA	Failed	—
	WG	Failed	—
Wei [33]	WA	Failed	—
	WG	Failed	—
Proposed Operators (q = 1)	WA	$S(c_1) = 0.0195, S(c_2) = 0.0382,$ $S(c_3) = 0.1232, S(c_4) = 0.2261$	$c_4 \geq c_3 \geq c_2 \geq c_1$
	WG	$S(c_1) = 0.0193, S(c_2) = 0.0379,$ $S(c_3) = 0.1228, S(c_4) = 0.2258$	$c_4 \geq c_3 \geq c_2 \geq c_1$
Proposed Operators (q = 2)	WA	$S(c_1) = 0.0401, S(c_2) = 0.0601,$ $S(c_3) = 0.2048, S(c_4) = 0.3494$	$c_4 \geq c_3 \geq c_2 \geq c_1$
	WG	$S(c_1) = 0.0396, S(c_2) = 0.0596,$ $S(c_3) = 0.2042, S(c_4) = 0.3488$	$c_4 \geq c_3 \geq c_2 \geq c_1$
Proposed Operators (q = 3)	WA	$S(c_1) = 0.0109, S(c_2) = 0.0109,$ $S(c_3) = 0.1891, S(c_4) = 0.3893$	$c_4 \geq c_3 \geq c_2 \geq c_1$
	WG	$S(c_1) = 0.0117, S(c_2) = 0.0117,$ $S(c_3) = 0.1883, S(c_4) = 0.3883$	$c_4 \geq c_3 \geq c_2 \geq c_1$

From the above analysis, we find that these existing operators fail to rank c_1, c_2, c_3, c_4 for the decision matrix of Table 1. However, our proposed operators with $q = 1$, $q = 2$, and $q = 3$ actually work well with the same best option c_4 .

Similarly, based on these operators, the comparative analysis of the proposed work for Table 4 is shown in Table 11.

Table 11. Comparative analysis of the proposed work with existing operators for Table 4.

Methods	Operators	Score Values	Ranking Results
Garg et al. [23]	WA	Failed	—
	WG	Failed	—
Liu et al. [24]	WA	Failed	—
	WG	Failed	—
Ullah et al. [25]	WA	Failed	—
	WG	Failed	—
Munir et al. [26]	WA	Failed	—
	WG	Failed	—
Ashraf and Abdullah [27]	WA	Failed	—
	WG	Failed	—
Ashraf et al. [28]	WA	Failed	—
	WG	Failed	—
Jin et al. [29]	WA	Failed	—
	WG	Failed	—
Wang et al. [30]	WA	Failed	—
	WG	Failed	—
Garg [31]	WA	Failed	—
	WG	Failed	—
Wei [32]	WA	Failed	—
	WG	Failed	—
Wei [33]	WA	Failed	—
	WG	Failed	—
Proposed Operators (q = 1)	WA	Failed	—
	WG	Failed	—
Proposed Operators (q = 2)	WA	$S(c_1) = 0.774, S(c_2) = 0.45,$ $S(c_3) = 0.774, S(c_4) = 0.634$	$c_1 \geq c_3 \geq c_4 \geq c_2$
	WG	$S(c_1) = 0.7734, S(c_2) = 0.5488,$ $S(c_3) = 0.7734, S(c_4) = 0.6327$	$c_1 \geq c_3 \geq c_4 \geq c_2$
Proposed Operators (q = 3)	WA	$S(c_1) = 0.812, S(c_2) = 0.48,$ $S(c_3) = 0.812, S(c_4) = 0.74$	$c_1 \geq c_3 \geq c_4 \geq c_2$
	WG	$S(c_1) = 0.834, S(c_2) = 0.483,$ $S(c_3) = 0.834, S(c_4) = 0.727$	$c_1 \geq c_3 \geq c_4 \geq c_2$

From the above analysis, we find that these existing operators and our proposed operators with $q = 1$ fail to rank c_1, c_2, c_3, c_4 for the decision matrix of Table 4. However, our proposed operators with $q = 2$ and $q = 3$ actually work well with the same best option c_1 .

Similarly, based on these operators, the comparative analysis of the proposed work for Table 7 is shown in Table 12.

Table 12. Comparative analysis of the proposed work with existing operators for Table 7.

Methods	Operators	Score Values	Ranking Results
Garg et al. [23]	WA	Failed	—
	WG	Failed	—
Liu et al. [24]	WA	Failed	—
	WG	Failed	—
Ullah et al. [25]	WA	Failed	—
	WG	Failed	—
Munir et al. [26]	WA	Failed	—
	WG	Failed	—
Ashraf and Abdullah [27]	WA	Failed	—
	WG	Failed	—
Ashraf et al. [28]	WA	Failed	—
	WG	Failed	—
Jin et al. [29]	WA	Failed	—
	WG	Failed	—
Wang et al. [30]	WA	Failed	—
	WG	Failed	—
Garg [31]	WA	Failed	—
	WG	Failed	—
Wei [32]	WA	Failed	—
	WG	Failed	—
Wei [33]	WA	Failed	—
	WG	Failed	—
Proposed Operators ($q = 1$)	WA	Failed	—
	WG	Failed	—
Proposed Operators ($q = 2$)	WA	Failed	—
	WG	Failed	—
Proposed Operators ($q = 7$)	WA	$S(c_1) = 0.1973, S(c_2) = 0.1081,$ $S(c_3) = 0.4822, S(c_4) = 0.0481$	$c_3 \geq c_1 \geq c_2 \geq c_4$
	WG	$S(c_1) = 0.1982, S(c_2) = 0.1072,$ $S(c_3) = 0.4812, S(c_4) = 0.0469$	$c_3 \geq c_1 \geq c_2 \geq c_4$

From the above analysis, we find that these existing operators and our proposed operators with $q = 1$ and $q = 2$ fail to rank c_1, c_2, c_3, c_4 for the decision matrix of Table 7. However, our proposed operators with $q = 7$ works well with the best option c_3 .

6. Conclusions

In general, the complex q -rung orthopair fuzzy set (CQROFS) is a wide generalization of fuzzy sets. However, CQROFS cannot handle well enough for some circumstances. In this paper, we propose the novel approach of complex T-spherical fuzzy set (CTSFS) with its operational laws where CTSFS composes the grade of truth, abstinence, and falsity with a condition in which the sum of q -power of the real part (also for imaginary part) of the truth, abstinence, and falsity grades is not exceeded from a unit interval. Using an example, we explain why the proposed CTSFS can handle some circumstances better than CFS, CIFS, CPFS, or even CQROFS. Furthermore, we construct the two aggregation operators—CTSFWA and CTSFWG—for examining the interrelationships among complex T-spherical fuzzy numbers (CTSFNs). We also give the properties of idempotent, monotonicity, and boundedness for CTSFNs. A MADM procedure using the CTSFWA and CTSFGA operators is created. To examine the proficiency and reliability of the proposed approach, we resolve a MADM problem and solve it

by using the proposed operators and some existing operators with comparisons. The results actually demonstrate that the CTSFWA and CTSFWG operators are well suited in the fuzzy environment with legitimacy and prevalence by contrasting existing operators. In our future work, we will extend these approaches to complex neutrosophic sets, complex neutrosophic hesitant fuzzy sets, picture hesitant fuzzy sets, and T-spherical hesitant fuzzy sets.

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