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# Rotating Melvin-like Universes and Wormholes in General Relativity 

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#### Abstract

We find a family of exact solutions to the Einstein-Maxwell equations for rotating cylindrically symmetric distributions of a perfect fluid with the equation of state $p=w \rho(|w|<1)$, carrying a circular electric current in the angular direction. This current creates a magnetic field along the $z$ axis. Some of the solutions describe geometries resembling that of Melvin's static magnetic universe and contain a regular symmetry axis, while some others (in the case $w>0$ ) describe traversable wormhole geometries which do not contain a symmetry axis. Unlike Melvin's solution, those with rotation and a magnetic field cannot be vacuum and require a current. The wormhole solutions admit matching with flat-space regions on both sides of the throat, thus forming a cylindrical wormhole configuration potentially visible for distant observers residing in flat or weakly curved parts of space. The thin shells, located at junctions between the inner (wormhole) and outer (flat) regions, consist of matter satisfying the Weak Energy Condition under a proper choice of the free parameters of the model, which thus forms new examples of phantom-free wormhole models in general relativity. In the limit $w \rightarrow 1$, the magnetic field tends to zero, and the wormhole model tends to the one obtained previously, where the source of gravity is stiff matter with the equation of state $p=\rho$.


Keywords: exact solutions; cylindrical symmetry; rotation; perfect fluid; wormholes

## 1. Introduction

Cylindrical symmetry is the second (after the spherical one) simplest space-time symmetry making it possible to obtain numerous exact solutions in general relativity and its extensions, characterizing local strong gravitational field configurations. One of the motivations of studying cylindrically symmetric configurations is the possible existence of such linearly extended structures as cosmic strings as well as the observed cosmic jets. A large number of static cylindrically symmetric solutions have been obtained and studied since the advent of general relativity, including vacuum, electrovacuum, perfect fluid and others, see reviews in [1-3] and references therein.

Important arguments is favor of the studies of cylindrically symmetric and rotating configurations come from cosmological observations. Thus, for instance, Birch [4] has reported the discovery of polarization anisotropy in radio signals from extragalactic sources which could be a signature of a slow rotation of the Universe. This gave rise to the emergence of numerous cosmological models with rotation, most of which possess cylindrical symmetry; see [5,6] and references therein. There are
indications of a distinguished direction in the Universe following from an analysis of the Cosmic Microwave Background [7] and the distribution of left-twirled and right-twirled spiral galaxies on the celestial sphere [8].

There are also reasons to try to include large-scale magnetic fields into cosmological models. A possible existence of a global magnetic field up to $10^{-15} \mathrm{G}$ may be suspected due to the observed correlated orientations of quasars distant from each other [9]. Various possible manifestations of primordial magnetic fields are discussed in the literature; see, e.g., [10] for a review. Among numerous anisotropic cosmologies with a large-scale magnetic field, admitting late-time isotropization, one can mention Bianchi type I [11] and Kantowski-Sachs models [12], the latter appearing beyond the horizon of a regular black hole with a radial magnetic field and a phantom scalar field.

Melvin's famous solution to the Einstein-Maxwell equations, an "electric or magnetic geon" [13], is a completely regular static, cylindrically symmetric solution with a longitudinal electric or magnetic field as the only source of gravity. It is a special case from a large set of static cylindrically symmetric Einstein-Maxwell fields, see more details in [3,14,15].

An important distinguishing feature of cylindrical symmetry as compared to the spherical one is the possible inclusion of rotation, avoiding complications inherent to the more realistic axial symmetry, not to mention the general nonsymmetric space-times. Accordingly, a great number of exact stationary (assuming rotation) solutions to the Einstein equations are known, with various sources of gravity: the cosmological constant [16-20]; scalar fields with different self-interaction potentials [21-23]; rigidly or differentially rotating dust [24-26], dust with electric charge [27] or a scalar field [28], fluids with different equations of state, above all, perfect fluids with $p=w \rho, w=$ const (in usual notations) [29-33], some kinds of anisotropic fluids [34-37] etc., see also references therein and the reviews [1,3].

In this paper we obtain rotating counterparts of the static cylindrically symmetric solutions to the Einstein-Maxwell equations with a longitudinal magnetic field. It turns out that such a field cannot exist without a source in the form of an electric current, and we find solutions where such a source is a perfect fluid with $p=w \rho$. Many features of these solutions are quite different from those of the static ones, in particular, their common feature is the emergence of closed timelike curves at large radii. Also, there is a family of wormhole solutions that do not have a symmetry axis but contain a throat as a minimum of the circular radius. As in our previous studies [21,22,38,39], we try to make such wormholes potentially observable from spatial infinity by joining outer flat-space regions at some junction surfaces and verify the validity of the Weak Energy Condition for matter residing on these surfaces.

The structure of the paper is as follows. Section 2 briefly describes the general formalism. In Section 3, we find solutions of the field equations. In Section 4, we discuss the properties of Melvin-like solutions, and in Section 5, the wormhole family. Section 6 contains some concluding remarks.

## 2. Basic Relations

We consider stationary cylindrically symmetric space-times with the metric

$$
\begin{equation*}
d s^{2}=\mathrm{e}^{2 \gamma(x)}\left[d t-E(x) \mathrm{e}^{-2 \gamma(x)} d \varphi\right]^{2}-\mathrm{e}^{2 \alpha(x)} d x^{2}-\mathrm{e}^{2 \mu(x)} d z^{2}-\mathrm{e}^{2 \beta(x)} d \varphi^{2} \tag{1}
\end{equation*}
$$

where $x^{0}=t \in \mathbb{R}, x^{1}=x, x^{2}=z \in \mathbb{R}$ and $x^{3}=\varphi \in[0,2 \pi)$ are the temporal, radial, longitudinal and angular (azimuthal) coordinates, respectively. The variable $x$ is here specified up to a substitution $x \rightarrow$ $f(x)$, therefore its range depends on both the geometry itself and the "gauge" (the coordinate condition). The off-diagonal component $E$ describes rotation, or the vortex component of the gravitational field. In the general case, this vortex gravitational field is determined by the 4-curl of the orthonormal tetrad field $e_{m}^{\mu}$ (Greek and Latin letters are here assigned to world and tetrad indices, respectively) [40,41]:

$$
\begin{equation*}
\omega^{\mu}=\frac{1}{2} \varepsilon^{\mu v \rho \sigma} e_{m \mu} \partial_{\rho} e_{\sigma}^{m} \tag{2}
\end{equation*}
$$

Kinematically, the axial vector $\omega^{\mu}$ is the angular velocity of tetrad rotation, it determines the proper angular momentum density of the gravitational field,

$$
\begin{equation*}
S^{\mu}(g)=\omega^{\mu} / \kappa, \quad \kappa=8 \pi G \tag{3}
\end{equation*}
$$

where $G$ is the Newtonian gravitational constant. In space-times with the metric (1) we have

$$
\begin{equation*}
\omega^{\mu}=\frac{1}{2} \delta^{\mu 2}\left(E \mathrm{e}^{-2 \gamma}\right)^{\prime} \mathrm{e}^{\gamma-\alpha-\beta-\mu} \tag{4}
\end{equation*}
$$

(a prime stands for $d / d x$ ), and it appears sufficient to consider its absolute value $\omega(x)=\sqrt{\omega^{\mu} \omega_{\mu}}$ that has the meaning of the angular velocity of a congruence of timelike curves (vorticity) [21,40,41],

$$
\begin{equation*}
\omega=\frac{1}{2}\left(E \mathrm{e}^{-2 \gamma}\right)^{\prime} \mathrm{e}^{\gamma-\beta-\alpha} \tag{5}
\end{equation*}
$$

Furthermore, in the reference frame comoving to matter as it rotates in the azimuthal $(\varphi)$ direction, the stress-energy tensor (SET) component $T_{0}^{3}$ is zero, therefore due to the Einstein equations the Ricci tensor component $R_{0}^{3} \sim\left(\omega \mathrm{e}^{2 \gamma+\mu}\right)^{\prime}=0$, which leads to [21]

$$
\begin{equation*}
\omega=\omega_{0} \mathrm{e}^{-\mu-2 \gamma}, \quad \omega_{0}=\text { const. } \tag{6}
\end{equation*}
$$

Then, according to (5),

$$
\begin{equation*}
E(x)=2 \omega_{0} \mathrm{e}^{2 \gamma(x)} \int \mathrm{e}^{\alpha+\beta-\mu-3 \gamma} d x \tag{7}
\end{equation*}
$$

Note that Equations (4)-(7) are valid for an arbitrary choice of the radial coordinate $x$. Preserving this arbitrariness, we can write the nonzero components of the Ricci $\left(R_{\mu}^{\nu}\right)$ tensor as

$$
\begin{align*}
& R_{0}^{0}=-\mathrm{e}^{-2 \alpha}\left[\gamma^{\prime \prime}+\gamma^{\prime}\left(\sigma^{\prime}-\alpha^{\prime}\right)\right]-2 \omega^{2} \\
& R_{1}^{1}=-\mathrm{e}^{-2 \alpha}\left[\sigma^{\prime \prime}+\sigma^{\prime 2}-2 U-\alpha^{\prime} \sigma^{\prime}\right]+2 \omega^{2} \\
& R_{2}^{2}=-\mathrm{e}^{-2 \alpha}\left[\mu^{\prime \prime}+\mu^{\prime}\left(\sigma^{\prime}-\alpha^{\prime}\right)\right] \\
& R_{3}^{3}=-\mathrm{e}^{-2 \alpha}\left[\beta^{\prime \prime}+\beta^{\prime}\left(\sigma^{\prime}-\alpha^{\prime}\right)\right]+2 \omega^{2} \\
& R_{3}^{0}=G_{3}^{0}=E \mathrm{e}^{-2 \gamma}\left(R_{3}^{3}-R_{0}^{0}\right) \tag{8}
\end{align*}
$$

where we are using the notations

$$
\begin{equation*}
\sigma=\beta+\gamma+\mu, \quad U=\beta^{\prime} \gamma^{\prime}+\beta^{\prime} \mu^{\prime}+\gamma^{\prime} \mu^{\prime} \tag{9}
\end{equation*}
$$

The Einstein equations may be written in two equivalent forms

$$
\begin{align*}
G_{\mu}^{v} & \equiv R_{\mu}^{v}-\frac{1}{2} \delta_{\mu}^{v} R=-\kappa T_{\mu}^{v} \quad \text { or }  \tag{10}\\
R_{\mu}^{v} & =-\kappa \widetilde{T}_{\mu}^{v} \equiv-\kappa\left(T_{\mu}^{v}-\frac{1}{2} \delta_{\mu}^{v} T\right) \tag{11}
\end{align*}
$$

$R$ being the Ricci scalar and $T$ the SET trace. We will mostly use the form (11), of the equations, but it is also necessary to write the constraint equation from (10), which contains only first-order derivatives of the metric and represents a first integral of the other equations:

$$
\begin{equation*}
G_{1}^{1}=\mathrm{e}^{-2 \alpha} U+\omega^{2}=-\kappa T_{1}^{1} \tag{12}
\end{equation*}
$$

Owing to the last line of (8) and its analogue for $T_{\mu}^{v}$, in the Einstein equations it is sufficient to solve the diagonal components, and then their only off-diagonal component holds automatically [21].

As is evident from (8), the diagonal components of both the Ricci $\left(R_{\mu}^{\nu}\right)$ and the Einstein $\left(G_{\mu}^{\nu}\right)$ tensors split into the corresponding tensors for the static metric (the metric (1) with $E=0$ ) plus a contribution containing $\omega$ [21]:

$$
\begin{array}{ll}
R_{\mu}^{v}={ }_{s} R_{\mu}^{v}+\omega R_{\mu}^{v}, & { }_{\omega} R_{\mu}^{v}=\omega^{2} \operatorname{diag}(-2,2,0,2) \\
G_{\mu}^{v}={ }_{s} G_{\mu}^{v}+\omega G_{\mu}^{v}, & { }_{\omega} G_{\mu}^{v}=\omega^{2} \operatorname{diag}(-3,1,-1,1) \tag{14}
\end{array}
$$

where ${ }_{s} R_{\mu}^{\nu}$ and ${ }_{s} G_{\mu}^{\nu}$ correspond to the static metric. It turns out that the tensors ${ }_{s} G_{\mu}^{\nu}$ and ${ }_{\omega} G_{\mu}^{\nu}$ (each separately) obey the conservation law $\nabla_{\alpha} G_{\mu}^{\alpha}=0$ in terms of this static metric. Therefore, the tensor $\omega G_{\mu}^{v} / \kappa$ may be interpreted as the SET of the vortex gravitational field. It possesses quite exotic properties (thus, the effective energy density is $-3 \omega^{2} / \kappa<0$ ), which favor the existence of wormholes, and indeed, a number of wormhole solutions with the metric (1) have already been obtained [21,22,39,41] with sources in the form of scalar fields, isotropic or anisotropic fluids. Further on we will obtain one more solution of this kind, now supported by a perfect fluid and a magnetic field due to an electric current. Let us mention that an alternative extension of static solutions to rotating ones, with a combination of electric and magnetic fields and a cosmological constant, was obtained in [42].

## 3. Solutions with A Perfect Fluid and A Magnetic Field

### 3.1. The Electromagnetic Field. A No-Go Theorem

Consider a longitudinal (z-directed) magnetic field, corresponding to the 4-potential

$$
\begin{equation*}
A_{\mu}=(0,0,0, \Phi(x)) \tag{15}
\end{equation*}
$$

so that the only nonzero components of the Maxwell tensor $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$ are $F_{13}=-F_{31}=\Phi^{\prime}(x)$. The nonzero contravariant components $F^{\mu v}$ are

$$
\begin{equation*}
F^{13}=\mathrm{e}^{-2 \alpha-2 \beta} \Phi^{\prime}, \quad F^{01}=E \mathrm{e}^{-2 \alpha-2 \beta-2 \gamma} \Phi^{\prime} \tag{16}
\end{equation*}
$$

The magnetic field magnitude (magnetic induction) $B$ is determined by $B^{2}=F^{13} F_{13}$, so that the electromagnetic field invariant is $F_{\mu \nu} F^{\mu \nu}=2 B^{2}$.

No-go theorem. It can be shown that a free longitudinal magnetic field is incompatible with a nonstatic $(\omega \neq 0)$ metric (1). This follows from solving the Maxwell equations, which, for $F^{\mu v}$ of the form (16) read

$$
\begin{equation*}
\left(\sqrt{-g} F^{13}\right)^{\prime}=0, \quad\left(\sqrt{-g} F^{01}\right)^{\prime}=0 \tag{17}
\end{equation*}
$$

where

$$
g=\operatorname{det}\left(g_{\mu v}\right), \quad \sqrt{-g}=\mathrm{e}^{\alpha+\beta+\gamma+\mu}
$$

Equation (17) are integrated to give, respectively,

$$
\begin{equation*}
\Phi^{\prime} \mathrm{e}^{-\alpha-\beta+\gamma+\mu}=h_{1}, \quad E \Phi^{\prime} \mathrm{e}^{-\alpha-\beta-\gamma+\mu}=h_{2} \tag{18}
\end{equation*}
$$

where $h_{1}, h_{2}=$ const $\neq 0$. From (18), it follows $E \mathrm{e}^{-2 \gamma}=h_{2} / h_{1}$, whence $\left(E \mathrm{e}^{-2 \gamma}\right)^{\prime}=0$, and according to (6), $\omega=0$. We have shown that a free longitudinal magnetic field cannot support a vortex gravitational field with the metric (1).

Let us also note that in the case $E \mathrm{e}^{-2 \gamma}=E_{1}=$ const, the term $E \mathrm{e}^{-2 \gamma} d \varphi$ is eliminated from (1) by introducing the new time coordinate $t^{\prime}=t-E_{1} \varphi$, making the metric explicitly static.

### 3.2. The Fluid

To circumvent the above no-go theorem, that is, to avoid the relation $E \mathrm{e}^{-2 \gamma}=$ const, let us introduce a source of the magnetic field in the form of an electric current density $J^{\mu}=\rho_{e} u^{\mu}$, where $\rho_{e}$ is the effective electric charge density (If we introduce a real nonzero charge density, it becomes necessary to consider, in addition, a Coulomb electric field, which will make the problem hardly tractable. We therefore consider an azimuthal electric current as if in a coil, in a neutral medium like a conductor with free electrons and ions at rest.), and $u^{\mu}$ is the 4 -velocity satisfying the usual normalization condition $u_{\mu} u^{\mu}=1$. We will assume that the effective charge distribution is at rest in our rotating reference frame, so that

$$
\begin{equation*}
u^{\mu}=\left(\mathrm{e}^{-\gamma}, 0,0,0\right), \quad J^{\mu}=\left(\rho_{e} \mathrm{e}^{-\gamma}, 0,0,0\right) . \tag{19}
\end{equation*}
$$

As usual, the electric charge conservation equation $\nabla_{\mu} J^{\mu}$ holds automatically due to the Maxwell equations $\nabla_{v} F^{\mu \nu}=J^{\mu}$.

Two nontrivial Maxwell equations now read

$$
\begin{align*}
& \left(\sqrt{-g} F^{13}\right)^{\prime}=0  \tag{20}\\
& \frac{1}{\sqrt{-g}}\left(\sqrt{-g} F^{01}\right)^{\prime}=\rho_{e} \mathrm{e}^{-\gamma} \tag{21}
\end{align*}
$$

Integrating Equation (20), we obtain, as before,

$$
\begin{equation*}
\Phi^{\prime}=h \mathrm{e}^{\alpha+\beta-\gamma-\mu}, \quad h=\text { const } \tag{22}
\end{equation*}
$$

and substituting this $\Phi^{\prime}$ to Equation (21) with (16), we arrive at the following expression for $\rho_{e}$ :

$$
\begin{equation*}
\rho_{e}=h \omega_{0} \mathrm{e}^{-2 \mu-3 \gamma} \tag{23}
\end{equation*}
$$

On the other hand, the electric charges should have a material carrier, for which we will assume a perfect fluid with a barotropic equation of state and postulate a constant ratio of the effective charge density $\rho_{e}$ to energy density $\rho$ :

$$
\begin{equation*}
\rho_{e} / \rho=A=\text { const } ; \quad \rho / \rho=w=\text { const } \tag{24}
\end{equation*}
$$

$p$ being the fluid pressure. We do not fix the value of $w$ but later on we will obtain a restriction on it. The fluid must obey the conservation law $\nabla_{\nu} T_{\mu}^{v}=0$, which gives in our comoving reference frame

$$
\begin{equation*}
p^{\prime}+(p+\rho) \gamma^{\prime}=0 \tag{25}
\end{equation*}
$$

which, for $w \neq 0$, leads to the expression

$$
\begin{equation*}
\rho=\rho_{0} \mathrm{e}^{-\gamma(w+1) / w}, \quad \rho_{0}=\text { const. } \tag{26}
\end{equation*}
$$

Comparing (23) and (26), taking into account the assumption $\rho_{e} / \rho=A=$ const, we obtain a relation between the metric coefficients $\mathrm{e}^{2 \gamma}$ and $\mathrm{e}^{2 \mu}$ :

$$
\begin{equation*}
A \rho_{0} \mathrm{e}^{2 \mu}=h \omega_{0} \mathrm{e}^{-\gamma(w+1) /(2 w)} \tag{27}
\end{equation*}
$$

In the case $w=0$ (zero pressure), the conservation law (25) simply leads to $\gamma=$ const.

### 3.3. Solution of the Einstein Equations

To address the Einstein equations, let us write the expressions for the SETs of the perfect fluid and the electromagnetic field. For the fluid we have

$$
\begin{equation*}
T_{\mu}^{v}[f]=\rho \operatorname{diag}(1,-w,-w,-w) \tag{28}
\end{equation*}
$$

For the electromagnetic field SET we have the standard expression

$$
T_{\mu}^{v}[e]=\frac{1}{16 \pi}\left(-F_{\mu \alpha} F^{v \alpha}+4 \delta_{\mu}^{v} F_{\alpha \beta} F^{\alpha \beta}\right)
$$

which in our case leads to

$$
\begin{equation*}
T_{\mu}^{\nu}[e]=\frac{B^{2}}{8 \pi} \operatorname{diag}(1 .-1,1,-1) \oplus T_{3}^{0}[e] \tag{29}
\end{equation*}
$$

where $B^{2}=h^{2} \mathrm{e}^{-2 \gamma-2 \mu}$, and the only off-diagonal component

$$
\begin{equation*}
T_{3}^{0}[e]=-\frac{1}{4 \pi} h \Phi^{\prime} \mathrm{e}^{-2 \gamma} \tag{30}
\end{equation*}
$$

does not affect the solution process, as mentioned in a remark after Equation (12).
Thus far all relations and expressions were written in terms of an arbitrary radial coordinate $x$. However, to solve the Einstein equations it is helpful, at last, to choose the "gauge", and by analogy with our previous studies we will use the harmonic radial coordinate corresponding to

$$
\begin{equation*}
\alpha=\beta+\gamma+\mu \tag{31}
\end{equation*}
$$

With the above expressions for the SET and Equation (8), the noncoinciding components of Equations (11) and (12), taking into account the expressions (6) for $\omega$ and (26) for $\rho$, may be written as

$$
\begin{align*}
& \gamma^{\prime \prime}=-2 \omega_{0}^{2} \mathrm{e}^{2 \beta-2 \gamma}+\frac{1+3 w}{2} K \mathrm{e}^{2 \beta-\gamma}+G h^{2} \mathrm{e}^{2 \beta},  \tag{32}\\
& \mu^{\prime \prime}=\frac{w-1}{2} K \mathrm{e}^{2 \beta-\gamma}+G h^{2} \mathrm{e}^{2 \beta},  \tag{33}\\
& \beta^{\prime \prime}=2 \omega_{0}^{2} \mathrm{e}^{2 \beta-2 \gamma}+\frac{w-1}{2} K \mathrm{e}^{2 \beta-\gamma}-G h^{2} \mathrm{e}^{2 \beta},  \tag{34}\\
& \beta^{\prime} \gamma^{\prime}+\beta^{\prime} \mu^{\prime}+\gamma^{\prime} \mu^{\prime}=-\omega_{0}^{2} \mathrm{e}^{2 \beta-2 \gamma}+w K \mathrm{e}^{2 \beta-\gamma}+G h^{2} \mathrm{e}^{2 \beta}, \tag{35}
\end{align*}
$$

where $K=\kappa h \omega_{0} / A$ (recall that $\kappa=8 \pi G$ ).
Now, combining Equations (32) and (33) with (27), we arrive at the algebraic equation for $\gamma$ with constant coefficients,

$$
\begin{equation*}
4 \omega_{0}^{2}(1-2 w) \mathrm{e}^{-2 \gamma}+\left(8 w^{2}-3 w-1\right) K \mathrm{e}^{-\gamma}+2(4 w-1) G h^{2}=0 \tag{36}
\end{equation*}
$$

from which it follows $\gamma=$ const, and we can without loss of generality put $\gamma \equiv 0$ by choosing a time scale. Then Equation (27) implies $\mu=$ const which also allows us to put $\mu \equiv 0$ by choosing the scale along $z$; from (27) we then obtain a relation among the constants: $h \omega_{0}=A \rho_{0}$, so that, in particular, $K=\kappa \rho_{0}$.

With $\mathrm{e}^{\gamma}=\mathrm{e}^{\mu}=1$, from Equations (32) and (33) we find

$$
\begin{equation*}
G h^{2}=\frac{1}{2}(1-w) K, \quad \omega_{0}^{2}=\frac{1}{2}(1+w) K \tag{37}
\end{equation*}
$$

which leads to a conclusion on the range of $w$ :

$$
\begin{equation*}
-1<w<1 \tag{38}
\end{equation*}
$$

With (37) it is directly verified that Equations (35) and (36) also hold. All our constant parameters may be expressed in terms of two of them, for example, $\omega_{0}$ and $w$ :

$$
\begin{equation*}
G h^{2}=\frac{1-w}{1+w} \omega_{0}^{2}, \quad \kappa \rho_{0}=\frac{2 \omega_{0}^{2}}{1+w^{2}}, \quad A=\frac{\rho_{e}}{\rho}=4 \pi \sqrt{G\left(1-w^{2}\right)} \tag{39}
\end{equation*}
$$

We see that in our system not only $\mu=\gamma=0$, but also both densities $\rho$ and $\rho_{e}$ as well as the angular velocity $\omega$ are constant. It is also of interest that the two limiting cases of the equation of state, $w=1$ (maximum stiffness compatible with causality) and $w=-1$ (the cosmological constant) are excluded in the present system. In both these cases, static and stationary cylindrically symmetric solutions without an electromagnetic field are well known [3,17,18,21,23,37,39].

The remaining differential Equation (34) has the Liouville form

$$
\begin{equation*}
\beta^{\prime \prime}=\frac{4 w}{w+1} \omega_{0}^{2} \mathrm{e}^{2 \beta} \tag{40}
\end{equation*}
$$

and has the first integral

$$
\begin{equation*}
\beta^{\prime 2}=\frac{4 w}{w+1} \omega_{0}^{2} \mathrm{e}^{2 \beta}+k^{2} \operatorname{sign} k \tag{41}
\end{equation*}
$$

with $k=$ const. The further integration depends on the signs of $w$ and $k$ :

$$
\begin{array}{ll}
\text { 1. } w<0, k>0, & \mathrm{e}^{\beta}=\frac{k}{m \cosh (k x)} ; \\
\text { 2. } w>0, k>0: & \mathrm{e}^{\beta}=\frac{k}{m \sinh (k x)} ; \\
\text { 3. } w>0, k=0: & \mathrm{e}^{\beta}=\frac{1}{m x} ; \\
\text { 4. } w>0, k<0: & \mathrm{e}^{\beta}=\frac{|k|}{m \cos (|k| x)} \tag{45}
\end{array}
$$

where we have denoted $m=\left(\frac{4|w| \omega_{0}^{2}}{w+1}\right)^{1 / 2}$.
In the previously excluded case $w=0$ (dustlike matter), the equality $\gamma=$ const is immediately obtained from (25), $\mu=$ const then follows from (27), and as before, without loss of generality, we can put $\mu=\gamma=0$. Instead of (40), we obtain $\beta^{\prime \prime}=0$ whence we can write

$$
\begin{equation*}
\mathrm{e}^{\beta}=r_{0} \mathrm{e}^{k x}, \quad r_{0}, k=\mathrm{const} . \tag{46}
\end{equation*}
$$

In all cases the off-diagonal metric function $E$ is easily obtained as

$$
\begin{equation*}
E(x)=2 \omega_{0} \int \mathrm{e}^{2 \beta} d x \tag{47}
\end{equation*}
$$

## 4. Melvin-Like Universes

Melvin's electric or magnetic geon [13] is among the most well-known static, cylindrically symmetric solutions to the Einstein-Maxwell equations; it is a special solution from a large class of static, cylindrically symmetric solutions with radial, azimuthal and longitudinal electric and/or magnetic fields; see, e.g., [1,3,14]. Its metric may be written in the form [3,14]

$$
\begin{equation*}
d s^{2}=\left(1+q^{2} x^{2}\right)^{2}\left(d t^{2}-d x^{2}-d z^{2}\right)-\frac{x^{2}}{\left(1+q^{2} x^{2}\right)^{2}} d \varphi^{2} \tag{48}
\end{equation*}
$$

where $x \geq 0$, and the magnetic (let us take it for certainty) field magnitude is

$$
B=B_{z}=2 q\left(1+q^{2} x^{2}\right)^{-2}
$$

with $q=$ const characterizing the effective current that might be its source. However, this solution describes a purely field configuration existing without any massive matter, electric charges or currents. Both the metric and the magnetic field are regular on the axis $x=0$. The other "end", $x \rightarrow \infty$, is infinitely far away (the distance $\int \sqrt{-g_{x x}} d x$ diverges), the magnetic field vanishes there, and the circular radius $r=\sqrt{-g_{\varphi \varphi}}$ also tends to zero, so that the whole configuration is closed in nature, without spatial infinity, and with finite total magnetic field energy per unit length along the $z$ axis.

As we saw in Section 3.1, such a free magnetic field cannot support a rotating counterpart of Melvin's solution, but Einstein-Maxwell solutions with a longitudinal magnetic field are obtained in the presence of perfect fluids with electric currents. Let us briefly discuss their main features.

In all cases under consideration, the magnetic field is directed along the $z$ axis and has the constant magnitude $B=h$, while the metric has the form

$$
\begin{equation*}
d s^{2}=(d t-E d \varphi)^{2}-\mathrm{e}^{2 \beta} d x^{2}-d z^{2}-\mathrm{e}^{2 \beta} d \varphi^{2} \tag{49}
\end{equation*}
$$

and $E$ is determined byEquation (47). Note that both $\varphi$ and $x$ are dimensionless while $t, z$ and $\mathrm{e}^{\beta}$ have the dimension of length.

## Dustlike Matter, Equation (46)

Let us begin with the case $w=0$. For $E(x)$ we find

$$
\begin{align*}
& E=E_{0}+\frac{\omega_{0} r_{0}^{2}}{k} \mathrm{e}^{2 k x}=E_{0}+\frac{\omega_{0}}{k} r^{2} \\
& r=r_{0} \mathrm{e}^{k x}, \quad E_{0}=\text { const } \tag{50}
\end{align*}
$$

where $E_{0}$ is an integration constant. In terms of the coordinate $r$, the metric reads

$$
\begin{equation*}
d s^{2}=\left(d t-E d \varphi^{2}\right)^{2}-k^{-2} d r^{2}-d z^{2}-r^{2} d \varphi^{2} \tag{51}
\end{equation*}
$$

The symmetry axis $r=0$ is regular in the case $E_{0}=0, k=1$ (The axis regularity conditions require $[1,3,43]$ finite values of the curvature invariants plus local flatness (sometimes also called "elementary flatness") as a correct circumference to radius ratio for small circles around the axis, which in our case leads to the condition $\mathrm{e}^{-2 \alpha} R^{\prime 2} \rightarrow 1$, where $R=\sqrt{-g_{33}}$.). Also, in this case

$$
\begin{equation*}
g_{33}=E^{2}-r^{2}=r^{2}\left(-1+\omega_{0}^{2} r^{2}\right) \tag{52}
\end{equation*}
$$

changes its sign at $r=\omega^{-1}$, and at larger $r$ the lines of constant $t, r, z$ (coordinate circles) are timelike, thus being closed timelike curves (CTCs) violating causality.

Solution 1, Equation (42)
For $w<0$, with (42), for $E(x)$ we calculate

$$
\begin{equation*}
E=E_{0}+\frac{2 k \omega_{0}}{m^{2}} \tanh k x \tag{53}
\end{equation*}
$$

The metric has the form

$$
\begin{equation*}
d s^{2}=(d t-E d \varphi)^{2}-d z^{2}-\frac{k^{2}}{m^{2} \cosh ^{2}(k x)}\left(d x^{2}+d \varphi^{2}\right) \tag{54}
\end{equation*}
$$

where, for convenience, we have rearranged the terms with $d z^{2}$ and $d x^{2}$ as compared to (49).
For $g_{33}$, similarly to (52), again putting $E_{0}=0$ and recalling the definition of $m$, we obtain

$$
\begin{equation*}
g_{33}=-\frac{k^{2}}{m^{2}}\left[1-\frac{1}{|w|} \tanh ^{2} k x\right] \tag{55}
\end{equation*}
$$

In this solution $x \in \mathbb{R}$, and at both extremes $x \rightarrow \pm \infty$ we have $r=\mathrm{e}^{\beta} \rightarrow 0$, i.e., these are two centers of symmetry (or poles) on the $(x, \varphi) 2$-surface, or two symmetry axes from the viewpoint of 3-dimensional space. However, as follows from (55), $g_{33}$ is positive (hence contains CTCs) where $|\tanh k x|>|w|$, that is, at large enough $|x|$, in circular regions around the two poles.

By choosing another value of the integration constant $E_{0}$ one can make one of the poles free from CTCs, at the expense of enlarging the CTC region around the other pole. One of the poles can even be made regular by a proper choice of the parameters. For example, choosing $E_{0}=2 k \omega_{0} / \mathrm{m}^{2}$, we obtain $E=0$ at $x=-\infty$, and it is easy to verify that the pole $x=-\infty$ is then regular under the condition $k=1$.

## Solution 2, Equation (43)

For $w>0, k>0$, with (43), for $E(x)$ we find

$$
\begin{equation*}
E=E_{0}-\frac{2 k \omega_{0}}{m^{2}} \operatorname{coth} k x \tag{56}
\end{equation*}
$$

The metric takes the form

$$
\begin{equation*}
d s^{2}=(d t-E d \varphi)^{2}-d z^{2}-\frac{k^{2}}{m^{2} \sinh ^{2}(k x)}\left(d x^{2}+d \varphi^{2}\right) \tag{57}
\end{equation*}
$$

It is convenient to introduce the new coordinate $y$ by substituting

$$
\begin{equation*}
\mathrm{e}^{-2 k x}=1-\frac{2 k}{y} \tag{58}
\end{equation*}
$$

after which we obtain

$$
\begin{equation*}
r^{2}=\mathrm{e}^{2 \beta}=\frac{y}{m^{2}}(y-2 k) . \quad E=E_{0}-\frac{2 \omega_{0}}{m^{2}} y \tag{59}
\end{equation*}
$$

The range $x>0$ is converted to $y \geq 2 k$, where $y=2 k$ is the axis of symmetry. The metric now has the form

$$
\begin{equation*}
d s^{2}=(d t-E d \varphi)^{2}-\frac{d y^{2}}{m y(y-2 k)}-d z^{2}-\frac{y}{m^{2}}(y-2 k) d \varphi^{2} \tag{60}
\end{equation*}
$$

Assuming $E_{0}=0$, for $g_{33}$ it is then easy to obtain the expression

$$
\begin{equation*}
g_{33}=\frac{y}{m^{2}}\left(\frac{y}{w}+2 k\right)>0 \tag{61}
\end{equation*}
$$

which means that CTCs are present everywhere, and actually this space-time has an incorrect signature, $(+--+)$ instead of $(+---)$.

However, with nonzero values of $E_{0}$ it becomes possible to get rid of CTCs in some part of space. Thus, choosing $E_{0}$ in such a way that $E=0$ at some $y_{0}>2 k$, we will obtain the normal sign $g_{33}<0$ in some range of $y$ around $y_{0}$.

## Solution 3, Equation (44)

In the case $w>0, k=0$, with (44), it is convenient to use the coordinate $r=1 /(m x)$, and then we obtain

$$
\begin{equation*}
E=E_{0}+\frac{2 \omega_{0}}{r}, \quad d s^{2}=(d t-E d \varphi)^{2}-\frac{d r^{2}}{m^{2} r^{2}}-d z^{2}-r^{2} d \varphi^{2} \tag{62}
\end{equation*}
$$

and assuming $E_{0}=0$, we arrive at

$$
\begin{equation*}
g_{33}=-r^{2}\left(1-\frac{2 \omega_{0}^{2}}{m^{2}}\right)=r^{2} \frac{1-w}{2 w}>0 \tag{63}
\end{equation*}
$$

We again obtain a configuration with an incorrect signature, possessing CTCs at all $r$. However, as in the previous case, by choosing $E_{0}$ so that $E=0$ at some $r=r_{0}$ we can provide a CTC-free region in a thick layer around $r=r_{0}$.

## 5. Wormholes

With the solution (45) for $r=\mathrm{e}^{\beta}$, the range of $x$ is $x \in(-\pi /(2 \bar{k}), \pi /(2 \bar{k}))$, where $\bar{k}=-k>0$, and we see that $r \rightarrow \infty$ on both ends, confirming the wormhole nature of this configuration, where $x=0$ is the wormhole throat (minimum of $r$ ). Substituting $y=\bar{k} \tan \bar{k} x$, we obtain the metric in the form

$$
\begin{equation*}
d s^{2}=(d t-E d \varphi)^{2}-\frac{d y^{2}}{m^{2}\left(\bar{k}^{2}+y^{2}\right)}-d z^{2}-\frac{\bar{k}^{2}+y^{2}}{m^{2}} d \varphi^{2} \tag{64}
\end{equation*}
$$

where $y \in \mathbb{R}$, and $y=0$ is the throat; furthermore,

$$
\begin{equation*}
E=E_{0}+\frac{2 \omega_{0}}{m^{2}} y \tag{65}
\end{equation*}
$$

and for $g_{33}$ in the case $E_{0}=0$ (which makes the solution symmetric with respect to $y=0$ ) it follows

$$
\begin{equation*}
g_{33}=-\frac{\bar{k}^{2}}{m^{2}}+\frac{1-w}{2 w} \frac{y^{2}}{m^{2}} . \tag{66}
\end{equation*}
$$

The expression (66) shows that CTCs are absent around the throat, at $y^{2}<2 \bar{k}^{2} w /(1-w)$, while at larger $|y|$ the CTCs emerge.

Let us note that in the limit $w \rightarrow 1$, so that the fluid EoS tends to that of maximally stiff matter, the magnetic field disappears $(h \rightarrow 0$ according to (37)), and the whole solution tends to the one obtained in [39] for a cylindrical wormhole with stiff matter.

As always with rotating cylindrical wormhole solutions, these wormholes do not have a flat-space asymptotic behavior at large $|x|$, which makes it impossible to interpret them as objects that can be observed from regions with small curvature. To overcome this problem, it has been suggested [21] to cut out of the obtained wormhole solution a regular region, containing a throat, and to place it between two flat regions, thus making the whole system manifestly asymptotically flat. However, to interpret such a "sandwich" as a single space-time, it is necessary to identify the internal and external metrics on the junction surfaces $\Sigma_{+}$and $\Sigma_{-}$, which should be common for these regions. The internal region will be described in the present case by (64), (65)). Furthermore, since the internal metric contains rotation, the external Minkowski metric should also be taken in a rotating reference frame.

Thus we take the Minkowski metric in cylindrical coordinates, $d s_{\mathrm{M}}^{2}=d t^{2}-d X^{2}-d z^{2}-X^{2} d \varphi^{2}$, and convert it to a rotating reference frame with angular velocity $\Omega=$ const by substituting $\varphi \rightarrow$ $\varphi+\Omega t$, so that

$$
\begin{equation*}
d s_{\mathrm{M}}^{2}=d t^{2}-d X^{2}-d z^{2}-X^{2}(d \varphi+\Omega d t)^{2} \tag{67}
\end{equation*}
$$

In the notations of (1), the relevant quantities in (67) are

$$
\begin{align*}
& \mathrm{e}^{2 \gamma}=1-\Omega^{2} X^{2}, \quad \mathrm{e}^{2 \beta}=\frac{X^{2}}{1-\Omega^{2} X^{2}} \\
& E=\Omega X^{2}, \quad \omega=\frac{\Omega}{1-\Omega^{2} X^{2}} . \tag{68}
\end{align*}
$$

This stationary metric admits matching with the internal metric at any $|X|<1 /|\Omega|$, inside the "light cylinder" $|X|=1 /|\Omega|$ on which the linear rotational velocity coincides with the speed of light.

Making use of the symmetry of (64), let us assume that the internal region is $-y_{*}<y<y_{*}$, so that the junction surfaces $\Sigma_{ \pm}$are situated at $y= \pm y_{*}$, to be identified with $X_{ \pm}= \pm X_{*}$ in Minkowski space, respectively, so that the external flat regions are $X<-X_{*}$ and $X>X_{+}$. Matching is achieved if we identify there the two metrics, so that

$$
\begin{equation*}
[\beta]=0, \quad[\mu]=0, \quad[\gamma]=0, \quad[E]=0, \tag{69}
\end{equation*}
$$

where, as usual, the brackets $[f]$ denote a discontinuity of any function $f$ across the surface. Under the conditions (69), we can suppose that the coordinates $t, z, \phi$ are the same in the whole space. At the same time, there is no need to adjust the choice of radial coordinates on different sides of the junction surfaces since the quantities involved in all matching conditions are insensitive to possible reparametrizations of $y$ or $X$.

Having identified the metrics, we certainly did not adjust their normal derivatives, whose jumps are well known to determine the properties of matter filling a junction surface $\Sigma$ and forming there a thin shell. The SET $S_{a}^{b}$ of such a thin shell is calculated using the Darmois-Israel formalism [44-46], and in the present case of a timelike surface, $S_{a}^{b}$ is related to the extrinsic curvature $K_{a}^{b}$ of $\Sigma$ as

$$
\begin{equation*}
S_{a}^{b}=-(8 \pi G)^{-1}\left[\tilde{K}_{a}^{b}\right], \quad \tilde{K}_{a}^{b}:=K_{a}^{b}-\delta_{a}^{b} K_{c}^{c}, \tag{70}
\end{equation*}
$$

where the indices $a, b, c=0,2,3$. The general expressions for nonzero components of $\tilde{K}_{a}^{b}$ for surfaces $x=$ const in the metric (1) are [39]

$$
\begin{align*}
& \tilde{K}_{00}=-\mathrm{e}^{-\alpha+2 \gamma}\left(\beta^{\prime}+\mu^{\prime}\right) \\
& \tilde{K}_{03}=-\frac{1}{2} \mathrm{e}^{-\alpha} E^{\prime}+E \mathrm{e}^{-\alpha}\left(\beta^{\prime}+\gamma^{\prime}+\mu^{\prime}\right) \\
& \tilde{K}_{22}=\mathrm{e}^{-\alpha+2 \mu}\left(\beta^{\prime}+\gamma^{\prime}\right) \\
& \tilde{K}_{33}=\mathrm{e}^{-\alpha+2 \beta}\left(\gamma^{\prime}+\mu^{\prime}\right)+\mathrm{e}^{-\alpha-2 \gamma}\left[E E^{\prime}-E^{2}\left(\beta^{\prime}+2 \gamma^{\prime}+\mu^{\prime}\right)\right] \tag{71}
\end{align*}
$$

From (71) it is straightforward to find $S_{b}^{a}$ on the surfaces $\Sigma_{ \pm}$. However, our interest is not in finding these quantities themselves but, instead, a verification of whether or not the resulting SET $S_{b}^{a}$ satisfies the WEC. Let us use for this purpose the necessary and sufficient conditions obtained in a general form in [38], see also a detailed description in [39]. These conditions are

$$
\begin{align*}
& a+c+\sqrt{(a-c)^{2}+4 d^{2}} \geq 0  \tag{72}\\
& a+c+\sqrt{(a-c)^{2}+4 d^{2}}+2 b \geq 0  \tag{73}\\
& a+c \geq 0 \tag{74}
\end{align*}
$$

where

$$
\begin{array}{ll}
a=-\left[\mathrm{e}^{-\alpha}\left(\beta^{\prime}+\mu^{\prime}\right)\right], & b=\left[\mathrm{e}^{-\alpha}\left(\beta^{\prime}+\gamma^{\prime}\right)\right] \\
c=\left[\mathrm{e}^{-\alpha}\left(\gamma^{\prime}+\mu^{\prime}\right)\right], & d=-[\omega] \tag{75}
\end{array}
$$

Let us discuss, for certainty, the conditions on $\Sigma_{+}: y=y_{*}, X=X_{*}$ with our metrics (64), (65) and (67). Among the matching conditions (69), $[\mu]=0$ holds automatically, while to satisfy the condition $[\gamma]=0$ we will rescale the time coordinate in the internal region according to

$$
\begin{equation*}
t=\sqrt{P} \tilde{t}, \quad P:=1-\Omega^{2} X^{2} \tag{76}
\end{equation*}
$$

and use the new coordinate $\tilde{t}$, with which, instead of $E$, we must use $E \sqrt{P}$ in all formulas. The remaining two conditions (69) yield

$$
\begin{equation*}
\Omega X^{2}=\frac{2 \omega_{0}}{m} y \sqrt{P}, \quad \frac{k^{2}+y^{2}}{m^{2}}=\frac{X^{2}}{P} \tag{77}
\end{equation*}
$$

where, without risk of confusion, we omit the asterisk at $X$ and $y$. With these conditions, there are four independent parameters of the system, for example, we can choose as such parameters

$$
\begin{equation*}
X, y, P, n=\frac{2 \omega_{0}^{2}}{m}=\frac{w+1}{2 w} \tag{78}
\end{equation*}
$$

The other parameters are expressed in their terms as

$$
\begin{equation*}
\Omega=\frac{\sqrt{1-P}}{X}, \quad \omega_{0}=\frac{n y \sqrt{P}}{X \sqrt{1-P}}, \quad k^{2}=\frac{y^{2}(2 n-1+P)}{1-P} \tag{79}
\end{equation*}
$$

Now we can calculate the quantities (75), with $[f]=f_{\text {out }}-f_{\text {in }}$ on $\Sigma_{+}$:

$$
\begin{equation*}
a=\frac{y P^{3 / 2}-1}{P X}, \quad b=\frac{1-y \sqrt{P}}{X}, \quad c=-\frac{1-P}{P X}, \quad d=\frac{n y P^{3 / 2}-1+P}{P X \sqrt{1-P}} . \tag{80}
\end{equation*}
$$

It can be easily verified that the conditions (72)-(74) are satisfied as long as

$$
\begin{equation*}
y \geq \frac{2-P}{P^{3 / 2}} \tag{81}
\end{equation*}
$$

in full analogy with the corresponding calculation in [39].
We have shown that under the condition (81) the WEC holds on $\Sigma_{+}$. Now, what changes on the surface $\Sigma_{-}$specified by $X=-X_{*}<0$ and $y=-y_{*}<0$, where we must take $[f]=f_{\text {out }}-f_{\text {in }}$ for any function $f$ ? As in [39], it can be verified that the parameters $a, b, c$ do not change from (80) if we replace $X$ with $|X|$ (we denote, as before, $y=y_{*}>0$ ). For $d=-[\omega]$ there will be another expression since, according to (69), $\Omega\left(\Sigma_{-}\right)=-\Omega\left(\Sigma_{+}\right)$, while in the internal solution $\omega\left(\Sigma_{-}\right)=\omega\left(\Sigma_{+}\right)$, hence on $\Sigma_{-}$

$$
d \mapsto d_{-}=-\frac{n|y| P^{3 / 2}+1-P}{|X| P \sqrt{1-P}}
$$

so that $\left|d_{-}\right|>|d|$, making it even easier to satisfy the WEC requirements. As a result, the WEC holds under the same condition (81), providing a wormhole model which is completely phantom-free.

We can also notice that from (79) it follows $y_{*}^{2}<k^{2}$, therefore, $y^{2}<k^{2}$ in the whole internal region, which is thus free from CTCs.

There is one more point to bear in mind: since there is a $z$-directed magnetic field in the internal region, we must suppose that there are some surface currents on $\Sigma_{ \pm}$in the $\varphi$ direction. Their values can be easily calculated using the Maxwell equations $\nabla_{\nu} F^{\mu \nu}=J^{\mu}$. Indeed, say, $\Sigma_{+}\left(\begin{array}{ll}x=x_{*}\end{array}\right)$ separates the region where $F^{\mu \nu}=0$ from the one with nonzero $F^{\mu \nu}$, therefore at their junction we have $J^{\mu}=-\delta\left(x-x_{*}\right) J^{\mu}\left(x_{*}-0\right)$, so that the surface current is $J^{a}=-\left.J^{\mu}\left(x_{*}-0\right)\right|_{\mu=a}$. Similarly,
on $\Sigma_{-}\left(x=-x_{*}\right)$ we obtain $J^{a}=\left.J^{\mu}\left(-x_{*}+0\right)\right|_{\mu=a}$. In our wormhole configurations we obtain, according to (23), (24), (27) and taking into account that $\gamma=\mu \equiv 0$,

$$
\begin{equation*}
J^{a}=\left(J^{0}, 0,0\right), \quad J^{0}\left(\Sigma_{ \pm}\right)=\mp h \omega_{0} \tag{82}
\end{equation*}
$$

Thus the surface currents on $\Sigma_{ \pm}$have only the temporal component, i.e., they are comoving to the matter and current in the internal region.

As is the case with the internal wormhole solution, in the limit $w \rightarrow 1$ (hence $n \rightarrow 1$ ) the whole twice asymptotically flat construction tends to the one obtained in [39] with a stiff matter source.

## 6. Concluding Remarks

We have obtained a family of stationary cylindrically symmetric solutions to the Einstein-Maxwell equations in the presence of perfect fluids with $p=w \rho,|w|<1$. Some of them (Solutions 1-3) contain a symmetry axis which can be made regular by properly choosing the solution parameters. The only geometry of closed type belongs to Solution 1, Equations (42) and (53)-(55). Unlike Melvin's solution and like all other solutions with rotation, it inevitably contains a region where $g_{33}>0$, so that the coordinate circles parametrized by the angle $\varphi$ are timelike, violating causality.

The wormhole models discussed here are of interest as new examples of phantom-free wormholes in general relativity, respecting the WEC. As in other known examples [38,39], such a result is achieved owing to the exotic properties of vortex gravitational fields with cylindrical symmetry, and their asymptotic behavior making them potentially observable from flat or weakly curved regions of space is provided by joining flat regions on both sides of the throat. Such a complex structure is necessary because asymptotic flatness at large circular radii cannot be achieved in any cylindrical solutions with rotation. The present family of models with a magnetic field, parametrized by the equation-of-state parameter $w<1$, tends to the one obtained in [39] in the limit $w \rightarrow 1$, in which the magnetic field vanishes.

Let us mention that other static or stationary wormhole models with proper asymptotic behavior and matter sources respecting the WEC have been obtained in extensions of general relativity, such as the Einstein-Cartan theory [47,48], Einstein-Gauss-Bonnet gravity [49], multidimensional gravity including brane worlds [50,51], theories with nonmetricity [52], Horndeski theories [53], etc.

One can also notice that the same trick as was used with wormhole models, that is, joining a flat region taken in a rotating reference frame, can be used as well with solutions possessing a symmetry axis. It is important that in all such cases the surface to be used as a junction should not contain CTCs (in other words, there should be, as usual, $g_{33}<0$ ) because $g_{33}<0$ in the admissible part of flat space, while $g_{33}$ taken from the external and internal regions should be identified at the junction. In this way one can obtain completely CTC-free models of extended cosmic strings with rotation.

A possible observer can be located far from such an extended string or wormhole configuration and be at rest in a nonrotating frame in flat space, other than the one used for the object construction. A question of interest is that of their observational appearance. If such a stringlike object does not emit radiation of its own, it can undoubtedly manifest itself by gravitational lensing in the same way as is discussed for cosmic strings (certainly if there is the corresponding angular deficit in the external, locally flat region); see, e.g., [54-56] and references therein. Moreover, possible signals scattered in the strong field region can carry information of interest on the nature and motion of rotating matter that forms such objects.

An evident further development of this study can be a search for other rotating configurations with electromagnetic fields, possibly including radiation in different directions in the spirit of [57], where radial, azimuthal and longitudinal radiation flows were considered as sources of gravity in space-times with the metric (1). Another set of problems concerns electrostatics in the fields of extended strings or wormholes with sources including electromagnetism. As follows from [58], even in simpler,
partly conical cylindrical geometries with thin shells electrostatics turns out to be rather interesting and complex.
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