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Rephasing Invariant for Three-Neutrino Oscillations Governed by a Non-Hermitian Hamiltonian

Dmitry V. Naumov, Vadim A. Naumov * and Dmitry S. Shkirmanov

JINR, Dubna 141980, Russia; dnaumov@jinr.ru (D.V.N.); shkirmanov@theor.jinr.ru (D.S.S.)
* Correspondence: vnaumov@theor.jinr.ru

Received: 18 June 2020; Accepted: 13 July 2020; Published: 3 August 2020

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Abstract: Time-reversal symmetry is broken for mixed and possibly unstable Dirac neutrino propagation through absorbing media. This implies that interplay between the neutrino mixing, refraction, absorption and/or decay can be described by non-Hermitian quantum dynamics. We derive an identity which sets up direct connection between the fundamental neutrino parameters (mixing angles, CP-violating phase, mass-squared splittings) in vacuum and their effective counterparts in matter.

Keywords: neutrino oscillations in matter; rephasing invariant; neutrino absorption

1. Introduction

High-energy neutrinos, unique messengers of the most violent processes that occurred during the evolution of the Universe, are under extensive study by the modern neutrino telescopes (see reference [1] for a comprehensive recent review and further references). The propagation of these particles through dense matter requires a theoretical consideration accounting for two major phenomena. (i) The quantum coherence and decoherence, most clearly manifested in the neutrino oscillation phenomenon, firmly established experimentally [2–10]. The corresponding theoretical approaches rely on either quantum mechanical [11–13] or quantum field theory [14–20] considerations. (ii) Neutrino production, inelastic interactions, and possible decays, typically considered by the classical transport theory [21–23].

In this paper we consider a more particular aspect of the full problem—propagation of high-energy neutrinos in dense environment with accounting for neutrino masses, mixing, CP violation, refraction, and absorption. We do not consider neutrino energy loss through neutral-current (NC) interactions and charged-current (CC) induced reaction chains, but of course we take into account disappearance of the neutrinos due to all these processes. In other words, the formalism does not predict the energy spectrum transformation due to the energy losses. This is acceptable in the case of sufficiently narrow boundary energy spectrum or nearly-monochromatic neutrino source, when we are interested in the flavor evolution at the same energy as on the boundary or in the source (e.g., annihilating non-relativistic WIMS). Since, in this statement of the problem, neutrinos simply disappear with time (due to both CC and NC interactions), the time-reversal symmetry is broken and the neutrino flavor evolution can be described within a non-Hermitian formulation of quantum mechanics. The inclusion of the neutrino energy loss effects is of course very important in more general and practically interesting conditions, but it will also require the more universal formalism, like quantum kinetic equations or a hybrid (approximate) technique based on the non-Hermitian dynamics and classical transport theory. One of the simplest realization of the hybrid approach is in the replacement of the mean free paths Λ_α (see Section 2) to effective functions $\widetilde{\Lambda}_\alpha$ derived from solution of the classical transport equations [22] for the given initial spectrum/source and given profiles of density and composition of the medium along the neutrino beam direction. Such a method conserves the generic results of the present study. The approach based on the the non-Hermitian dynamics has been considered earlier in reference [24], for a generic three-level system and latter in reference [25], for a simplified two-flavor mixing model, which included either the mixing between the active (standard) or active and sterile neutrinos; see also references [26,27] for recent developments and further references. Here we follow these studies and consider the Standard Model's three-neutrino species.

Consideration of the neutrino oscillation phenomenon in the simplest adiabatic regime usually requires a diagonalization of the corresponding Hamiltonian. The instantaneous eigenstates are defined not uniquely but up to certain ("rephasing") transformations, keeping the observable transition and survival probabilities $P_{\alpha\beta} \equiv P_{\nu_\alpha \to \nu_\beta}$ invariant. This is discussed in greater details in Section 3. An important class of observables invariant under the same transformations is known as flavor-symmetric Jarlskog invariants, introduced by Jarlskog [28] for quarks. In the three-generation case, the nine Jarlskog invariants are equal and uniquely determine the amount of CP violation in the quark sector of the Standard Electroweak Model. Similar rephasing invariants determine the amount of CP violation in the lepton sector. In present work, we found an extension of the Jarlskog invariants for the dissipating three-neutrino system; this is one of the results of this study.

As was first pointed out by Wolfenstein [29], the neutrino mixing is modified when neutrinos propagate through normal C-asymmetric matter, owing to the CC forward scattering of electron neutrinos on electrons in matter. In some circumstances, these ghostly interactions may drastically modify the neutrino oscillation pattern [30,31]. It is however interesting that a nontrivial observable proportional to the Jarlskog invariant, J, is also a "matter invariant". More precisely, in references [32,33] (see also reference [34] for a relevant result), an identity has been found which relates the products of J and neutrino squared-mass splittings, $\Delta m_{ij} = m_i^2 - m_j^2$, in vacuum and in matter:

$$J\Delta m_{23}^2 \Delta m_{31}^2 \Delta m_{12}^2 = \widetilde{J}\Delta \widetilde{m}_{23}^2 \Delta \widetilde{m}_{31}^2 \Delta \widetilde{m}_{12}^2. \tag{1}$$

Here tilde marks the quantities perturbed by the matter. This identity has proved to be useful in various phenomenological and mathematical aspects of the oscillating neutrino propagation in matter [35–80]. The main result of the present work is a generalization of the identity (1) to the case of the neutrino propagation in absorbing media, which can be described by non-Hermitian dynamics. Since the quantities in the RHS of Equation (1) are defined as instantaneous functions, the adiabaticity conditions are not formally essential (so we do not study the corresponding constraints). However, the actual usage of the generalized identity is mainly reasonable in the environments where the neutrino flavors evolve adiabatically or quasi-adiabatically. It is also pertinent to note that the adiabatic solution can be adapted to form the basis of a numerical method: by dividing the medium into a number of layers with slowly varying densities, the solution is obtained as chronological product of the (non-unitary) evolution operators for each layer [25]. Though, our primary interest is motivated by the neutrino oscillation phenomenon, the obtained identity has a much wider range of applicability relevant to arbitrary quantum three-level system governed by a non-Hermitian Hamiltonian.

The paper is organized as follows. The master equation and appropriate theoretical framework are considered in Section 2. In Section 3 we introduce two "mixing matrices" for a generic three-level quantum dissipative system, describing, in particular, the neutrino mixing, refraction, decay, and absorption due to standard or nonstandard inelastic neutrino-matter interactions. We show that these matrices are not uniquely defined. In Section 4 we study the generalized "rephasing" and "dynamic" invariants constructed from the elements of the mixing matrices and of the Hamiltonian matrix, respectively. Then we put forward the relation generalizing the identity (1). The proof of this relation is delivered in Appendix A. Finally, we draw the summary in Section 5. Some auxiliary information is summarized in Appendix B.

2. Master Equation

The Schrödinger equation

$$i\frac{d}{dt}|\nu_f(t)\rangle = \mathbf{H}(t)|\nu_f(t)\rangle \tag{2}$$

describes the time evolution of the three-neutrino state

$$|\nu_f(t)\rangle = \big(|\nu_e(t)\rangle, |\nu_\mu(t)\rangle, |\nu_\tau(t)\rangle\big)^T \tag{3}$$

governed by a Hamiltonian $\mathbf{H}(t)$. The bold face is used for matrices in what follows. The flavor ν_α ($\alpha = e, \mu, \tau$) and mass ν_i ($i = 1, 2, 3$) eigenstates are related to each other as

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} V_{\alpha i} |\nu_i\rangle. \tag{4}$$

This definition differs from that used in quantum filed theoretical (QFT) description. Their relationship is given by $V_{\text{QFT}} = V^*_{\text{QM}}$. Since the observables are flavor changing probabilities $P_{\alpha\beta}(t) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2$, it is convenient to rewrite Equation (2) as one for the corresponding amplitudes

$$S_{\beta\alpha}(t) = \langle\nu_\beta|\nu_\alpha(t)\rangle \tag{5}$$

as follows

$$i\frac{d}{dt}\mathbf{S}(t) = \mathbf{H}(t)\mathbf{S}(t) = \Big[\mathbf{V}\mathbf{H}_0\mathbf{V}^\dagger + \mathbf{W}(t)\Big]\,\mathbf{S}(t), \quad (\mathbf{S}(0) = \mathbf{1})\,, \tag{6}$$

where $\mathbf{S}(t)$ is a matrix with elements $S_{\alpha\beta}(t)$ (evolution operator), $\mathbf{V}$ is the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix with elements $V_{\alpha i}$, $\mathbf{H}_0$, and $\mathbf{W}(t)$ are the free and neutrino-matter interaction Hamiltonians, respectively,

$$\mathbf{H}_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}, \quad \mathbf{W}(t) = -p_\nu \begin{pmatrix} n_e(t)-1 & 0 & 0 \\ 0 & n_\mu(t)-1 & 0 \\ 0 & 0 & n_\tau(t)-1 \end{pmatrix}, \tag{7}$$

$E_i = \sqrt{p_\nu^2 + m_i^2} \simeq p_\nu + m_i^2/2p_\nu$ and m_i are, respectively, the total energies and masses of the neutrino mass eigenstates, and $n_\alpha(t)$ are the complex indices of refraction; where we assume, as usual, that neutrinos are ultrarelativistic, $p_\nu^2 \simeq E_\nu^2 \gg \max\left(m_i^2\right)$. In normal matter, the functions n_α are linear with respect to the densities of scatterers. The same is also true for hot media under the assumption that introduction of a finite temperature does not break the coherent condition [81]. With these assumptions

$$n_\alpha(t) = 1 + \frac{2\pi N_0 \rho(t)}{p_\nu^2} \sum_k Y_k(t) f_{\nu_\alpha k}(0), \tag{8}$$

where $N_0 = 6.022 \times 10^{23}\,\text{cm}^{-3}$, $f_{\nu_\alpha k}(0)$ is the amplitude for the ν_α zero-angle scattering from particle k ($k = e, p, n, \ldots$), $\rho(t)$ is the density of the matter (in g/cm^3) and $Y_k(t)$ is the number of particles k per AMU in the point t of the medium. The optical theorem says (see, e.g., reference [82]):

$$\text{Im}\left[f_{\nu_\alpha k}(0)\right] = \frac{p_\nu}{4\pi}\,\sigma^{\text{tot}}_{\nu_\alpha k}(p_\nu)\,, \tag{9}$$

where $\sigma^{\text{tot}}_{\nu_\alpha k}(p_\nu)$ is the total cross section for $\nu_\alpha k$ scattering due to both CC and NC interactions. This implies that

$$p_\nu \text{Im}\left[n_\alpha(t)\right] = \frac{N_0\rho(t)}{2} \sum_k Y_k(t)\sigma^{\text{tot}}_{\nu_\alpha k}(p_\nu) = \frac{1}{2\Lambda_\alpha(t)}, \tag{10}$$

where $\Lambda_\alpha(t)$ is the (energy dependent) mean free path of neutrino ν_α in the point t of the medium.

It is convenient to transform Equation (6) into the one with a traceless Hamiltonian. For this purpose we define the matrix

$$\widetilde{\mathbf{S}}(t) = \exp\left\{\frac{i}{3}\int_0^t \text{Tr}\left[\mathbf{H}_0 + \mathbf{W}(t')\right] dt'\right\} \mathbf{S}(t). \tag{11}$$

After substituting Equation (11) into Equation (6), we have

$$i\frac{d}{dt}\widetilde{\mathbf{S}}(t) = \mathbf{H}(t)\widetilde{\mathbf{S}}(t), \quad \widetilde{\mathbf{S}}(0) = \mathbf{1}, \tag{12}$$

where

$$\mathbf{H}(t) = \begin{pmatrix} \mathcal{W}_e - q_e & \mathcal{H}_\tau & \mathcal{H}_\mu^* \\ \mathcal{H}_\tau^* & \mathcal{W}_\mu - q_\mu & \mathcal{H}_e \\ \mathcal{H}_\mu & \mathcal{H}_e^* & \mathcal{W}_\tau - q_\tau \end{pmatrix}. \tag{13}$$

The constants $\mathcal{W}_\alpha$ and $\mathcal{H}_\alpha$ are determined by the elements of the PMNS matrix, $\mathbf{V} = \|V_{\alpha i}\|$, and by the neutrino masses m_i:

$$\begin{gathered} \mathcal{W}_\alpha = \sum_i |V_{\alpha i}|^2 \Delta_i, \quad \mathcal{H}_\alpha = \sum_i \eta_\alpha^{\beta\gamma} V_{\beta i} V_{\gamma i}^* \Delta_i, \\ \Delta_i = \frac{m_i^2 - \langle m^2 \rangle}{2p_\nu}, \quad \langle m^2 \rangle = \frac{1}{3}\sum_i m_i^2. \end{gathered} \tag{14}$$

The PMNS matrix is usually parameterized in terms of three mixing angles and the CP-violating (Dirac) phase (see Appendix B); the two additional phases present in the Majorana case do not affect the neutrino oscillation pattern in matter. Here and below, the symbol $\eta_\alpha^{\beta\gamma}$ is defined to be 1 if the triplet (α, β, γ) is a cyclic permutation of the indices (e, μ, τ) and zero otherwise.

The traceless Hamiltonian (13) depends on the distance $L = t$ through the set of optical potentials, $\mathbf{q} = (q_e, q_\mu, q_\tau)$, related to the neutrino indices of refraction, $n_\alpha(t)$, for a given medium:

$$q_\alpha(t) = p_\nu[n_\alpha(t) - \langle n(t) \rangle], \quad \langle n(t) \rangle = \frac{1}{3}\sum_\alpha n_\alpha(t). \tag{15}$$

It is seen from Equation (15) that evolution of the neutrino flavors in arbitrary medium depends on no more than two independent potentials $q_\alpha(t)$ due to the identity

$$\sum_\alpha q_\alpha(t) = 0.$$

In general, the indices of refraction $n_\alpha(t)$ and thus optical potentials $q_\alpha(t)$ are complex functions (see below). Owing to radiative electroweak contributions, the real parts of the potentials for different neutrino flavors α differ in magnitude, in both normal cold media [83,84] and hot CP-symmetric plasma (such as the early Universe) [81]. The imaginary parts of the potentials are given by

$$\operatorname{Im} q_\alpha(t) = \frac{1}{2}\left[\frac{1}{\Lambda_\alpha(t)} - \frac{1}{\Lambda(t)}\right], \quad \frac{1}{\Lambda(t)} = \frac{1}{3}\sum_\alpha \frac{1}{\Lambda_\alpha(t)}, \tag{16}$$

and are in general nonzero functions of neutrino energy and distance. This makes the Hamiltonian (13) non-Hermitian.

The neutrino flavor changing oscillation probabilities are just the squared absolute values of the elements of the evolution matrix $\mathbf{S}(t)$,

$$P\left[\nu_\alpha(0) \to \nu_{\alpha'}(t)\right] \equiv P_{\alpha\alpha'}(t) = |S_{\alpha'\alpha}(t)|^2. \tag{17}$$

Taking into account Equations (7), (10), (11) and (17) yields

$$P_{\alpha\alpha'}(t) = A(t)\left|\widetilde{S}_{\alpha'\alpha}(t)\right|^2, \tag{18}$$

where
$$A(t) = \exp\left[-\int_0^t \frac{dt'}{\Lambda(t')}\right]. \tag{19}$$

This factor accounts for the attenuation (due to inelastic scattering) of all flavors in the mean. It is apparent that in the absence of mixing and refraction (that is an appropriate approximation at superhigh energies),
$$\widetilde{S}_{\alpha'\alpha} = \delta_{\alpha'\alpha} \exp\left[-\int_0^t \operatorname{Im} q_\alpha(t')dt'\right]$$
and, according to Equations (18) and (19), the survival and transition probabilities reduce to the "classical limit":
$$P_{\alpha\alpha'}(t) = \delta_{\alpha\alpha'} \exp\left[-\int_0^t \frac{dt'}{\Lambda_\alpha(t')}\right].$$

Owing to the complex potentials q_α, the Hamiltonian in Equation (13) is non-Hermitian and the evolution matrix $\widetilde{\mathbf{S}}(t)$ is non-unitary. It is apparent that the matrix $\mathbf{H}(t)$ becomes Hermitian when one neglects differences in the mean free paths of neutrinos of different flavors. In this case, Equation (12) reduces to one describing the standard Mikheev–Smirnov–Wolfenstein (MSW) mechanism [29–31]. Clearly, this approximation may not be good for very thick environments and/or very high neutrino energies.

At essentially all energies, the CC total cross sections for e or μ production in the neutrino and antineutrino interaction with nucleons are well above the one for the τ-lepton production,
$$\sigma^{\text{CC}}_{\nu_{e,\mu}N} > \sigma^{\text{CC}}_{\nu_\tau N}, \quad \sigma^{\text{CC}}_{\overline{\nu}_{e,\mu}N} > \sigma^{\text{CC}}_{\overline{\nu}_\tau N}.$$

This is because of large value of the τ-lepton mass, m_τ, which leads to several consequences (see, e.g., references [85,86] and references therein):

(i) high neutrino energy threshold for τ production;
(ii) sharp shrinkage of the phase spaces for the CC interactions of ν_τ and $\overline{\nu}_\tau$ with protons, neutrons, and nuclei;
(iii) kinematic correction factors ($\propto m_\tau^2$) to the nucleon structure functions (the corresponding structures are negligible for the electron production and small for the muon production);
(iv) the differences $\sigma^{\text{CC}}_{\nu_{e,\mu}N} - \sigma^{\text{CC}}_{\nu_\tau N}$ and $\sigma^{\text{CC}}_{\overline{\nu}_{e,\mu}N} - \sigma^{\text{CC}}_{\overline{\nu}_\tau N}$ are relatively slow varying functions of (anti)neutrino energy, having gently sloping maxima in the range of 10–100 PeV and vanishing at super-high energies.

Since the Standard Model NC interactions are universal for all neutrino flavors, it is clear from Equation (16), that the NC contributions to the total cross sections are canceled out from $\operatorname{Im} q_\alpha$ and thus $\operatorname{Im}\left(q_{e,\mu} - q_\tau\right) > 0$ at all energies. However, nonstandard NC interactions may be in general different for different flavors and thus contribute to both real and imaginary parts of the potentials q_α. Moreover, flavor-changing interactions (see, e.g., references [87,88] and references therein) would contribute to the non-diagonal elements of the Hamiltonian making these t-dependent.

Similar situation, although in different and rather narrow energy range, holds for $\overline{\nu}_e$ interaction with electrons. This is a particular case for the C-asymmetric media (planets, stars, astrophysical jets, etc.) because of the W-boson resonance formed in the neighborhood of $E_\nu^{\text{res}} = m_W^2/2m_e \approx 6.33$ PeV through the reactions
$$\overline{\nu}_e e^- \to W^- \to \text{hadrons} \quad \text{and} \quad \overline{\nu}_e e^- \to W^- \to \overline{\nu}_\ell \ell^- \quad (\ell = e, \mu, \tau).$$

Just at the resonance peak, $\sigma^{\text{tot}}_{\overline{\nu}_e e} \approx 250\, \sigma^{\text{tot}}_{\overline{\nu}_e N}$ (see, e.g., references [89–91] and references therein).

We conclude this section by explicitly emphasizing that the master equation to be solved is given by Equation (12) and the relevant definitions are given by Equations (8) and (13)–(15).

3. Mixing Matrices In Matter

Solution of the master equation (12) in adiabatic approximation has been found in reference [24]. In the present study we do not use the explicit form of that solution. Moreover, below we will consider an abstract Hamiltonian, which is a 3×3 complex matrix $\mathbf{H}$ describing a generic 3-level quantum system with dissipation (through absorption, friction, decay, etc.); such a Hamiltonian may, in particular, be used to describe the nonstandard neutrino interactions and decay. Below, keeping in mind our particular problem (3ν oscillation in absorbing matter) we will use specific notation. In the most general case the Hamiltonian $\mathbf{H}$ depends on time through a set of real parameters $(x_1(t), \ldots, x_s(t)) \equiv \mathbf{x}(t)$. We define these parameters in such a way that $x_k(t) = 0$ in vacuum; in our particular case, $\mathbf{x} = \mathbf{q}$ and this condition holds automatically.

Let us now define the two "mixing matrices" $\mathbf{V}^{(\mathrm{m})}(\mathbf{x})$ and $\overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x})$ by the equations

$$\mathbf{H}(\mathbf{x})\mathbf{V}^{(\mathrm{m})}(\mathbf{x}) = \mathbf{V}^{(\mathrm{m})}(\mathbf{x})\mathbf{E}(\mathbf{x}), \quad \mathbf{H}^{\dagger}(\mathbf{x})\overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x}) = \overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x})\mathbf{E}^{\dagger}(\mathbf{x}), \tag{20}$$

with

$$\mathbf{E}(\mathbf{x}) = \mathrm{diag}\left(\mathcal{E}_{N_1}(\mathbf{x}), \mathcal{E}_{N_2}(\mathbf{x}), \mathcal{E}_{N_3}(\mathbf{x})\right). \tag{21}$$

The solution to Equations (20) can be found in two steps. First, one have to find the eigenvalues and eigenvectors of the matrices $\mathbf{H}$ and $\mathbf{H}^{\dagger}$,

$$\mathbf{H}(\mathbf{x})|N;\mathbf{x}\rangle = \mathcal{E}_N(\mathbf{x})|N;\mathbf{x}\rangle, \quad \mathbf{H}^{\dagger}(\mathbf{x})\overline{|N;\mathbf{x}\rangle} = \mathcal{E}_N^{*}(\mathbf{x})\overline{|N;\mathbf{x}\rangle}, \tag{22}$$

where

$$|N;\mathbf{x}\rangle = \begin{pmatrix} U_{Ne}(\mathbf{x}) \\ U_{N\mu}(\mathbf{x}) \\ U_{N\tau}(\mathbf{x}) \end{pmatrix}, \qquad \overline{|N;\mathbf{x}\rangle} = \begin{pmatrix} \overline{U}_{Ne}(\mathbf{x}) \\ \overline{U}_{N\mu}(\mathbf{x}) \\ \overline{U}_{N\tau}(\mathbf{x}) \end{pmatrix}, \tag{23}$$

with $N = -1, 0, +1$ or simply $-, 0, +$. For simplicity we will neglect possible degeneracy of the energy levels. Then the eigenvectors form a complete biorthonormal set:

$$\overline{\langle N';\mathbf{x}|}N;\mathbf{x}\rangle = \delta_{NN'}, \quad \sum_N |N;\mathbf{x}\rangle\overline{\langle N;\mathbf{x}|} = \mathbf{I}, \tag{24}$$

or, in the component-wise notation,

$$\sum_\alpha \overline{U}^{\,*}_{N'\alpha}(\mathbf{x})U_{N\alpha}(\mathbf{x}) = \delta_{NN'}, \quad \sum_N \overline{U}^{\,*}_{N\alpha}(\mathbf{x})U_{N\beta}(\mathbf{x}) = \delta_{\alpha\beta}. \tag{25}$$

Second, from simple algebra it follows that the matrices

$$\begin{aligned} \mathbf{U}(\mathbf{x}) &\equiv \|U_{\alpha j}(\mathbf{x})\| = \left(|N_1;\mathbf{x}\rangle, |N_2;\mathbf{x}\rangle, |N_3;\mathbf{x}\rangle\right), \\ \overline{\mathbf{U}}(\mathbf{x}) &\equiv \|\overline{U}_{\alpha j}(\mathbf{x})\| = \left(\overline{|N_1;\mathbf{x}\rangle}, \overline{|N_2;\mathbf{x}\rangle}, \overline{|N_3;\mathbf{x}\rangle}\right), \end{aligned} \tag{26}$$

satisfy Equations (20) and thus diagonalize the Hamiltonian matrix $\mathbf{H}$.

The solutions (26) are not however unique. In most general case, the following products

$$\mathbf{V}^{(\mathrm{m})}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathbf{D}^{\dagger}(\mathbf{x}), \quad \overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x}) = \overline{\mathbf{U}}(\mathbf{x})\overline{\mathbf{D}}(\mathbf{x}),$$

with arbitrary diagonal and nonsingular matrices $\mathbf{D}(\mathbf{x})$ and $\overline{\mathbf{D}}(\mathbf{x})$, also satisfy Equation (20). This freedom implies that not all elements of the mixing matrices $\mathbf{V}^{(\mathrm{m})}(\mathbf{x})$ and $\overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x})$ are physically observable. Recall that the eigenvectors have been built so that

$$\mathbf{U}(\mathbf{0}) = \overline{\mathbf{U}}(\mathbf{0}) = \mathbf{V}$$

and the following obvious conditions are assumed: $U_{i\alpha}(\mathbf{0}) = U_{N_i\alpha}(\mathbf{0}) = \overline{U}_{i\alpha}(\mathbf{0}) = \overline{U}_{N_i\alpha}(\mathbf{0}) = V_{i\alpha}$. Equations (20) and (21) are universal, i.e., they are hold true for any medium and for any value of the neutrino momentum. In particular, they are hold for vacuum. Therefore

$$\mathbf{V}^{(\mathrm{m})}(\mathbf{0}) = \overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{0}) = \mathbf{V}, \tag{27}$$

where $\mathbf{V}$ is the vacuum mixing matrix ("correspondence principle"). Hence, according to Equation (27), the matrices $\mathbf{D}(\mathbf{x})$ and $\overline{\mathbf{D}}(\mathbf{x})$ must satisfy the condition

$$\mathbf{D}(\mathbf{0}) = \overline{\mathbf{D}}(\mathbf{0}) = \mathbf{I}.$$

As a less trivial limiting case, let us consider a medium, where the imaginary part of the optic potentials can be neglected (this standard approximation is true in essence for any media if its thickness is much smaller than the neutrino mean free path). In this case the eigenvalues $\mathcal{E}_N(\mathbf{x})$ are real and the following inequalities are valid [32]:

$$\mathcal{E}_-(\mathbf{x}) \leq \mathcal{E}_0(\mathbf{x}) \leq \mathcal{E}_+(\mathbf{x}).$$

Considering these limiting cases one finds that the numeration of the diagonal elements in (21) (i.e., the one-to-one congruence $N_i \Leftrightarrow i$) is given by the neutrino mass hierarchy. For example, $N_1 = -1$, $N_2 = 0$, $N_3 = +1$ for the "natural hierarchy", $m_1^2 > m_2^2 > m_3^2$ but $N_1 = +1$, $N_2 = 0$, $N_3 = -1$ for the following case: $m_3^2 < m_2^2 < m_1^2$; other cases can be derived similarly. Thus, to simplify formulas, we will use the notation $\mathcal{E}_{N_i}(\mathbf{x}) = E_i(\mathbf{x})$, when it is suitable.

According to Equations (25) and (26)

$$\sum_\alpha U_{\alpha i}^* \overline{U}_{\alpha j} = \sum_\alpha \overline{U}_{\alpha i}^* U_{\alpha j} = \delta_{ij}, \tag{28}$$

or, equivalently,

$$\mathbf{U}^\dagger(\mathbf{x})\overline{\mathbf{U}}(\mathbf{x}) = \overline{\mathbf{U}}^\dagger(\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{I}. \tag{29}$$

It is reasonable to impose the same constraint on the mixing matrices:

$$\left[\mathbf{V}^{(\mathrm{m})}(\mathbf{x})\right]^\dagger \overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x}) = \left[\overline{\mathbf{V}}^{(\mathrm{m})}(\mathbf{x})\right]^\dagger \mathbf{V}^{(\mathrm{m})}(\mathbf{x}) = \mathbf{I}.$$

Then

$$\mathbf{D}^\dagger(\mathbf{x})\overline{\mathbf{D}}(\mathbf{x}) = \overline{\mathbf{D}}^\dagger(\mathbf{x})\mathbf{D}^\dagger(\mathbf{x}) = \mathbf{I}$$

and therefore

$$\mathbf{D}(\mathbf{x}) = \mathrm{diag}\left(e^{-a_1+ib_1}, e^{-a_2+ib_2}, e^{-a_3+ib_3}\right),$$
$$\overline{\mathbf{D}}(\mathbf{x}) = \mathrm{diag}\left(e^{+a_1+ib_1}, e^{+a_2+ib_2}, e^{+a_3+ib_3}\right),$$

where $a_k = a_k(\mathbf{x})$ and $b_k = b_k(\mathbf{x})$ are arbitrary real functions which vanish at $\mathbf{x} = \mathbf{0}$.

As is generally known (see for example [92]), the vacuum mixing matrix for Majorana neutrinos may be written in the form $\mathbf{V}\mathbf{D}^{\mathrm{M}}$, where

$$\mathbf{D}^{\mathrm{M}} = \mathrm{diag}\left(e^{i\delta_1^{\mathrm{M}}}, e^{i\delta_2^{\mathrm{M}}}, e^{i\delta_3^{\mathrm{M}}}\right)$$

and δ_k^{M} are the (real) *CP*-violating parameters (strictly speaking, in the three-neutrino case only two "Majorana parameters" δ_k^{M} are independent [92,93]). By analogy, one may call the functions $\delta_k(\mathbf{x}) = b_k(\mathbf{x}) + ia_k(\mathbf{x})$ and $\overline{\delta}_k(\mathbf{x}) = b_k(\mathbf{x}) - ia_k(\mathbf{x})$ the Majorana phases in matter. Just as in the vacuum case, these phases play no part in neutrino oscillations at relativistic energies [92,93]. Here they merely show the ambiguity in the definition of the mixing matrices in matter. The additional CP-violating Majorana phases are always associated with effects whose magnitude is suppressed by the factor $\left(m_i^{\mathrm{M}}/E_\nu\right)^2$, where E_ν is the neutrino energy in the relevant process and m_i^{M} is the mass of the Majorana neutrino taking part in the process [93,94].

4. Rephasing Invariant In Matter

Let us introduce two sets of functions

$$J_{\alpha i}^{\pm}(\mathbf{x}) = \frac{1}{2}\eta_\alpha^{\beta\gamma}\eta_i^{jk}V_{\beta j}^{(\mathrm{m})}(\mathbf{x})V_{\gamma k}^{(\mathrm{m})}(\mathbf{x})\left[\overline{V}_{\beta k}^{(\mathrm{m})}(\mathbf{x})\overline{V}_{\gamma j}^{(\mathrm{m})}(\mathbf{x})\right]^* \pm \frac{1}{2}\eta_\alpha^{\beta\gamma}\eta_i^{jk}\overline{V}_{\beta j}^{(\mathrm{m})}(\mathbf{x})\overline{V}_{\gamma k}^{(\mathrm{m})}(\mathbf{x})\left[V_{\beta k}^{(\mathrm{m})}(\mathbf{x})V_{\gamma j}^{(\mathrm{m})}(\mathbf{x})\right]^*,$$

which provide the straightforward generalization of the rephasing invariants considered in references [32,33,95,96] (see also reference [34,97] for the Dirac neutrino case or in reference [94] for the Majorana neutrino case (Cheng [94] considered so called second-class rephasing invariant which contains the Majorana phases).

First of all, the functions $J_{\alpha i}^{\pm}(\mathbf{x})$ are independent of the Majorana phases, $\delta_k(\mathbf{x})$, $\overline{\delta}_k(\mathbf{x})$, i.e., these functions are independent of the $\mathbf{D}(\mathbf{x})$ and $\overline{\mathbf{D}}(\mathbf{x})$ matrices. This fact elucidates the term "rephasing invariant". Therefore the functions $J_{\alpha i}^{\pm}(\mathbf{x})$ can be rewritten as

$$J_{\alpha i}^{\pm}(\mathbf{x}) = \frac{1}{2}\eta_\alpha^{\beta\gamma}\eta_i^{jk}U_{\beta j}(\mathbf{x})U_{\gamma k}(\mathbf{x})\overline{U}_{\beta k}^*(\mathbf{x})\overline{U}_{\gamma j}^*(\mathbf{x}) \pm \frac{1}{2}\eta_\alpha^{\beta\gamma}\eta_i^{jk}\overline{U}_{\beta j}(\mathbf{x})\overline{U}_{\gamma k}(\mathbf{x})U_{\beta k}^*(\mathbf{x})U_{\gamma j}^*(\mathbf{x}). \tag{30}$$

Let us rewrite Equation (29) in the form

$$\mathbf{U}^\dagger(\mathbf{x}) = \overline{\mathbf{U}}^{-1}(\mathbf{x}), \quad \overline{\mathbf{U}}^\dagger(\mathbf{x}) = \mathbf{U}^{-1}(\mathbf{x}),$$

or, in terms of the matrix elements,

$$\begin{aligned} U_{\alpha i}^*(\mathbf{x}) &= |\overline{\mathbf{U}}|^{-1}\eta_\alpha^{\beta\gamma}\eta_i^{jk}\left(\overline{U}_{\beta j}\overline{U}_{\gamma k} - \overline{U}_{\beta k}\overline{U}_{\gamma j}\right), \\ \overline{U}_{\alpha i}^*(\mathbf{x}) &= |\mathbf{U}|^{-1}\eta_\alpha^{\beta\gamma}\eta_i^{jk}\left(U_{\beta j}U_{\gamma k} - U_{\beta k}U_{\gamma j}\right). \end{aligned} \tag{31}$$

Using these identities, one finds from Equation (30) that the real functions

$$\begin{aligned} \mathrm{Re}J_{\alpha i}^{-}(\mathbf{x}) &= +\frac{1}{2}\mathrm{Re}\left(U_{e1}U_{\mu 2}U_{\tau 3}|\overline{\mathbf{U}}^\dagger| - \overline{U}_{e1}\overline{U}_{\mu 2}\overline{U}_{\tau 3}|\mathbf{U}^\dagger|\right) \equiv R(\mathbf{x}), \\ \mathrm{Im}J_{\alpha i}^{+}(\mathbf{x}) &= -\frac{1}{2}\mathrm{Im}\left(U_{e1}U_{\mu 2}U_{\tau 3}|\overline{\mathbf{U}}^\dagger| + \overline{U}_{e1}\overline{U}_{\mu 2}\overline{U}_{\tau 3}|\mathbf{U}^\dagger|\right) \equiv I(\mathbf{x}), \end{aligned} \tag{32}$$

as well as their complex combination

$$J(\mathbf{x}) = I(\mathbf{x}) + iR(\mathbf{x}) \tag{33}$$

are independent of indices α and i. Clearly, in the Hermitian case $J_{\alpha i}^{-} = 0$ and therefore $J = I = -\mathrm{Im}\left(U_{e1}U_{\mu 2}U_{\tau 3}|\mathbf{U}^\dagger|\right)$.

We consider now the following constructions:

$$\mathcal{J}(\mathbf{x}) = \frac{1}{2i}\left[\prod_\alpha \eta_\alpha^{\beta\gamma}H_{\beta\gamma}(\mathbf{x}) - \prod_\alpha \eta_\alpha^{\beta\gamma}H_{\gamma\beta}(\mathbf{x})\right], \tag{34}$$

$$\mathcal{P}(\mathbf{x}) = \prod_{\alpha} \eta_{\alpha}^{\beta\gamma} |H_{\beta\gamma}(\mathbf{x})|, \quad \overline{\mathcal{P}}(\mathbf{x}) = \prod_{\alpha} \eta_{\alpha}^{\beta\gamma} |H_{\gamma\beta}(\mathbf{x})|, \tag{35}$$

$$\varphi(\mathbf{x}) = \sum_{\alpha} \eta_{\alpha}^{\beta\gamma} \arg H_{\beta\gamma}(\mathbf{x}), \quad \overline{\varphi}(\mathbf{x}) = \sum_{\alpha} \eta_{\alpha}^{\beta\gamma} \arg H_{\gamma\beta}^{*}(\mathbf{x}). \tag{36}$$

It is easy to show that

$$\mathcal{J}(\mathbf{x}) = \Im(\mathbf{x}) + i\Re(\mathbf{x}),$$

where

$$\Im(\mathbf{x}) = \frac{1}{2}\left[\mathcal{P}(\mathbf{x}) \sin\varphi(\mathbf{x}) + \overline{\mathcal{P}}(\mathbf{x}) \sin\overline{\varphi}(\mathbf{x})\right],$$
$$\Re(\mathbf{x}) = \frac{1}{2}\left[\overline{\mathcal{P}}(\mathbf{x}) \cos\overline{\varphi}(\mathbf{x}) - \mathcal{P}(\mathbf{x}) \cos\varphi(\mathbf{x})\right].$$

In the absence of flavor-changing neutral currents the off-diagonal matrix elements of the Hamiltonian are time independent and thus $\mathcal{J}$ is a complex constant ("dynamic invariant"). In the most general case the following theorem holds true:

$$\mathcal{J}(\mathbf{x}) = \varsigma J(\mathbf{x}) \prod_{L} \eta_{L}^{MN} \left[\mathcal{E}_M(\mathbf{x}) - \mathcal{E}_N(\mathbf{x})\right], \tag{37}$$

where ς is the parity of the cyclic permutation $\begin{pmatrix} -1 & 0 & +1 \\ N_1 & N_2 & N_3 \end{pmatrix}$. The proof of this theorem is given in Appendix A. The obtained identity is very general and does not depend on explicit form of the eigenvalues and eigenvectors, but the full the derivation of these quantities is discussed in detail in reference [24].

To gain a further insight into the identity (37), it is instructive to consider an example of neutrino propagation in matter governed by the Hamiltonian (7). Then, it is seen that $\mathcal{P}(\mathbf{q}) = \overline{\mathcal{P}}(\mathbf{q})$, $\varphi(\mathbf{q}) = \overline{\varphi}(\mathbf{q})$ and these quantities are time independent:

$$\mathcal{P}(\mathbf{q}) = \overline{\mathcal{P}}(\mathbf{q}) = \left|\sum_i V_{\mu i} V_{\tau i}^{*} \Delta_i\right| \cdot \left|\sum_i V_{e i} V_{\mu i}^{*} \Delta_i\right| \cdot \left|\sum_i V_{\tau i} V_{e i}^{*} \Delta_i\right|,$$

$$\varphi(\mathbf{q}) = \overline{\varphi}(\mathbf{q}) = \arg \sum_i \left(V_{\mu i} V_{\tau i}^{*} + V_{e i} V_{\mu i}^{*} + V_{\tau i} V_{e i}^{*}\right) \Delta_i.$$

Therefore, $\Re = 0$ and $\mathcal{J} = \Im = \mathcal{P} \sin\varphi$ in this case. It can be verified that the LHS of Equation (37) is exactly the product the Jarlskog invariant (see Appendix B)

$$J_0 = J(\mathbf{0}) = \frac{1}{8} \sin\delta \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \tag{38}$$

and the factor

$$\prod_i \eta_i^{jk} \frac{m_j^2 - m_k^2}{2p_\nu} = \prod_i \eta_i^{jk} \frac{\Delta m_{jk}^2}{2p_\nu}.$$

Let us define the effective (complex) masses $\widetilde{m}_i = \widetilde{m}_i(\mathbf{q})$ of the neutrino mass eigenstates in matter by

$$E_i = \mathcal{E}_{N_i} \stackrel{\text{def}}{=} \frac{\widetilde{m}_i^2 - \langle \widetilde{m}^2 \rangle}{2p_\nu}, \quad \langle \widetilde{m}^2 \rangle = \frac{1}{3} \sum_i \widetilde{m}_i^2,$$

where we used the obvious identity $\sum_i E_i = 0$ and analogy with the vacuum case (see Equation (14)). Then Equation (37) can be written as

$$J(\mathbf{0}) \Delta m_{23}^2 \Delta m_{31}^2 \Delta m_{12}^2 = J(\mathbf{q}) \Delta\widetilde{m}_{23}^2(\mathbf{q}) \Delta\widetilde{m}_{31}^2(\mathbf{q}) \Delta\widetilde{m}_{12}^2(\mathbf{q}). \tag{39}$$

The obtained identity is evidently a generalization of the relation (1) to the case of neutrino-absorbing environments. Remarkably that the effective masses are complex functions but the RHS of Equation (39) is proved to be real. The form of Equation (39) confirms that Equations (32) and (33) provide a non-Hermitian extension of the usual rephasing invariant.

5. Summary

In this paper we considered three-neutrino oscillations in thick (including neutrino opaque) media by using the non-Hermitian quantum dynamics framework, which describes the interplay between neutrino mixing, refraction and absorption. We proved an identity which relates (through a product of splitting of the complex energy levels) a rephasing invariant in vacuum and absorbing matter. These findings might be of certain interest in studies of soft-spectrum, high-energy neutrino propagation through Earth or astrophysical objects (jets, blast waves, etc.) whose thickness along the neutrino beam is comparable to or larger than the neutrino mean free path.

Author Contributions: Conceptualization, D.V.N. and V.A.N.; methodology, D.V.N. and V.A.N.; validation, D.V.N., V.A.N., and D.S.S.; writing—original draft preparation, D.V.N., V.A.N., and D.S.S.; writing—review and editing, D.V.N., V.A.N., D.S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PMNS	Pontecorvo-Maki-Nakagawa-Sakata (mixing matrix)
MSW	Mikheev-Smirnov-Wolfenstein (mechanism, equation)
KM	Kobayashi-Maskawa (representation of mixing matrix)
CK	Chau-Keung (representation of mixing matrix)
CC	Charged Current
NC	Neutral Current
AMU	Atomic Mass Unit
CP	Charge Parity
LHS	Left-Hand Side
RHS	Right-Hand Side
QED	Quod Erat Demonstrandum (Lat.)

Appendix A. Proof of The Theorem

Using the definitions for the mixing matrices one can easily show that

$$\mathcal{J} = \frac{1}{2i}\prod_{\alpha}\eta_{\alpha}^{\beta\gamma}\sum_{i}U_{\beta i}\overline{U}^{*}_{\gamma i}E_i - \frac{1}{2i}\prod_{\alpha}\eta_{\alpha}^{\beta\gamma}\sum_{i}\overline{U}^{*}_{\beta i}U_{\gamma i}E_i,$$

where we omitted argument $\mathbf{x}$ for short. Denote

$$G_{ijk} = U_{ei}U_{\mu j}U_{\tau k}, \quad \overline{G}_{ijk} = \overline{U}_{ei}\overline{U}_{\mu j}\overline{U}_{\tau k}.$$

Then $\mathcal{J}$ can be written as

$$\frac{1}{2i}\sum_{ijk}E_iE_jE_k\left(G_{ijk}\overline{G}^{*}_{kij} - \overline{G}^{*}_{ijk}G_{kij}\right).$$

It can be shown from here that

$$\mathcal{J} = \frac{1}{2i}\sum_{i}\eta_i^{jk}E_i^2\left(C_i^jE_j + C_i^kE_k\right), \tag{A1}$$

where

$$C_i^j = G_{iij}\overline{G}^*_{jii} + G_{iji}\overline{G}^*_{iij} + G_{jii}\overline{G}^*_{iji} - \overline{G}^*_{iij}G_{jii} - \overline{G}^*_{iji}G_{iij} - \overline{G}^*_{jii}G_{iji}$$

and the coefficients C_i^k are defined in a similar way. To derive Equation (A1) it has been taken into account that the terms in the sum over i, j, k with $i = j = k$, as well as with the i, j, and k which unequal to each other, are vanish. This statement is apparent for the term

$$\frac{1}{2i}\sum_i E_i^3 \left(G_{iii}\overline{G}^*_{iii} - \overline{G}^*_{iii}G_{iii}\right).$$

As regards the term

$$\frac{1}{2i}{\sum_{ijk}}' E_iE_jE_k \left(G_{ijk}\overline{G}^*_{kij} - \overline{G}^*_{ijk}G_{kij}\right)$$

(where prime indicates that all indices are different), it can be rewritten in the following form:

$$-\frac{i}{2}|\mathbf{H}|\sum_i \eta_i^{jk}\left(G_{ijk}\overline{G}^*_{kij} + G_{ikj}\overline{G}^*_{jik} - \overline{G}^*_{ijk}G_{kij} - \overline{G}^*_{ikj}G_{jik}\right), \tag{A2}$$

where it is taken into account that

$$E_1E_2E_3 = \mathcal{E}_-\mathcal{E}_0\mathcal{E}_+ = |\mathbf{H}|$$

By applying sequentially the identities (31), one can transform the term (A2) to the following form:

$$-\frac{i}{2}|\mathbf{H}|\sum_i \eta_i^{jk}\left(G_{ijk}|\overline{\mathbf{U}}|^* - \overline{G}^*_{ijk}|\mathbf{U}|\right)\left(U_{\tau j}\overline{U}^*_{\tau j} - U_{\tau k}\overline{U}^*_{\tau k}\right),$$

However, according to Equation (32),

$$\frac{1}{2}\eta_i^{jk}\left(G_{ijk}|\overline{\mathbf{U}}|^* - \overline{G}^*_{ijk}|\mathbf{U}|\right) = R - iI = -iJ.$$

Hence the term (A2) vanishes.

Next, using the identities (28), (31), and definition (32) yields

$$\eta_i^{jk}C_i^j = -2iJ, \quad \eta_i^{jk}C_i^k = 2iJ.$$

Direct substituting into Equation (A1) then gives

$$\mathcal{J} = -J\sum_i \eta_i^{jk}E_i^2(E_j - E_k) = J\prod_i \eta_i^{jk}(E_j - E_k) = \varsigma J\prod_L \eta_L^{MN}\left(\mathcal{E}_M - \mathcal{E}_N\right).$$

QED.

Appendix B. Rephasing Invariant In Vacuum

The imaginary part of the rephasing invariant in vacuum (Jarlskog invariant) may be written in terms of the mixing angles and CP-violating Dirac phase dependent of the parametrization of the PMNS mixing matrix. For example, in the Kobayashi–Maskawa (KM) representation [98],

$$\mathbf{V}^{(\mathrm{KM})} = \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ -s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix}$$

(where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ for $i = 1, 2, 3$; $0 < \theta_i < \pi/2$, $-\pi < \delta \leq \pi$, $\det \mathbf{V}^{(\mathrm{KM})} = -e^{i\delta}$),

$$J_0^{(\mathrm{KM})} = \sin\delta \sin\theta_1 \prod_i \sin 2\theta_i$$

In the now more conventional Chau–Keung (CK) representation [99],

$$\mathbf{V}^{(\mathrm{CK})} = \begin{pmatrix} c_{12}c_{31} & s_{12}c_{31} & s_{31}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{31}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{31}e^{i\delta} & s_{23}c_{31} \\ s_{12}s_{23} - c_{12}c_{23}s_{31}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{31}e^{i\delta} & c_{23}c_{31} \end{pmatrix}$$

(where $s_{jk} = \sin\theta_{jk}$ and $c_{jk} = \cos\theta_{jk}$ for $j, k = 1, 2, 3$; $0 < \theta_{jk} < \pi/2$ ($\theta_{jk} \equiv \theta_{jk}$), $0 \leq \delta < 2\pi$, $\det \mathbf{V}^{(\mathrm{CK})} = 1$),

$$J_0^{(\mathrm{CK})} = \sin\delta \cos\theta_{31} \prod_i \eta_i^{jk} \sin 2\theta_{jk}$$

Here the symbol η_i^{jk} has the same sense as $\eta_\alpha^{\beta\gamma}$. Details about the interconnection between the KM and CK representations can be found in references [99–101].

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