

Review

A Spacetime Symmetry Approach to Relativistic Quantum Multi-Particle Entanglement

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Received: 30 June 2020; Accepted: 24 July 2020; Published: 30 July 2020



Abstract: A Lorentz transformation group $SO(m, n)$ of signature (m, n) , $m, n \in \mathbb{N}$, in m time and n space dimensions, is the group of pseudo-rotations of a pseudo-Euclidean space of signature (m, n) . Accordingly, the Lorentz group $SO(1, 3)$ is the common Lorentz transformation group from which special relativity theory stems. It is widely acknowledged that special relativity and quantum theories are at odds. In particular, it is known that entangled particles involve Lorentz symmetry violation. We, therefore, review studies that led to the discovery that the Lorentz group $SO(m, n)$ forms the symmetry group by which a multi-particle system of m entangled n -dimensional particles can be understood in an extended sense of relativistic settings. Consequently, we enrich special relativity by incorporating the Lorentz transformation groups of signature $(m, 3)$ for all $m \geq 2$. The resulting enriched special relativity provides the common symmetry group $SO(1, 3)$ of the $(1 + 3)$ -dimensional spacetime of individual particles, along with the symmetry group $SO(m, 3)$ of the $(m + 3)$ -dimensional spacetime of multi-particle systems of m entangled 3-dimensional particles, for all $m \geq 2$. A unified parametrization of the Lorentz groups $SO(m, n)$ for all $m, n \in \mathbb{N}$, shakes down the underlying matrix algebra into elegant and transparent results. The special case when $(m, n) = (1, 3)$ is supported experimentally by special relativity. It is hoped that this review article will stimulate the search for experimental support when $(m, n) = (m, 3)$ for all $m \geq 2$.

Keywords: Einstein gyrogroups; Galilei transformations of signature (m, n) ; Lorentz transformations of signature (m, n) ; pseudo-Euclidean spaces; quantum multi-particle entanglement; special relativity

1. Introduction

Nature organizes itself using the language of symmetries. Thus, in particular, the underlying symmetry group by which Einstein's special relativity theory can be understood is the Lorentz group $SO_c(1, 3)$. A physical system obeys the Lorentz symmetry if the relevant laws of physics are invariant under Lorentz transformations. Lorentz symmetry is one of the cornerstones of modern physics. However, it is known that entangled particles in relativistic quantum mechanics involve Lorentz symmetry violation [1–5]. Indeed, several explorers exploit entangled particles to observe Lorentz symmetry violation; see, for instance, [6–11].

Quantum entanglement [12] was named by Einstein, “spooky action at a distance.” It is a physical phenomenon that occurs when groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others, even when the particles are separated by a large distance. Instead, a quantum state must be described for a system of particles as a whole.

Understanding entanglement in relativistic settings has been a key question in relativistic quantum mechanics. Some results show that entanglement is observer-dependent [2]. The aim of this review article is, therefore, (i) to present results that demonstrate in extended relativistic settings that the Lorentz transformation groups $SO_c(m, n)$, $m, n \in \mathbb{N}$, are the missing symmetry groups of multi-particle systems of entangled particles; and (ii) to stimulate the search for experimental support when $m \geq 2$ and $n = 3$, in addition to the available experimental support when $(m, n) = (1, 3)$ that Einstein's

special theory of relativity provides. To achieve those goals, we place the Lorentz groups $SO_c(m, n)$ for all $m, n \in \mathbb{N}$ under the same umbrella, so that the incorporation of $SO_c(m, 3)$ into special relativity for all $m \geq 2$ is naturally suggested.

The Lorentz group $SO(m, n)$ of signature (m, n) , $m, n \in \mathbb{N}$, also known as the group of pseudo-rotations in a pseudo-Euclidean space of signature (m, n) , is well-known in algebra [13] and in physics [14]. It descends to the common homogeneous, proper, orthochronous Lorentz transformation group $SO(1, 3)$ of Einstein's special relativity theory in the special case when $(m, n) = (1, 3)$. Following studies in [15–19], we realize the Lorentz group $SO_c(m, n)$ parametrically, calling the elements of $SO_c(m, n)$ Lorentz transformations of signature (m, n) , or in short, (m, n) -Lorentz transformations, in m time dimensions and n space dimensions. Then, the Lorentz transformation group $SO_c(1, 3)$ is the special relativistic Lorentz transformation group $SO(1, 3)$ with $c > 0$ being an arbitrarily fixed constant that, in physical applications, represents the vacuum speed of light. Accordingly, $SO_{c=1}(m, n) = SO(m, n)$.

The parametric realization (147) of each Lorentz group $SO_c(m, n)$, $m, n \in \mathbb{N}$, in Theorem 15 forms the tool that we employ in this paper. It allows us to associate Lorentz groups of signature (m, n) to Galilei groups of same signature. Indeed, by means of the additive decomposition (155) of the Lorentz bi-boost, a Lorentz bi-boost of signature (m, n) is associated to a corresponding Galilei bi-boost of same signature (m, n) in such a way that (i) a Galilei bi-boost can be recovered from its corresponding Lorentz bi-boost; and, conversely, (ii) a Lorentz bi-boost can be recovered from its corresponding Galilei bi-boost.

A Lorentz (Galilei) bi-boost of signature (m, n) is a Lorentz (Galilei) transformation of signature (m, n) without rotations. We will see that in some obvious sense, a Galilei bi-boost can be said to be a *Galilei multi-boost*, and hence, a Lorentz bi-boost can be said to be a *Lorentz multi-boost* as well.

Unlike Lorentz groups, Galilei groups of any signature are intuitively clear, as shown in Sections 19–20. The resulting clear, physical interpretation of a Galilei transformation of signature (m, n) induces, by means of the additive decomposition (155), a corresponding physical interpretation for its associated Lorentz transformation of signature (m, n) . Following the induced interpretation for any $m, n \in \mathbb{N}$, the Lorentz transformation group $SO_c(m, n)$ turns out to be the symmetry group of the $(m + n)$ -dimensional spacetime of multi-particle systems that consist of m entangled n -dimensional particles.

It is now clear why quantum entanglement involves a violation of the Lorentz symmetry group $SO_c(1, 3)$. The symmetry group that controls multi-particle entanglement of $m \geq 2$ particles is $SO_c(m, 3)$ rather than $SO_c(1, 3)$, as we demonstrate in this article in terms of mathematical analogies and patterns.

2. Einstein Velocity Addition

Einstein's special relativity stems from a transformation law, the Lorentz transformation, for the spacetime coordinates of a point particle. Guided by mathematical patterns and by analogies with the intuitively clear Galilei transformation, we extend both Galilei and Einstein boosts of particles to multi-boosts, called bi-boosts, of multi-particle systems. This extension enriches special relativity to a relativistic theory that accommodates multi-particle entanglement. The journey to our *enriched special relativity theory* begins with the review of Einstein velocity addition law of relativistically admissible velocities and its gyroformalism.

Let $c > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n -space, $n \in \mathbb{N}$, endowed with the common vector addition, $+$, and inner product, \cdot . The space of all n -dimensional relativistically admissible velocities is the c -ball

$$\mathbb{R}_c^n = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}. \quad (1)$$

Einstein velocity addition is a binary operation, \oplus , in the c -ball \mathbb{R}_c^n given by [20–24]

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}, \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. In physical applications $n = 3$, but in geometry $n \in \mathbb{N}$. Here, $\gamma_{\mathbf{u}}$ is the Lorentz gamma factor,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \geq 1, \tag{3}$$

where $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball \mathbb{R}_c^n inherits from its ambient space \mathbb{R}^n , and $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v}^2$. A nonempty set with a binary operation is a *groupoid*, so that the pair (\mathbb{R}_c^n, \oplus) is an *Einstein groupoid*.

The constant $c > 0$ represents the vacuum speed of light. In the Euclidean–Newtonian limit of large c , $c \rightarrow \infty$, the ball \mathbb{R}_c^n expands to the whole of its ambient space \mathbb{R}^n , as we see from (1), and the Einstein addition \oplus in \mathbb{R}_c^n descends to vector addition $+$ in \mathbb{R}^n , as we see from (2) and (3).

When the nonzero vectors \mathbf{u} and \mathbf{v} in the ball \mathbb{R}_c^n of \mathbb{R}^n are parallel in \mathbb{R}^n , $\mathbf{u} \parallel \mathbf{v}$, that is, $\mathbf{u} = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{R} \setminus 0$, Einstein addition (2) descends to the Einstein addition of parallel velocities,

$$\mathbf{u} \oplus \mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{1}{c^2} \mathbf{u} \cdot \mathbf{v}}, \quad \mathbf{u} \parallel \mathbf{v}, \tag{4}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, which was partially confirmed experimentally by the Fizeau’s 1851 experiment [25].

The *restricted Einstein addition* (4) is both commutative and associative, so that it is a group operation, as Einstein noted in [26]; see [27], p. 142. However, the general Einstein addition is neither commutative nor associative, so that it is not a group operation. Rather, it is a gyrocommutative gyrogroup operation, a rich algebraic structure discovered more than 80 years later, in 1988 [20,28–30], which we review in Section 5.

Einstein addition (2) with $n = 3$ was introduced by Einstein in his 1905 paper [26] ([27], p. 141) that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2) are presented.

We use the notation $\mathbf{u} \ominus \mathbf{v} = \mathbf{u} \oplus (-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v} \ominus \mathbf{v} = \mathbf{0}$ and

$$\ominus \mathbf{v} = \mathbf{0} \ominus \mathbf{v} = -\mathbf{v}. \tag{5}$$

Einstein addition and subtraction satisfy the equations

$$\ominus(\mathbf{u} \oplus \mathbf{v}) = \ominus \mathbf{u} \ominus \mathbf{v} \tag{6}$$

and

$$\ominus \mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{v}) = \mathbf{v} \tag{7}$$

for all \mathbf{u}, \mathbf{v} in the ball \mathbb{R}_c^n . Identity (6) expresses the *gyroautomorphic inverse property* of Einstein addition, and Identity (7) expresses the *left cancellation law* of Einstein addition.

Einstein addition does not obey the naive right counterpart of the left cancellation law (7) since, in general,

$$(\mathbf{u} \oplus \mathbf{v}) \ominus \mathbf{v} \neq \mathbf{u}. \tag{8}$$

This seeming lack of a *right cancellation law* of Einstein addition is remedied in (76) by the incorporation of a second, dual binary operation, \boxplus .

Einstein addition and the gamma factor are linked by the *gamma identity*,

$$\gamma_{\mathbf{u} \oplus \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} \right), \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. The gamma identity signaled the emergence of hyperbolic geometry into special relativity ([31], p. 79).

The structure of Einstein addition gives rise to our *gyrolanguage* in which we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and in nonassociative algebra. The prefix “gyro” stems from “gyration”, which is the mathematical abstraction of the special relativistic effect known as “Thomas precession”. In gyrolanguage, thus, Einstein addition is a gyroaddition. The gamma factor (3) associated with Einstein gyroaddition will be extended in (129) to the bi-gamma pair associated with Einstein bi-gyroaddition as a step in our program to extend the Lorentz group $SO_c(1, 3)$ to $SO_c(m, n)$ for all $m, n \in \mathbb{N}$ under the same umbrella.

A useful identity that follows immediately from (3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2}. \tag{10}$$

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u} \oplus \mathbf{v}\| = \|\mathbf{v} \oplus \mathbf{u}\|, \tag{11}$$

in general, it is noncommutative,

$$\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}, \tag{12}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Einstein addition is nonassociative as well since, in general,

$$(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}), \tag{13}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

As an application of the gamma identity (9), we present the Einstein gyrotriangle inequality in the following theorem.

Theorem 1. (Gyrotriangle Inequality ([19] Theorem 2.1)).

$$\|\mathbf{u} \oplus \mathbf{v}\| \leq \|\mathbf{u}\| \oplus \|\mathbf{v}\| \tag{14}$$

for all \mathbf{u}, \mathbf{v} in an Einstein groupoid (\mathbb{R}_c^n, \oplus) .

Proof. By the gamma identity (9) and the Cauchy–Schwarz inequality [32], we have

$$\gamma_{\|\mathbf{u}\| \oplus \|\mathbf{v}\|} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\|\mathbf{u}\| \|\mathbf{v}\|}{c^2} \right) \geq \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} \right) = \gamma_{\mathbf{u} \oplus \mathbf{v}} = \gamma_{\|\mathbf{u} \oplus \mathbf{v}\|} \tag{15}$$

for all \mathbf{u}, \mathbf{v} in an Einstein groupoid (\mathbb{R}_c^n, \oplus) , $n \in \mathbb{N}$. The function $\gamma_{\mathbf{x}} = \gamma_{\|\mathbf{x}\|}$ is a monotonically increasing function of $\|\mathbf{x}\|$, $0 \leq \|\mathbf{x}\| < c$. Hence (15) implies (14). \square

The gyrotriangle inequality (14) will be extended from Einstein gyrovector spaces $(\mathbb{R}_c^n, \oplus, \otimes)$ to Einstein bi-gyrovector spaces $(\mathbb{R}_c^{n \times m}, \oplus_E, \otimes)$ of any signature (m, n) , $m, n \in \mathbb{N}$, in (175).

Remark 1. (Einstein Addition of Signature (m, n)). Einstein addition \oplus in (2) is of signature $(1, n)$. It will be generalized in Section 22 to the Einstein addition $\oplus_E = \oplus_{E, (m, n), c}$ of signature (m, n) in the ball

$\mathbb{R}_c^{n \times m}$ for all $m, n \in \mathbb{N}$. Einstein addition \oplus in (2) will, then, be recognized as the Einstein addition $\oplus_{E,(1,n),c}$ of signature $(1, n)$ in the ball $\mathbb{R}_c^{n \times 1} = \mathbb{R}_c^n$, that is, $\oplus = \oplus_E = \oplus_{E,(1,n),c}$.

3. The Gyroformalism of Einstein Addition

Vector addition, $+$, in \mathbb{R}^n is both commutative and associative; that is,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} && \text{Commutative Law} \\ \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} && \text{Associative Law} \end{aligned} \quad (16)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, \oplus , in \mathbb{R}_c^n is neither commutative nor associative, where the deviation from both commutativity and associativity is controlled by gyrations, as evidenced from (18).

Gyrations $\text{gyr}[\mathbf{u}, \mathbf{v}] \in \text{Aut}(\mathbb{R}_c^n, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, are given in terms of Einstein addition by the *gyrator equation*

$$\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u} \oplus \mathbf{v}) \oplus \{\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})\} \quad (17)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$. Equation (17) presents the application to \mathbf{w} of the gyration $\text{gyr}[\mathbf{u}, \mathbf{v}]$ generated by \mathbf{u} and \mathbf{v} . Gyrations are automorphisms of the Einstein groupoid (\mathbb{R}_c^n, \oplus) .

An automorphism of a groupoid (S, \oplus) is a bijective map f of S onto itself that respects the binary operation, that is, $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid (S, \oplus) forms a group, denoted by $\text{Aut}(S, \oplus)$, where the group operation is given by automorphism composition. To emphasize that the gyrations of an Einstein groupoid (\mathbb{R}_c^n, \oplus) are automorphisms of the groupoid, gyrations are also called *gyroautomorphisms*.

Possessing their own rich structure, gyrations measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the following list of identities [20,21,31]:

$$\begin{aligned} \mathbf{u} \oplus \mathbf{v} &= \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v} \oplus \mathbf{u}) && \text{Gyrocommutative Law} \\ \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= (\mathbf{u} \oplus \mathbf{v}) \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\ (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \mathbf{u} \oplus (\mathbf{v} \oplus \text{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law} \\ \text{gyr}[\mathbf{u} \oplus \mathbf{v}, \mathbf{v}] &= \text{gyr}[\mathbf{u}, \mathbf{v}] && \text{Left Reduction Property} \\ \text{gyr}[\mathbf{u}, \mathbf{v} \oplus \mathbf{u}] &= \text{gyr}[\mathbf{u}, \mathbf{v}] && \text{Right Reduction Property} \\ \text{gyr}[\ominus \mathbf{u}, \ominus \mathbf{v}] &= \text{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Even Property} \\ (\text{gyr}[\mathbf{u}, \mathbf{v}])^{-1} &= \text{gyr}[\mathbf{v}, \mathbf{u}] && \text{Gyration Inversion Law} \end{aligned} \quad (18)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

It is clear from the gyrocommutative and gyroassociative laws in (18) that the departure of Einstein addition, \oplus , from commutativity and associativity is controlled by gyrations.

The properties of Einstein gyroaddition and resulting gyrations in (18) give rise to a mathematical formalism and its associated gyrolanguage. These properties will be extended in Section 22 to Einstein bi-gyroaddition and bi-gyrations of signature (m, n) for all $m, n \in \mathbb{N}$, where the special signature $(m, n) = (1, 3)$ corresponds to Einstein's special relativity in $m = 1$ time dimension and $n = 3$ space dimensions.

4. Gyrations

Owing to its nonassociativity, Einstein addition gives rise in (17) to gyrations,

$$\text{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}_c^n \rightarrow \mathbb{R}_c^n, \quad (19)$$

of an Einstein groupoid (\mathbb{R}_c^n, \oplus) for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Gyrations, in turn, are automorphisms of (\mathbb{R}_c^n, \oplus) that regulate Einstein addition, \oplus , endowing it with the rich structure of a gyrocommutative gyrogroup,

as we will see in Section 5, and a gyrovector space, as shown, for instance, in [19–21,31,33–38]. A differential geometric approach to gyrogroups and gyrovector spaces is found in [39,40].

In Definition 3, the left reduction property is elevated to the *reduction axiom*. As noted by F. Chatelin in [41], the reduction axiom triggers remarkable reduction in complexity as, for instance, in (72).

Gyrations are expressed in (17) in terms of Einstein addition. Explicitly, they are given by the equation

$$\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{A\mathbf{u} + B\mathbf{v}}{D}, \tag{20}$$

where

$$\begin{aligned} A &= -\frac{1}{c^2} \frac{\gamma_{\mathbf{u}}^2}{(\gamma_{\mathbf{u}} + 1)} (\gamma_{\mathbf{v}} - 1)(\mathbf{u} \cdot \mathbf{w}) + \frac{1}{c^2} \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} (\mathbf{v} \cdot \mathbf{w}) \\ &\quad + \frac{2}{c^4} \frac{\gamma_{\mathbf{u}}^2 \gamma_{\mathbf{v}}^2}{(\gamma_{\mathbf{u}} + 1)(\gamma_{\mathbf{v}} + 1)} (\mathbf{u} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{w}) \\ B &= -\frac{1}{c^2} \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \{ \gamma_{\mathbf{u}} (\gamma_{\mathbf{v}} + 1)(\mathbf{u} \cdot \mathbf{w}) + (\gamma_{\mathbf{u}} - 1) \gamma_{\mathbf{v}} (\mathbf{v} \cdot \mathbf{w}) \} \\ D &= \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} (1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}) + 1 = \gamma_{\mathbf{u} \oplus \mathbf{v}} + 1 \geq 2 \end{aligned} \tag{21}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$, as shown in [19].

In the three special cases when (i) $\mathbf{u} = \mathbf{0}$, or (ii) $\mathbf{v} = \mathbf{0}$, or (iii) \mathbf{u} and \mathbf{v} are parallel in \mathbb{R}^n , $\mathbf{u} \parallel \mathbf{v}$, we have $A\mathbf{u} + B\mathbf{v} = \mathbf{0}$, so that $\text{gyr}[\mathbf{u}, \mathbf{v}]$ is trivial; that is,

$$\begin{aligned} \text{gyr}[\mathbf{0}, \mathbf{v}]\mathbf{w} &= \mathbf{w} \\ \text{gyr}[\mathbf{u}, \mathbf{0}]\mathbf{w} &= \mathbf{w} \\ \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} &= \mathbf{w}, \quad \mathbf{u} \parallel \mathbf{v}, \end{aligned} \tag{22}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, where $\mathbf{u} \parallel \mathbf{v}$ in the third equation, and for all $\mathbf{w} \in \mathbb{R}^n$.

Following (20) we have

$$\text{gyr}[\mathbf{v}, \mathbf{u}](\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}) = \mathbf{w} \tag{23}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n, \mathbf{w} \in \mathbb{R}^n$, that is,

$$\text{gyr}[\mathbf{v}, \mathbf{u}]\text{gyr}[\mathbf{u}, \mathbf{v}] = I \tag{24}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, where I denotes the identity map.

Hence, gyrations are invertible linear maps of \mathbb{R}^n , the inverse, $\text{gyr}^{-1}[\mathbf{u}, \mathbf{v}]$, of $\text{gyr}[\mathbf{u}, \mathbf{v}]$ being $\text{gyr}[\mathbf{v}, \mathbf{u}]$. We thus have the gyration inversion property

$$\text{gyr}^{-1}[\mathbf{u}, \mathbf{v}] = \text{gyr}[\mathbf{v}, \mathbf{u}] \tag{25}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

Gyrations keep the inner product invariant; that is,

$$\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b} \tag{26}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Hence, $\text{gyr}[\mathbf{u}, \mathbf{v}]$ is an *isometry* of \mathbb{R}_c^n , keeping the norm of elements of the ball \mathbb{R}_c^n invariant.

$$\|\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\| = \|\mathbf{w}\|. \tag{27}$$

Accordingly, $\text{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball \mathbb{R}_c^n about its origin for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

The invertible map $\text{gyr}[\mathbf{u}, \mathbf{v}]$ of \mathbb{R}_c^n respects Einstein addition in \mathbb{R}_c^n :

$$\text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a} \oplus \mathbf{b}) = \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} \tag{28}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Hence, $\text{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of the Einstein groupoid (\mathbb{R}_c^n, \oplus) .

Example 1. As an example that illustrates the use of the invariance of the norm under gyrations, we note that

$$\|\ominus \mathbf{u} \oplus \mathbf{v}\| = \|\mathbf{u} \ominus \mathbf{v}\| = \|\ominus \mathbf{v} \oplus \mathbf{u}\|. \tag{29}$$

Indeed, we have the following chain of equations, which are numbered for subsequent derivation.

$$\begin{aligned} \|\ominus \mathbf{u} \oplus \mathbf{v}\| &\stackrel{(1)}{=} \|\ominus(\ominus \mathbf{u} \oplus \mathbf{v})\| \\ &\stackrel{(2)}{=} \|\mathbf{u} \ominus \mathbf{v}\| \\ &\stackrel{(3)}{=} \|\text{gyr}[\mathbf{u}, \ominus \mathbf{v}](\ominus \mathbf{v} \oplus \mathbf{u})\| \\ &\stackrel{(4)}{=} \|\ominus \mathbf{v} \oplus \mathbf{u}\| \end{aligned} \tag{30}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Derivation of the numbered equalities in (30):

1. Follows from the result that $\ominus \mathbf{w} = -\mathbf{w}$, so that $\|\ominus \mathbf{w}\| = \|-\mathbf{w}\| = \|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{R}_c^n$.
2. Follows from the automorphic inverse property (6) of Einstein addition.
3. Follows from the gyrocommutative law of Einstein addition.
4. Follows from (27).

5. From Einstein Velocity Addition to Gyrogroups

Guided by analogies with groups, the key features of Einstein groupoids (\mathbb{R}_c^n, \oplus) , $n = 1, 2, 3, \dots$, suggest the formal gyrogroup definition in which gyrogroups form a most natural generalization of groups.

Definition 1. (Binary Operations). A binary operation $+$ in a set S is a function $+: S \times S \rightarrow S$. We use the notation $a + b$ to denote $+(a, b)$ for any $a, b \in S$.

Definition 2. (Groupoids, Automorphisms). A groupoid $(S, +)$ is a nonempty set, S , with a binary operation, $+$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective self-map of S which respects its groupoid operation; that is, $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in S$. The automorphisms of a groupoid $(S, +)$ form a group denoted by $\text{Aut}(S, +)$.

Definition 3. (Gyrogroup ([31] Definition 2.5)). A groupoid (G, \oplus) is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0 , called a left identity, satisfying

$$(G1) \quad 0 \oplus a = a$$

for all $a \in G$. There is an element $0 \in G$ satisfying axiom (G1) such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a , satisfying

$$(G2) \quad \ominus a \oplus a = 0.$$

Moreover, for any $a, b, c \in G$ there exists an automorphism $\text{gyr}[a, b] \in \text{Aut}(G, \oplus)$ such that the binary operation obeys the left gyroassociative law

$$(G3) \quad a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c.$$

The automorphism $\text{gyr}[a, b]$ of G is called the gyroautomorphism, or the gyration, of G generated by $a, b \in G$. The operator $\text{gyr} : G \times G \rightarrow \text{Aut}(G, \oplus)$ is called the gyrator of G . Finally, the gyroautomorphism $\text{gyr}[a, b]$ generated by any $a, b \in G$ obeys the left reduction axiom

$$(G4) \quad \text{gyr}[a, b] = \text{gyr}[a \oplus b, b].$$

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups split up into gyrocommutative and non-gyrocommutative ones.

Definition 4. (Gyrocommutative Gyrogroup ([31], Definition 2.6)). A gyrogroup (G, \oplus) is gyrocommutative if its binary operation obeys the gyrocommutative law

$$(G5) \quad a \oplus b = \text{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$.

The abstract gyrocommutative gyrogroup is an algebraic structure derived from Einstein addition \oplus in \mathbb{R}_c^n . Indeed, Einstein groupoids (\mathbb{R}_c^n, \oplus) , $n \in \mathbb{N}$, are gyrocommutative gyrogroups. Gyrogroups, both gyrocommutative and nongyrocommutative abound in group theory, as demonstrated in [42] and [43]. A finite, non-gyrocommutative gyrogroup of order 16, K_{16} , is presented in ([20] Figures 2.1–2.2, p. 41).

The way to approach gyrogroups via Möbius addition in the ball rather than Einstein addition in the ball is found in [44]. In full analogy with groups, there are topological gyrogroups, studied in [45,46]. Gyrogroups share remarkable analogies with groups studied, for instance, in [47–53].

6. Gyrogroup Cooperation (Coaddition)

In order to capture analogies with groups, we introduce into the abstract gyrogroup (G, \oplus) a second binary operation, \boxplus , called the gyrogroup cooperation, or coaddition. The two binary operations \oplus and \boxplus in G are dual to each other in some sense [20].

Definition 5. (Gyrogroup Cooperation (Coaddition) ([31] Definition 2.7). Let (G, \oplus) be a gyrogroup. The gyrogroup cooperation (or, coaddition), \boxplus , is a second binary operation in G related to the gyrogroup operation (or, addition), \oplus , by the equation

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b \tag{31}$$

for all $a, b \in G$.

Naturally, we use the notation $a \boxminus b = a \boxplus (\ominus b)$ where $\boxminus b = \ominus b = -b$, so that

$$a \boxminus b = a \ominus \text{gyr}[a, b]b. \tag{32}$$

The gyrogroup cooperation is commutative if and only if the gyrogroup operation is gyrocommutative, as stated in ([31] Theorem 3.4, p. 50). In particular, Einstein coaddition \boxplus is commutative since Einstein addition \oplus is gyrocommutative.

As a concrete example of (31), let us calculate Einstein coaddition \boxplus . By substituting into (31) both (i) Einstein addition in (2), and (ii) Einstein gyration $\text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$ in (17), lengthy, but straightforward, algebra reveals that Einstein coaddition \boxplus is given explicitly by the equation ([31] p. 81).

$$\mathbf{u} \boxplus \mathbf{v} = \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}) - 1} (\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) = 2 \otimes \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \tag{33}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^k$ where $2 \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v}$. Einstein coaddition (33) of two summands is extended to k summands, $k \geq 2$, in ([54] Equations (6.23) and (6.84)).

With the emergence of Einstein coaddition, the dream of the famous mathematician Émile Borel to create a commutative variant of Einstein addition came true [55].

7. First Gyrogroup Properties

It is clear how to define a right identity and a right inverse in a gyrogroup. The existence of such elements is not presumed since the existence of a unique identity and a unique inverse, both left and right, is a consequence of the gyrogroup axioms. This result is stated in the following theorem, along with other important basic results in gyrogroup theory.

Theorem 2. (First Gyrogroup Properties ([31] Theorem 2.8)). *Let (G, \oplus) be a gyrogroup. For any elements $a, b, c, x \in G$ we have:*

1. *If $a \oplus b = a \oplus c$, then $b = c$ (general left cancellation law; see Item (9) below).*
2. *$\text{gyr}[0, a] = I$ for any left identity 0 in G .*
3. *$\text{gyr}[x, a] = I$ for any left inverse x of a in G .*
4. *$\text{gyr}[a, a] = I$*
5. *There is a left identity which is a right identity.*
6. *There is only one left identity.*
7. *Every left inverse is a right inverse.*
8. *There is only one left inverse, $\ominus a$, of a , and $\ominus(\ominus a) = a$.*
9. *The left cancellation law: $\ominus a \oplus (a \oplus b) = b$.*
10. *The gyrator equation: $\text{gyr}[a, b]x = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus x)\}$.*
11. *$\text{gyr}[a, b]0 = 0$.*
12. *$\text{gyr}[a, b](\ominus x) = \ominus \text{gyr}[a, b]x$.*
13. *$\text{gyr}[a, 0] = I$.*

Proof. The proof of each item of the theorem follows:

1. Let x be a left inverse of a corresponding to a left identity, 0 , in G . We have $x \oplus (a \oplus b) = x \oplus (a \oplus c)$, implying

$$(x \oplus a) \oplus \text{gyr}[x, a]b = (x \oplus a) \oplus \text{gyr}[x, a]c \quad (34)$$

by left gyroassociativity. Since 0 is a left identity, $\text{gyr}[x, a]b = \text{gyr}[x, a]c$. Since automorphisms are bijective, $b = c$.

2. By left gyroassociativity we have for any left identity 0 of G ,

$$a \oplus x = 0 \oplus (a \oplus x) = (0 \oplus a) \oplus \text{gyr}[0, a]x = a \oplus \text{gyr}[0, a]x. \quad (35)$$

Hence, by Item (1) above we have $x = \text{gyr}[0, a]x$ for all $x \in G$ so that $\text{gyr}[0, a] = I$, I being the trivial (identity) map.

3. By the left reduction property and by Item (2) above we have

$$\text{gyr}[x, a] = \text{gyr}[x \oplus a, a] = \text{gyr}[0, a] = I. \quad (36)$$

4. That follows from an application of the left reduction property and Item (2) above. Thus,

$$I = \text{gyr}[0, a] = \text{gyr}[0 \oplus a, a] = \text{gyr}[a, a]. \quad (37)$$

5. Let x be a left inverse of a corresponding to a left identity, 0 , of G . Then by left gyroassociativity and Item (3) above,

$$x \oplus (a \oplus 0) = (x \oplus a) \oplus \text{gyr}[x, a]0 = 0 \oplus 0 = 0 = x \oplus a. \quad (38)$$

Hence, by (1), $a \oplus 0 = a$ for all $a \in G$ so that 0 is a right identity.

6. Suppose 0 and 0^* are two left identities, one of which, say 0 , is also a right identity. Then $0 = 0^* \oplus 0 = 0^*$.
7. Let x be a left inverse of a . Then

$$x \oplus (a \oplus x) = (x \oplus a) \oplus \text{gyr}[x, a]x = 0 \oplus x = x = x \oplus 0, \quad (39)$$

by left gyroassociativity, (G2) of Definition 3 and Items (3), (5), (6) above. By Item (1) we have $a \oplus x = 0$ so that x is a right inverse of a .

8. Suppose x and y are left inverses of a . By Item (7) above, they are also right inverses, so $a \oplus x = 0 = a \oplus y$. By Item (1), $x = y$. Let $\ominus a$ be the resulting unique inverse of a . Then $\ominus a \oplus a = 0$ so that the inverse $\ominus(\ominus a)$ of $\ominus a$ is a .
9. By left gyroassociativity and by Item (3) we have

$$\ominus a \oplus (a \oplus b) = (\ominus a \oplus a) \oplus \text{gyr}[\ominus a, a]b = b. \quad (40)$$

10. By an application of the left cancellation law in Item (9) to the left gyroassociative law (G3) in Definition 3 we obtain the result in Item (10).
11. We obtain Item (11) from Item (10) with $x = 0$.
12. Since $\text{gyr}[a, b]$ is an automorphism of (G, \oplus) , we have from Item (11)

$$\text{gyr}[a, b](\ominus x) \oplus \text{gyr}[a, b]x = \text{gyr}[a, b](\ominus x \oplus x) = \text{gyr}[a, b]0 = 0 \quad (41)$$

and hence the result.

13. We obtain Item (13) from Item (10) with $b = 0$, and a left cancellation, Item (9).

□

8. Elements of Gyrogroup Theory

Einstein gyrogroups (G, \oplus) possess the *gyroautomorphic inverse property*, according to which $\ominus(a \oplus b) = \ominus a \oplus b$ for all $a, b \in G$. In general, however, $\ominus(a \oplus b) \neq \ominus a \oplus b$ in some gyrogroups. Hence, the following theorem is interesting.

Theorem 3. (Gyrosom Inversion Law ([19] Theorem 2.17)). For any two elements a, b of a gyrogroup (G, \oplus) we have the gyrosom inversion law

$$\ominus(a \oplus b) = \text{gyr}[a, b](\ominus b \ominus a). \quad (42)$$

Proof. By the gyrator equation in Theorem 2 (10) and a left cancellation, given in Theorem 2 (9), we have

$$\begin{aligned} \text{gyr}[a, b](\ominus b \ominus a) &= \ominus(a \oplus b) \oplus (a \oplus (b \oplus (\ominus b \ominus a))) \\ &= \ominus(a \oplus b) \oplus (a \oplus a) \\ &= \ominus(a \oplus b). \end{aligned} \quad (43)$$

□

Theorem 4. ([19] Theorem 2.18). For any two elements, a and b , of a gyrogroup (G, \oplus) , we have

$$\begin{aligned} \text{gyr}[a, b]b &= \ominus\{\ominus(a \oplus b) \oplus a\} \\ \text{gyr}[a, \ominus b]b &= \ominus(a \oplus b) \oplus a. \end{aligned} \quad (44)$$

Proof. The first identity in (44) follows from Theorem 2 (10) with $x = \ominus b$, and Theorem 2 (12), and the second part of Theorem 2 (8). The second identity in (44) follows from the first one by replacing b by $\ominus b$, noting Theorem 2 (12). □

A nested gyroautomorphism is a gyration generated by points that depend on another gyration. Thus, for instance, some gyrations in (45)–(47) are nested.

Theorem 5. ([19] Theorem 2.19). *Any three elements a, b, c of a gyrogroup (G, \oplus) satisfy the nested gyroautomorphism identities*

$$\text{gyr}[a, b \oplus c] \text{gyr}[b, c] = \text{gyr}[a \oplus b, \text{gyr}[a, b]c] \text{gyr}[a, b] \quad (45)$$

$$\text{gyr}[a \oplus b, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b] = I \quad (46)$$

$$\text{gyr}[a, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b] = I \quad (47)$$

and the gyroautomorphism product identities

$$\text{gyr}[\ominus a, a \oplus b] \text{gyr}[a, b] = I \quad (48)$$

$$\text{gyr}[b, a \oplus b] \text{gyr}[a, b] = I. \quad (49)$$

Proof. By two successive applications of the left gyroassociative law in two different ways, we obtain the following two chains of equations for all $a, b, c, x \in G$,

$$\begin{aligned} a \oplus (b \oplus (c \oplus x)) &= a \oplus ((b \oplus c) \oplus \text{gyr}[b, c]x) \\ &= (a \oplus (b \oplus c)) \oplus \text{gyr}[a, b \oplus c] \text{gyr}[b, c]x \end{aligned} \quad (50)$$

and

$$\begin{aligned} a \oplus (b \oplus (c \oplus x)) &= (a \oplus b) \oplus \text{gyr}[a, b](c \oplus x) \\ &= (a \oplus b) \oplus (\text{gyr}[a, b]c \oplus \text{gyr}[a, b]x) \\ &= ((a \oplus b) \oplus \text{gyr}[a, b]c) \oplus \text{gyr}[a \oplus b, \text{gyr}[a, b]c] \text{gyr}[a, b]x \\ &= (a \oplus (b \oplus c)) \oplus \text{gyr}[a \oplus b, \text{gyr}[a, b]c] \text{gyr}[a, b]x. \end{aligned} \quad (51)$$

By comparing the extreme right-hand sides of these two chains of equations, and by employing the left cancellation law, Theorem 2 (1), we obtain the identity

$$\text{gyr}[a, b \oplus c] \text{gyr}[b, c]x = \text{gyr}[a \oplus b, \text{gyr}[a, b]c] \text{gyr}[a, b]x \quad (52)$$

for all $x \in G$, thereby verifying (45).

In the special case when $c = \ominus b$, (45) descends to (46); note that the left-hand side of (45) becomes trivial owing to Items (2) and (3) of Theorem 2.

Identity (47) results from the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} I &\stackrel{(1)}{\cong} \text{gyr}[a \oplus b, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b] \\ &\stackrel{(2)}{\cong} \text{gyr}[(a \oplus b) \ominus \text{gyr}[a, b]b, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b] \\ &\stackrel{(3)}{\cong} \text{gyr}[a \oplus (b \ominus b), \ominus \text{gyr}[a, b]b] \text{gyr}[a, b] \\ &\stackrel{(4)}{\cong} \text{gyr}[a, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b]. \end{aligned} \quad (53)$$

Derivation of the numbered equalities in (53):

1. Follows from (46).
2. Follows from (1) by the left reduction property.
3. Follows from (2) by the left gyroassociative law. Indeed, an application of the left gyroassociative law to the first entry of the left gyration in (3) gives the first entry of the left gyration in (2); that is, $a \oplus (b \ominus b) = (a \oplus b) \ominus \text{gyr}[a, b]b$.
4. Follows from (3) immediately, since $b \ominus b = 0$.

To verify (48) we consider the special case of (45) when $b = \ominus a$, obtaining

$$\text{gyr}[a, \ominus a \oplus c] \text{gyr}[\ominus a, c] = \text{gyr}[0, \text{gyr}[a, \ominus a]c] \text{gyr}[a, \ominus a] = I \quad (54)$$

where the second identity in (54) follows from Items (2) and (3) of Theorem 2. Through replacing a by $\ominus a$ and c by b in (54), we obtain (48).

Finally, (49) is derived from (48) by an application of the left reduction property to the first gyroautomorphism in (48) followed by a left cancellation, Theorem 2 (9). Accordingly,

$$\begin{aligned} I &= \text{gyr}[\ominus a, a \oplus b] \text{gyr}[a, b] \\ &= \text{gyr}[\ominus a \oplus (a \oplus b), a \oplus b] \text{gyr}[a, b] \\ &= \text{gyr}[b, a \oplus b] \text{gyr}[a, b], \end{aligned} \quad (55)$$

as desired. \square

The nested gyroautomorphism identity (47) in Theorem 5 allows the equation that defines the coaddition \boxplus to be dualized with its corresponding equation in which the roles of the binary operations \boxplus and \oplus are interchanged, as shown in the following theorem.

Theorem 6. (Operation-Cooperation Duality Symmetry ([19] Theorem 2.20)). *Let (G, \oplus) be a gyrogroup with cooperation \boxplus given in Definition 5 by the equation*

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b. \quad (56)$$

Then

$$a \oplus b = a \boxplus \text{gyr}[a, b]b \quad (57)$$

for all $a, b \in G$.

Proof. Let a and b be any two elements of G . By (56) and (47) we have

$$\begin{aligned} a \boxplus \text{gyr}[a, b]b &= a \oplus \text{gyr}[a, \ominus \text{gyr}[a, b]b] \text{gyr}[a, b]b \\ &= a \oplus b, \end{aligned} \quad (58)$$

thus verifying (57). \square

Identities (56) and (57) exhibit a duality symmetry between the binary operations \oplus and \boxplus , associated with a duality symmetry between the gyrations $\text{gyr}[a, b]$ and $\text{gyr}[a, \ominus b]$. A remarkable duality symmetry solely between the gyrations $\text{gyr}[a, b]$ and $\text{gyr}[a, \ominus b]$ is presented in ([20], Theorem 4.22).

Using the notation $a \boxminus b = a \boxplus (\ominus b)$, we have in a gyrogroup (G, \oplus) , by (56) and Theorem 2 (12).

$$\begin{aligned} a \boxminus b &= a \boxplus (\ominus b) \\ &= a \oplus \text{gyr}[a, b](\ominus b) \\ &= a \ominus \text{gyr}[a, b]b. \end{aligned} \quad (59)$$

Hence,

$$a \boxminus a = a \ominus a = 0. \tag{60}$$

Identity (60) implies the equality between the inverses of $a \in G$ with respect to \oplus and \boxplus ,

$$\boxminus a = \ominus a. \tag{61}$$

Theorem 7. ([19] Theorem 2.21). *Let (G, \oplus) be a gyrogroup. Then*

$$(\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \tag{62}$$

for all $a, b, c \in G$.

Proof. By the left gyroassociative law and the left cancellation law, and using the notation $d = \ominus b \oplus c$, we have

$$\begin{aligned} (\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b](\ominus b \oplus c) &= (\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b]d \\ &= \ominus a \oplus (b \oplus d) \\ &= \ominus a \oplus (b \oplus (\ominus b \oplus c)) \\ &= \ominus a \oplus c. \end{aligned} \tag{63}$$

□

Theorem 8. (Left Gyrotranslation Theorem ([19] Theorem 2.22)). *Let (G, \oplus) be a gyrogroup. Then*

$$\ominus(\ominus a \oplus b) \oplus (\ominus a \oplus c) = \text{gyr}[\ominus a, b](\ominus b \oplus c) \tag{64}$$

for all $a, b, c \in G$.

Proof. Identity (64) is a rearrangement of (62) obtained by left-gyroadding the term $\ominus(\ominus a \oplus b)$ to both sides of (62) followed by a left cancellation. □

The significance of Identity (64) stems from the analogy it shares with its group counterpart, $-(-a + b) + (-a + c) = -b + c$ in any group with group operation $+$.

9. Basic Gyrogroup Equations

The basic equations of gyrogroup theory are

$$a \oplus x = b \tag{65}$$

and

$$x \oplus a = b, \tag{66}$$

$a, b, x \in G$, each equation for the unknown x in a gyrogroup (G, \oplus) .

Theorem 9 below asserts that each of the two basic equations possesses a unique solution. The unique solutions give rise in Section 10 to the basic gyrogroup cancellation laws.

Theorem 9. (Basic Gyrogroup Equations ([19] Theorem 2.23)). *Let (G, \oplus) be a gyrogroup, and let $a, b \in G$. The unique solution of the equation*

$$a \oplus x = b \tag{67}$$

in G for the unknown x is

$$x = \ominus a \oplus b \tag{68}$$

and the unique solution of the equation

$$x \oplus a = b \quad (69)$$

in G for the unknown x is

$$x = b \boxminus a. \quad (70)$$

Proof. Let x be a solution of the first basic equation, (65). Then we have by (65) and the left cancellation law, Theorem 2 (9), p. 9,

$$\ominus a \oplus b = \ominus a \oplus (a \oplus x) = x. \quad (71)$$

Hence, if a solution x of (65) exists then it must be given by $x = \ominus a \oplus b$, as we see from (71).

Conversely, $x = \ominus a \oplus b$ is, indeed, a solution of (65) as we see by substituting $x = \ominus a \oplus b$ into (65) and applying the left cancellation law in Theorem 2 (9). Hence, the gyrogroup equation (65) possesses the unique solution $x = \ominus a \oplus b$.

The solution of the second basic gyrogroup equation, (66), is quite different from that of the first, (65), owing to the noncommutativity of the gyrogroup operation. Let x be a solution of (66). Then we have the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} x &\stackrel{(1)}{\cong} x \oplus 0 \\ &\stackrel{(2)}{\cong} x \oplus (a \ominus a) \\ &\stackrel{(3)}{\cong} (x \oplus a) \oplus \text{gyr}[x, a](\ominus a) \\ &\stackrel{(4)}{\cong} (x \oplus a) \ominus \text{gyr}[x, a]a \\ &\stackrel{(5)}{\cong} (x \oplus a) \ominus \text{gyr}[x \oplus a, a]a \\ &\stackrel{(6)}{\cong} b \ominus \text{gyr}[b, a]a \\ &\stackrel{(7)}{\cong} b \boxminus a. \end{aligned} \quad (72)$$

Derivation of the numbered equalities in (72):

1. Follows from the existence of a unique identity element, 0, in the gyrogroup (G, \oplus) by Theorem 2, p. 9.
2. Follows from the existence of a unique inverse element $\ominus a$ of a in the gyrogroup (G, \oplus) by Theorem 2.
3. Follows from (2) by the left gyroassociative law in Axiom (G3) of gyrogroups in Definition 3, p. 7.
4. Follows from (3) by Theorem 2 (12).
5. Follows from (4) by the left reduction property (G4) of gyrogroups in Definition 3.
6. Follows from (5) by the assumption that x is a solution of (66).
7. Follows from (6) by (59).

Hence, if a solution x of (66) exists then it must be given by $x = b \boxminus a$, as we see from (72).

Conversely, $x = b \boxminus a$ is, indeed, a solution of (66), as we see from the following chain of equations:

$$\begin{aligned}
 x \oplus a &\stackrel{(1)}{=} (b \boxminus a) \oplus a \\
 &\stackrel{(2)}{=} (b \ominus \text{gyr}[b, a]a) \oplus a \\
 &\stackrel{(3)}{=} (b \ominus \text{gyr}[b, a]a) \oplus \text{gyr}[b, \ominus \text{gyr}[b, a]a] \text{gyr}[b, a]a \\
 &\stackrel{(4)}{=} b \oplus (\ominus \text{gyr}[b, a]a \oplus \text{gyr}[b, a]a) \\
 &\stackrel{(5)}{=} b \oplus 0 \\
 &\stackrel{(6)}{=} b.
 \end{aligned} \tag{73}$$

Derivation of the numbered equalities in (73):

1. Follows from the assumption that $x = b \boxminus a$.
 2. Follows from (1) by (59).
 3. Follows from (2) by Identity (47) of Theorem 5, according to which the gyration product applied to a in (3) is trivial; that is, $\text{gyr}[b, \ominus \text{gyr}[b, a]a] \text{gyr}[b, a] = I$.
 4. Follows from (3) by the left gyroassociative law. Indeed, an application of the left gyroassociative law to (4) results in (3).
 5. Follows from (4) since $\ominus \text{gyr}[b, a]a$ is the unique inverse of $\text{gyr}[b, a]a$.
 6. Follows from (5) since 0 is the unique identity element of the gyrogroup (G, \oplus) .
-

10. Basic Gyrogroup Cancellation Laws

Basic cancellation laws of gyrogroup theory are obtained from the basic equations of gyrogroups solved in Section 9. By substituting the solution (68) into its equation (Equation (67)), we obtain the left cancellation law

$$a \oplus (\ominus a \oplus b) = b \tag{74}$$

for all $a, b \in G$, stated in Theorem 2 (9).

Similarly, by substituting the solution (70) into its equation (Equation (69)), we get the first right cancellation law

$$(b \boxminus a) \oplus a = b \tag{75}$$

for all $a, b \in G$. The latter can be dualized, obtaining the second right cancellation law

$$(b \ominus a) \boxplus a = b \tag{76}$$

for all $a, b \in G$. Indeed, (76) follows from the chain of equations

$$\begin{aligned}
 b &= b \oplus 0 \\
 &= b \oplus (\ominus a \oplus a) \\
 &= (b \ominus a) \oplus \text{gyr}[b, \ominus a]a \\
 &= (b \ominus a) \oplus \text{gyr}[b \ominus a, \ominus a]a \\
 &= (b \ominus a) \boxplus a,
 \end{aligned} \tag{77}$$

where one employs the left gyroassociative law, the left reduction property, and the gyrogroup cooperation definition. Identities (74)–(76) form the three basic cancellation laws of gyrogroup theory ([19] Section 2.11).

11. Automorphisms and Gyroautomorphisms

In this section we find a commuting relation between automorphisms of a gyrogroup and its gyroautomorphisms.

Theorem 10. ([19] Theorem 2.24). *For any two elements a, b of a gyrogroup (G, \oplus) and any automorphism A of (G, \oplus) , $A \in \text{Aut}(G, \oplus)$,*

$$\text{Agyr}[a, b] = \text{gyr}[Aa, Ab]A. \quad (78)$$

Proof. For any three elements $a, b, x \in (G, \oplus)$ and any automorphism $A \in \text{Aut}(G, \oplus)$ we have by the left gyroassociative law,

$$\begin{aligned} (Aa \oplus Ab) \oplus \text{Agyr}[a, b]x &= A((a \oplus b) \oplus \text{gyr}[a, b]x) \\ &= A(a \oplus (b \oplus x)) \\ &= Aa \oplus (Ab \oplus Ax) \\ &= (Aa \oplus Ab) \oplus \text{gyr}[Aa, Ab]Ax. \end{aligned} \quad (79)$$

Hence, by a left cancellation,

$$\text{Agyr}[a, b]x = \text{gyr}[Aa, Ab]Ax \quad (80)$$

for all $x \in G$, implying (78). \square

Theorem 11. ([19] Theorem 2.25). *Let a, b be any two elements of a gyrogroup (G, \oplus) and let $A \in \text{Aut}(G)$ be an automorphism of G . Then*

$$\text{gyr}[a, b] = \text{gyr}[Aa, Ab] \quad (81)$$

if and only if the automorphisms A and $\text{gyr}[a, b]$ commute.

Proof. If $\text{gyr}[Aa, Ab] = \text{gyr}[a, b]$, then, by Theorem 10, the automorphisms $\text{gyr}[a, b]$ and A commute, $\text{Agyr}[a, b] = \text{gyr}[Aa, Ab]A = \text{gyr}[a, b]A$. Conversely, if $\text{gyr}[a, b]$ and A commute then, by Theorem 10, $\text{gyr}[Aa, Ab] = \text{Agyr}[a, b]A^{-1} = \text{gyr}[a, b]AA^{-1} = \text{gyr}[a, b]$. \square

As a simple, useful consequence of Theorem 11 we present the identity

$$\text{gyr}[\text{gyr}[a, b]a, \text{gyr}[a, b]b] = \text{gyr}[a, b]. \quad (82)$$

Theorem 12. ([19] Theorem 2.26). *A gyrogroup (G, \oplus) and its associated cogyrogroup (G, \boxplus) possess the same automorphism group,*

$$\text{Aut}(G, \boxplus) = \text{Aut}(G, \oplus). \quad (83)$$

Proof. Let $\tau \in \text{Aut}(G, \oplus)$. Then by Theorem 10

$$\begin{aligned} \tau(a \boxplus b) &= \tau(a \oplus \text{gyr}[a, \ominus b]b) \\ &= \tau a \oplus \tau \text{gyr}[a, \ominus b]b \\ &= \tau a \oplus \text{gyr}[\tau a, \ominus \tau b]\tau b \\ &= \tau a \boxplus \tau b, \end{aligned} \quad (84)$$

so that $\tau \in \text{Aut}(G, \boxplus)$, implying

$$\text{Aut}(G, \boxplus) \supseteq \text{Aut}(G, \oplus). \quad (85)$$

Conversely, let $\tau \in \text{Aut}(G, \boxplus)$. Then

$$0 \oplus \tau 0 = \tau 0 = \tau(0 \boxplus 0) = \tau 0 \boxplus \tau 0 = \tau 0 \oplus \tau 0 \quad (86)$$

implying

$$\tau 0 = 0. \quad (87)$$

Hence,

$$0 = \tau 0 = \tau(a \boxminus a) = \tau(a \boxplus (\boxminus a)) = \tau a \boxplus \tau(\boxminus a) \quad (88)$$

implying that $\tau(\boxminus a)$ is the unique inverse of τa in (G, \boxplus) ,

$$\tau(\boxminus a) = \boxminus \tau a, \quad (89)$$

so that, finally,

$$\tau(a \boxminus b) = \tau(a \boxplus (\boxminus b)) = \tau a \boxplus \tau(\boxminus b) = \tau a \boxplus (\boxminus \tau b) = \tau a \boxminus \tau b. \quad (90)$$

Owing to the second right cancellation law (76), and (90), we have

$$\tau a = \tau((a \oplus b) \boxminus b) = \tau(a \oplus b) \boxminus \tau b, \quad (91)$$

so that by the first right cancellation law (75),

$$\tau(a \oplus b) = \tau a \oplus \tau b. \quad (92)$$

Hence, $\tau \in \text{Aut}(G, \oplus)$, implying

$$\text{Aut}(G, \boxplus) \subseteq \text{Aut}(G, \oplus), \quad (93)$$

so that, by (85) and (93),

$$\text{Aut}(G, \boxplus) = \text{Aut}(G, \oplus), \quad (94)$$

as desired. \square

Theorem 12 enhances the duality symmetry that Einstein addition and coaddition share in Theorem 6 and in (75) and (76).

12. On the Rich Gyrostructure that Stems from Einstein Addition

The introduction of a coaddition, \boxplus , into the abstract gyrogroup (G, \oplus) is dictated by our wish to capture analogies with classical results.

Einstein addition, \oplus , admits scalar multiplication, \otimes , which turns Einstein gyrocommutative gyrogroups (\mathbb{R}_c^n, \oplus) to Einstein gyrovectors spaces $(\mathbb{R}_c^n, \oplus, \otimes)$. The latter are extended by abstraction to abstract gyrovectors spaces, which are fully analogous to vector spaces. Thus, in particular, (i) Einstein gyrovectors spaces form the algebraic setting for the Beltrami–Klein ball model of hyperbolic geometry; and (ii) Möbius gyrovectors spaces form the algebraic setting for the Poincaré ball model of hyperbolic geometry, just as (iii) vector spaces form the algebraic setting for the standard model of Euclidean geometry, as shown, for instance, in [20,21,31].

Within the frame of gyrovectors spaces we study concepts such as gyrolines, cogyrolines, gyroangles and cogyroangles, in full analogy with the study of their classical counterparts. Remarkably, along with Einstein addition, it is Einstein coaddition that enables us to capture analogies between

parallelograms in the Euclidean geometry of \mathbb{R}^n and gyroparallelograms in the hyperbolic geometry of the ball \mathbb{R}_c^n . For details, see [19–21,31,54,56–58]),

The importance of understanding the rich gyrostructure that stems from Einstein addition to our study of Lorentz symmetry groups rests on the fact that (i) relativistically admissible velocities in the ball \mathbb{R}_c^n parametrize the Lorentz groups; and (ii) compositions of Lorentz transformations involve both Einstein addition and gyroautomorphisms, as we will see in (103).

13. Galilei and Lorentz Boosts and Multi-boosts

In order to pave the road to extending the Galilei and Lorentz boosts to *multi-boosts* called bi-boosts of signature (m, n) , $m, n \in \mathbb{N}$, we present in this section the common Galilei and Lorentz boosts along with an additive decomposition (104) that associates Galilei and Lorentz boosts to each other.

A Galilei transformation without rotations is called a Galilei boost. The Galilei boost is a linear transformation of time-space coordinates in $(1 + 3)$ -dimensional time-space, represented by the matrix

$$G(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}. \tag{95}$$

The action of a Galilei boost $G(\mathbf{v})$ on the time-space coordinates $(t, \mathbf{x})^t$ of a particle with position \mathbf{x} at time t gives

$$G(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ v_1 t + x_1 \\ v_2 t + x_2 \\ v_3 t + x_3 \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix}, \tag{96}$$

where $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, and where exponent t denotes transposition.

Here \mathbf{v} represents velocity and $(t, \mathbf{x})^t$ represents a particle with position \mathbf{x} at time t . Accordingly, the Galilei boost $B(\mathbf{v})$ in (96) boosts a particle with position \mathbf{x} at time t to a boosted particle with position $\mathbf{x} + \mathbf{v}t$ at time t .

Extending from 3 to n space dimensions, the Galilei boost is represented by the $(n + 1) \times (n + 1)$ matrix

$$G(\mathbf{v}) = \begin{pmatrix} 1 & 0_{1,n} \\ \mathbf{v} & I_n \end{pmatrix} \tag{97}$$

and its action on time-space coordinates gives

$$G(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix}, \tag{98}$$

where $\mathbf{v} = (v_1, v_2, \dots, v_n)^t \in \mathbb{R}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)^t \in \mathbb{R}^n$ are column vectors. Here, I_n is the $n \times n$ identity matrix and $0_{m,n}$ is the $m \times n$ zero matrix.

Galilei boosts are parametrized by the velocity parameter \mathbf{v} , and the Galilei boost composition is given by velocity addition, that is,

$$G(\mathbf{v}_1)G(\mathbf{v}_2) = G(\mathbf{v}_1 + \mathbf{v}_2) \tag{99}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$. Obviously, the inverse of $G(\mathbf{v})$ is $G^{-1}(\mathbf{v}) = G(-\mathbf{v})$.

Anticipating the extension of the Galilei boost to m time and n space dimensions for all $m, n \in \mathbb{N}$, we call $G(\mathbf{v})$ in (97)–(99) a Galilei boost of signature $(1, n)$. The Galilei boosts of signature $(1, n)$

form a group, and the group of Galilei boosts of signature (1,3) forms the symmetry group of classical mechanics.

A Lorentz boost is a Lorentz transformation without rotations. The Lorentz boosts $B_c(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}_c^n$, that correspond to the Galilei boosts of signature (1, n) are represented by the $(n + 1) \times (n + 1)$ matrix

$$B_c(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & \frac{1}{c^2} \gamma_{\mathbf{v}} \mathbf{v}^t \\ \gamma_{\mathbf{v}} \mathbf{v} & I_n + \frac{1}{c^2} \frac{\gamma_{\mathbf{v}}^2}{1 + \gamma_{\mathbf{v}}} \mathbf{v} \mathbf{v}^t \end{pmatrix}, \tag{100}$$

parametrized by the velocity vector $\mathbf{v} \in \mathbb{R}_c^n$. The inverse of $B_c(\mathbf{v})$ is given by

$$B_c^{-1}(\mathbf{v}) = B_c(-\mathbf{v}). \tag{101}$$

We clearly have the limit in which a Lorentz boost descends to its Galilean counterpart,

$$\lim_{c \rightarrow \infty} B_c(\mathbf{v}) = G(\mathbf{v}) \tag{102}$$

for any $\mathbf{v} \in \mathbb{R}_c^n$.

The composition of two Galilei boosts of signature (1, n) is, again, a Galilei boost of signature (1, n), as we see from (99). In contrast, the composition of two Lorentz boosts of signature (1, n) is not a Lorentz boost. Rather, it is a Lorentz boost of signature (1, n) preceded (or, followed) by a space gyration (a gyration is a rotation known in special relativity as Thomas precession),

$$\begin{aligned} B_c(\mathbf{v}_1)B_c(\mathbf{v}_2) &= B_c(\mathbf{v}_1 \oplus \mathbf{v}_2) \begin{pmatrix} 1 & 0_{1,n} \\ 0_{n,1} & \text{gyr}[\mathbf{v}_1, \mathbf{v}_2] \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0_{1,n} \\ 0_{n,1} & \text{gyr}[\mathbf{v}_1, \mathbf{v}_2] \end{pmatrix} B_c(\mathbf{v}_2 \oplus \mathbf{v}_1), \end{aligned} \tag{103}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}_c^n$, where Einstein addition \oplus in the ball \mathbb{R}_c^n is given by (2), and the gyration $\text{gyr}[\mathbf{v}_1, \mathbf{v}_2]$ is given in terms of Einstein addition in Theorem 2 (10). The study of the Lorentz boost composition law (103) is presented in ([21] Theorem 4.10).

The Lorentz boost composition law (103) involves Einstein addition along with a gyration. As such, it demonstrates the important role that the gyroalgebra of Einstein addition plays in our understanding of the Lorentz transformation group of signature (1, n). The gyration $\text{gyr}[\mathbf{v}_1, \mathbf{v}_2]$ in (103) represents a space rotation. In the extension of our gyroformalism to Lorentz transformations of signature (m, n), $m, n \geq 2$, a second gyration, which represents a time rotation, emerges. Naturally, the pair of a space and a time gyration will be called a *bi-gyration*, as we will see in the sequel.

The Lorentz transformation group of signature (1, n) is the symmetry group of Einstein’s special relativity, where $n = 3$ in physical applications. In order to understand the Lorentz transformation $B_c(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}_c^n$, of signature (1, n) in terms of its associated Galilei transformation of signature (1, n), we consider the following *additive decomposition* of $B_c(\mathbf{v})$ ([19] Section 6.3.1),

$$B_c(\mathbf{v}) = \begin{pmatrix} 1 & 0_{1,n} \\ \mathbf{v} & I_n \end{pmatrix} + \frac{1}{c^2} \gamma_{\mathbf{v}} \begin{pmatrix} \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v}^2 & \mathbf{v}^t \\ \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v}^2 \mathbf{v} & \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v} \mathbf{v}^t \end{pmatrix} \tag{104}$$

which one can readily prove by means of (10), where $\mathbf{v}^2 = \mathbf{v}^t \mathbf{v} = \|\mathbf{v}\|^2$, for all $\mathbf{v} \in \mathbb{R}_c^n$.

The additive decomposition (104) of $B_c(\mathbf{v})$ has two components. It can be written as

$$B_c(\mathbf{v}) = G(\mathbf{v}) + \frac{1}{c^2}E(\mathbf{v}) \quad (105)$$

where (i) $G(\mathbf{v})$ is the Galilean component of $B_c(\mathbf{v})$, given in (97), and (ii) $(1/c^2)E(\mathbf{v})$, where

$$E(\mathbf{v}) = \gamma_{\mathbf{v}} \begin{pmatrix} \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v}^2 & \mathbf{v}^t \\ \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v}^2 \mathbf{v} & \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v} \mathbf{v}^t \end{pmatrix}, \quad (106)$$

$\mathbf{v} \in \mathbb{R}_c^n$, is the relativistic entanglement component of $B_c(\mathbf{v})$.

The interpretation of the parameter $\mathbf{v} \in \mathbb{R}_c^n$ of the Lorentz boost $B_c(\mathbf{v})$ as a velocity vector is induced by the interpretation of its Galilean component, the component that is intuitively clear. This interpretation demonstrates the importance of the additive decomposition (105) in understanding Lorentz boosts in terms of their Galilean counterparts.

The additive decomposition (105) of the Lorentz boost $B_c(\mathbf{v})$ tells us that the effects of a Lorentz boost $B_c(\mathbf{v})$ consist of Galilean effects due to the Galilean component $G(\mathbf{v})$, along with relativistic effects due to the relativistic entanglement component $(1/c^2)E(\mathbf{v})$. The relativistic effects are directly noticeable only in high speeds, owing to the presence of the factor $(1/c^2)$ in the relativistic entanglement component. Unlike the Galilean effects, which are intuitively clear, the relativistic effects are counterintuitive, involving entanglement of space and time coordinates of moving particles, as well as other relativistic effects, such as time dilation, length contraction and Thomas precession.

Accordingly, it is intuitively clear how to extend the Galilei boost from the common boost that acts on particles individually to a *multi-boost* that acts on several particles collectively. In contrast, it is not yet clear how to extend the Lorentz boost from the common boost that acts on particles individually to a *multi-boost* that acts on several particles collectively.

Fortunately, the intuitively clear Galilei multi-boosts come to the rescue. They will guide us in the search for the Lorentz multi-boosts that act on several particles collectively, rather than individually. Remarkably, the Lorentz multi-boosts that the Galilei multi-boosts suggest, will turn out in the sequel to be nothing else but the well-known Lorentz transformations of signature (m, n) , $m, n > 1$.

We, therefore, proceed with the study of Lorentz transformations of any signature (m, n) , $m, n \in \mathbb{N}$, before turning to study Galilei multi-boosts. Our goal is, accordingly, to discover how to extend the additive decomposition (104) of the Lorentz boost from signature $(1, n)$ to any signature (m, n) , $m, n \in \mathbb{N}$.

14. Pseudo-Euclidean Spaces and Lorentz Transformations of any Signature

We now face the task to discover that the rich gyroalgebra of Einstein addition and Lorentz transformations of signature $(1, n)$ that we have studied in Sections 2–13 within the framework of special relativity, survive unimpaired in their transition from signature $(1, n)$ to any signature (m, n) , $m, n \in \mathbb{N}$ ([19] Section 4.2), [59].

The study of pseudo-Euclidean spaces and pseudo-rotations is well-known in linear algebra [13] and in physics [14]. A pseudo-Euclidean space $\mathbb{R}^{m,n}$, $m, n \in \mathbb{N}$, of signature (m, n) is an $(m + n)$ -dimensional space with an orthogonal basis e_i , $i = 1, \dots, m + n$,

$$e_i \cdot e_j = \epsilon_i \delta_{ij}, \quad (107)$$

where δ_{ij} , $j = 1, \dots, m+n$, is the Kronecker delta, where

$$\epsilon_i = \begin{cases} +1, & i = 1, \dots, m \\ -\frac{1}{c^2}, & i = m+1, \dots, m+n, \end{cases} \quad (108)$$

and where c is an arbitrarily fixed positive constant.

Without loss of generality one may select $c = 1$. However, we prefer to view c as a free positive parameter, $0 < c < \infty$, in order to allow limits as $c \rightarrow \infty$ to be available, where the modern and new descends to the classical and familiar. An illustrative point in case is the Lorentz transformation of special relativity theory, which descends to the Galilei transformation of classical mechanics when the vacuum speed of light c tends to infinity, as shown in (149)–(155) with any signature (m, n) . The usefulness of viewing c as a positive parameter rather than selecting $c = 1$ is clearly demonstrated in the additive decomposition (155) of the Lorentz multi-boost, which plays an important role in our study.

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n) is equipped with an inner product of signature (m, n) . The inner product $\mathbf{x} \cdot \mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$,

$$\mathbf{x} = \sum_{i=1}^{m+n} x_i e_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{m+n} y_i e_i, \quad (109)$$

is given by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{m+n} \epsilon_i x_i y_i = \sum_{i=1}^m x_i y_i - \frac{1}{c^2} \sum_{i=m+1}^{m+n} x_i y_i. \quad (110)$$

Accordingly, the squared norm of $\mathbf{x} \in \mathbb{R}^{m,n}$ is

$$\|\mathbf{x}\|^2 = \mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^{m+n} \epsilon_i x_i^2 = \sum_{i=1}^m x_i^2 - \frac{1}{c^2} \sum_{i=m+1}^{m+n} x_i^2. \quad (111)$$

Let η be the $(m+n) \times (m+n)$ diagonal matrix

$$\eta = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -\frac{1}{c^2} I_n \end{pmatrix}, \quad (112)$$

where I_m is the $m \times m$ identity matrix and $0_{m,n}$ is the $m \times n$ zero matrix. Then, the matrix representation of the inner product (110) is given by

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t \eta \mathbf{y}, \quad (113)$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$ are the column vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+n} \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+n} \end{pmatrix}, \quad (114)$$

and exponent t denotes transposition.

Let Λ be an $(m+n) \times (m+n)$ matrix that leaves the inner product (113) invariant. Then, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$,

$$(\Lambda \mathbf{x})^t \eta \Lambda \mathbf{y} = \mathbf{x}^t \eta \mathbf{y}, \quad (115)$$

implying $\mathbf{x}^t \Lambda^t \eta \Lambda \mathbf{y} = \mathbf{x}^t \eta \mathbf{y}$, so that ([60] p. 193)

$$\Lambda^t \eta \Lambda = \eta. \quad (116)$$

The determinant of the matrix equation (116) yields

$$(\det \Lambda)^2 = 1, \quad (117)$$

noting that $\det(\Lambda^t \eta \Lambda) = (\det \Lambda^t)(\det \eta)(\det \Lambda)$ and $\det \Lambda^t = \det \Lambda$. Hence, $\det \Lambda = \pm 1$.

The special transformations Λ that can be reached continuously from the identity transformation of $\mathbb{R}^{m,n}$ constitute the special pseudo-orthogonal group $\text{SO}_c(m, n)$, called the group of pseudo-rotations of signature (m, n) . A pseudo-rotation of signature (m, n) is also called a Lorentz transformation of signature (m, n) , or in short, an (m, n) -Lorentz transformation. The Lorentz transformation of signature $(1, 3)$ turns out to be the common homogeneous, proper, orthochronous Lorentz transformation of Einstein's special theory of relativity [21]. The formal definition of $\text{SO}_c(m, n)$ follows.

Definition 6. (Lorentz Group $\text{SO}_c(m, n)$ of Signature (m, n) ([19] Definition 4.1)). Let $m, n \in \mathbb{N}$ be two positive integers. A linear transformation Λ of the pseudo-Euclidean space $\mathbb{R}^{m,n}$ is called a pseudo-rotation of signature (m, n) , or a Lorentz transformation of signature (m, n) , if it leaves the inner product (110) invariant, and if it can be reached continuously from the identity transformation of $\mathbb{R}^{m,n}$. The group of all Lorentz transformations of signature (m, n) , denoted by $\text{SO}_c(m, n)$, is called the Lorentz group of signature (m, n) , or the (m, n) -Lorentz group.

If $\Lambda \in \text{SO}_c(m, n)$ then its determinant is equal to 1, and the determinant of its first m rows and columns is positive ([61] p. 478).

15. Matrix Balls of Radius c

The concept of relativistically admissible velocities is extended in this section from velocity vectors in a ball $\mathbb{R}_c^n \subset \mathbb{R}^n$ of vectors to velocity matrices in a ball $\mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$ of matrices.

Definition 7. (Spectrum, Spectral Radius ([62] p. 35)). Let $\mathbb{R}^{n \times m}$ be the set of all real $n \times m$ matrices, $m, n \in \mathbb{N}$. The set of all complex numbers that are eigenvalues of a square matrix $A \in \mathbb{R}^{n \times n}$ is called the spectrum of A and is denoted by $\sigma(A)$. The spectral radius of A is the nonnegative number

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}. \quad (118)$$

The spectral radius $\rho(A)$ is thus the radius of the smallest disc centered at the origin of the complex plane that includes all the eigenvalues of A .

For Definition 8 below, we note that for any $V \in \mathbb{R}^{n \times m}$ the set of nonzero eigenvalues of $VV^t \in \mathbb{R}^{n \times n}$ equals the set of nonzero eigenvalues of $V^tV \in \mathbb{R}^{m \times m}$.

Definition 8. (Matrix Ball, Matrix Spectral Norm ([19] Definition 5.7)). For any $m, n \in \mathbb{N}$ and $c > 0$, the c -ball $\mathbb{R}_c^{n \times m}$ of the ambient space $\mathbb{R}^{n \times m}$ of all $n \times m$ real matrices is given by

$$\begin{aligned} \mathbb{R}_c^{n \times m} &= \{V \in \mathbb{R}^{n \times m} : \forall \lambda \in \sigma(VV^t), \sqrt{\lambda} < c\} \\ &= \{V \in \mathbb{R}^{n \times m} : \forall \lambda \in \sigma(V^tV), \sqrt{\lambda} < c\}. \end{aligned} \quad (119)$$

The matrix spectral norm $\|V\|$ of $V \in \mathbb{R}^{n \times m}$, or norm in short, is defined by

$$\begin{aligned} \|V\| &= \max\{\sqrt{\lambda} : \lambda \in \sigma(VV^t)\} \\ &= \max\{\sqrt{\lambda} : \lambda \in \sigma(V^tV)\}. \end{aligned} \tag{120}$$

It is clear from Definition 8 that

$$\mathbb{R}_c^{n \times m} = \{V \in \mathbb{R}^{n \times m} : \|V\| < c\}, \tag{121}$$

so that the matrix ball $\mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$ is a natural generalization of the ball $\mathbb{R}_c^n = \mathbb{R}_c^{n \times 1} \subset \mathbb{R}^{n \times 1} = \mathbb{R}^n$ in (1) from signature $(1, n)$ to any signature (m, n) , $m, n \in \mathbb{N}$.

Well-known properties of the matrix spectral norm are [62]:

$\ A\ \geq 0$	Nonnegative	(122)
$\ A\ = 0$ if and only if $A = 0$	Positive	
$\ rA\ = r \ A\ $	Homogeneity Property	
$\ A + B\ \leq \ A\ + \ B\ $	Triangle Inequality	
$\ AB\ \leq \ A\ \ B\ $	Submultiplicity	

for any $r \in \mathbb{R}$ and square matrices $A, B \in \mathbb{R}^{k \times k}$, $k \in \mathbb{N}$.

Theorem 13. ([19] Section 5.3, Theorem 5.17). For any $m, n \in \mathbb{N}$ and $c > 0$, let $V \in \mathbb{R}^{n \times m}$. Then, $V \in \mathbb{R}_c^{n \times m}$ if and only if the real matrix

$$\Gamma_V^L = \Gamma_{n,V,c}^L := \sqrt{I_n - c^{-2}VV^t}^{-1} \in \mathbb{R}^{n \times n} \tag{123}$$

exists. Similarly, $V \in \mathbb{R}_c^{n \times m}$ if and only if the real matrix

$$\Gamma_V^R = \Gamma_{m,V,c}^R := \sqrt{I_m - c^{-2}V^tV}^{-1} \in \mathbb{R}^{m \times m} \tag{124}$$

exists.

The obvious limits

$$\begin{aligned} \lim_{c \rightarrow \infty} \Gamma_V^L &= I_n \\ \lim_{c \rightarrow \infty} \Gamma_V^R &= I_m \end{aligned} \tag{125}$$

will prove useful in (152).

We use the abbreviated notation Γ_V^L and Γ_V^R in (123) and (124), rather than the full notation $\Gamma_{n,V,c}^L$ and $\Gamma_{m,V,c}^R$, since the values of m, n and c are always known from the context. We call Γ_V^L and Γ_V^R , respectively, the left and right gamma factors of signature (m, n) . Collectively, the pair (Γ_V^L, Γ_V^R) is called the *bi-gamma factor*.

In view of (125), it is interesting to note that for any fixed $m, n \in \mathbb{N}$ and $c > 0$, the left and right gamma factors of signature (m, n) satisfy the matrix identities ([19] Lemma 5.82)

$$\begin{aligned} \Gamma_V^L &= I_n + \frac{1}{c^2} \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} VV^t \\ \Gamma_V^R &= I_m + \frac{1}{c^2} \frac{(\Gamma_V^R)^2}{I_m + \Gamma_V^R} V^tV, \end{aligned} \tag{126}$$

for all $V \in \mathbb{R}_c^{n \times m}$.

In (126) we use the convenient *matrix division notation* A/B to denote either AB^{-1} or $B^{-1}A$ when no confusion may arise, that is, when the matrices A and B satisfy $AB^{-1} = B^{-1}A$. The identities in (126) prove useful in establishing the additive decomposition (155), which plays an important role in the interpretation of the Lorentz transformation of signature (m, n) , $m, n \in \mathbb{N}$.

In the special case when $m = 1$, the right gamma factor, Γ_V^R in (124), descends to the Lorentz gamma factor, γ_V in (3), of special relativity theory in a single time dimension and n space dimensions,

$$\Gamma_V^R = \frac{1}{\sqrt{1 - c^{-2}\|V\|^2}} =: \gamma_V, \quad (m = 1), \tag{127}$$

$V \in \mathbb{R}^{n \times 1} = \mathbb{R}^n$, where $\|V\|^2 = V^tV$.

It can be shown that, for $m = 1$, the left counterpart of (127) is given by Equation (5.172) in [19].

$$\Gamma_V^L = I_n + \frac{1}{c^2} \frac{\gamma_V^2}{1 + \gamma_V} VV^t, \quad (m = 1), \tag{128}$$

$V \in \mathbb{R}_c^{n \times 1} = \mathbb{R}_c^n$, which holds for $m = 1$ and all $n \in \mathbb{N}$.

16. Bi-Gamma Factor

Analogies with the gamma factor of special relativity theory in (3) and (127) suggest the definition of a *left gamma factor* Γ_V^L and a *right gamma factor* Γ_V^R by the equations

$$\begin{aligned} \Gamma_V^L &:= \sqrt{I_n - c^{-2}VV^t}^{-1} \in \mathbb{R}^{n \times n} \\ \Gamma_V^R &:= \sqrt{I_m - c^{-2}V^tV}^{-1} \in \mathbb{R}^{m \times m}, \end{aligned} \tag{129}$$

$V \in \mathbb{R}_c^{n \times m}$. It clearly follows from (129) that

$$\begin{aligned} \Gamma_{V^t}^L &= \sqrt{I_m - c^{-2}V^tV}^{-1} = \Gamma_V^R \\ \Gamma_{V^t}^R &= \sqrt{I_n - c^{-2}VV^t}^{-1} = \Gamma_V^L \end{aligned} \tag{130}$$

for all $V \in \mathbb{R}_c^{n \times m}$. Following (130) we have interesting matrix identities as, for instance, the relations

$$\begin{aligned} (\Gamma_V^L V)^t &= V^t \Gamma_{V^t}^R \\ (V \Gamma_V^R)^t &= \Gamma_{V^t}^L V^t \end{aligned} \tag{131}$$

for any $V \in \mathbb{R}_c^{n \times m}$.

The matrices Γ_V^L and Γ_V^R are symmetric. Furthermore, they are even in the parameter $V \in \mathbb{R}_c^{n \times m}$, that is,

$$\begin{aligned} \Gamma_{-V}^L &= \Gamma_V^L \\ \Gamma_{-V}^R &= \Gamma_V^R. \end{aligned} \tag{132}$$

Naturally, the pair (Γ_V^L, Γ_V^R) of a left and a right gamma factor is called a *bi-gamma factor*.

Theorem 14. (Bi-gamma Commuting Relation). For any two positive integers $m, n \in \mathbb{N}$, the left and right gamma factors are related by the commuting relation

$$\Gamma_V^L V = V \Gamma_V^R \tag{133}$$

for all $V \in \mathbb{R}_c^{n \times m}$.

Proof. For simplicity, we assume $c = 1$. For $c > 0$ the proof is similar. By (129) and (133), we have to prove the matrix identity

$$\sqrt{I_n - VV^t}^{-1} V = V \sqrt{I_m - V^t V}^{-1} \tag{134}$$

or, equivalently,

$$\sqrt{I_n - VV^t} V = V \sqrt{I_m - V^t V} \tag{135}$$

$V \in \mathbb{R}_c^{n \times m}$.

Let $V \in \mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$ be represented by its *singular value decomposition* (SVD),

$$V = O_n \begin{pmatrix} \Sigma_k & 0_{k,m-k} \\ 0_{n-k,k} & 0_{n-k,m-k} \end{pmatrix} O_m^t \tag{136}$$

where ([62] Section 7.3)

1. $k = \text{rank}(V)$ is the rank of V , $k \leq \min\{m, n\}$;
2. Σ_k is a $k \times k$ diagonal matrix of which the diagonal entries are positive;
3. $O_n \in O(n)$ and $O_m \in O(m)$, where $O(n)$ is the group of all $n \times n$ real orthogonal matrices.

The diagonal matrix

$$\Sigma_k^2 = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2), \tag{137}$$

$\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_k^2 > 0$, where $\sigma_i^2, i = 1, \dots, k$, are the positive eigenvalues of VV^t (or, equivalently, of $V^t V$), is determined uniquely by V in (136). The square roots $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0$ are the positive *singular values* of V . Contrasting Σ_k , the orthogonal matrices O_n and O_m in (136) are not determined uniquely by V .

Hence,

$$V^t = O_m \begin{pmatrix} \Sigma_k & 0_{k,n-k} \\ 0_{m-k,k} & 0_{m-k,n-k} \end{pmatrix} O_n^t, \tag{138}$$

so that, by (136) and (138),

$$VV^t = O_n \begin{pmatrix} \Sigma_k^2 & 0_{k,n-k} \\ 0_{n-k,k} & 0_{n-k,n-k} \end{pmatrix} O_n^t \tag{139}$$

and

$$V^t V = O_m \begin{pmatrix} \Sigma_k^2 & 0_{k,m-k} \\ 0_{m-k,k} & 0_{m-k,m-k} \end{pmatrix} O_m^t. \tag{140}$$

By means of (139) we have the chain of equations

$$\begin{aligned}
 \left\{ O_n \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} O_n^t \right\}^2 &= O_n \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} O_n^t \\
 &= O_n \begin{pmatrix} I_k - \Sigma_k^2 & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} O_n^t \\
 &= I_n - O_n \begin{pmatrix} \Sigma_k^2 & 0_{k,n-k} \\ 0_{n-k,k} & 0_{n-k,n-k} \end{pmatrix} O_n^t \\
 &= I_n - VV^t,
 \end{aligned} \tag{141}$$

implying

$$\sqrt{I_n - VV^t} = O_n \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} O_n^t. \tag{142}$$

Similarly,

$$\sqrt{I_m - V^tV} = O_m \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,m-k} \\ 0_{m-k,k} & I_{m-k} \end{pmatrix} O_m^t. \tag{143}$$

Hence, by (142) and (136) we have

$$\begin{aligned}
 \sqrt{I_n - VV^t} V &= O_n \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix} \begin{pmatrix} \Sigma_k & 0_{k,m-k} \\ 0_{n-k,k} & 0_{n-k,m-k} \end{pmatrix} O_m^t \\
 &= O_n \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} \Sigma_k & 0_{k,m-k} \\ 0_{n-k,k} & 0_{n-k,m-k} \end{pmatrix} O_m^t
 \end{aligned} \tag{144}$$

and by (143) and (136) we have

$$\begin{aligned}
 V \sqrt{I_m - V^tV} &= O_n \begin{pmatrix} \Sigma_k & 0_{k,m-k} \\ 0_{n-k,k} & 0_{n-k,m-k} \end{pmatrix} \begin{pmatrix} \sqrt{I_k - \Sigma_k^2} & 0_{k,m-k} \\ 0_{m-k,k} & I_{m-k} \end{pmatrix} O_m^t \\
 &= O_n \begin{pmatrix} \Sigma_k \sqrt{I_k - \Sigma_k^2} & 0_{k,m-k} \\ 0_{n-k,k} & 0_{n-k,m-k} \end{pmatrix} O_m^t.
 \end{aligned} \tag{145}$$

The extreme right-hand sides of (144) and (145) are equal, thereby implying (135), and the proof is complete. \square

Following (133), the left and right gamma factors are related by the first commuting relation in (146) below. The remaining commuting relations in (146) follow immediately from the first one; note that left and right gamma factors are symmetric matrices.

$$\begin{aligned}
 \Gamma_V^L V &= V \Gamma_V^R \\
 \Gamma_V^R V^t &= V^t \Gamma_V^L \\
 \Gamma_V^L V V^t &= V V^t \Gamma_V^L \\
 \Gamma_V^R V^t V &= V^t V \Gamma_V^R.
 \end{aligned} \tag{146}$$

17. V-Parametric Realization of Lorentz Transformations of Signature (m,n)

The following theorem realizes the (m, n) -Lorentz group $SO_c(m, n)$ parametrically. Each element $\Lambda \in SO_c(m, n)$ is a Lorentz transformation of signature (m, n) . It is parametrized by a velocity matrix parameter $V \in \mathbb{R}_c^{n \times m}$ in the ball, and two orientation parameters $O_m \in SO(m)$ and $O_n \in SO(n)$, according to the following Theorem 15. The term *velocity matrix* will be justified in the sequel, where we will see by analogies with Galilei transformations that the columns of a velocity matrix represent relative velocities of the constituents of a multi-particle system.

Theorem 15. (Lorentz Transformation Bi-gyration Decomposition and the V-Parametric Realization ([19] Theorem 5.20)). A matrix $\Lambda \in \mathbb{R}^{(m+n) \times (m+n)}$, $m, n \in \mathbb{N}$, is the matrix representation of a Lorentz transformation Λ of signature (m, n) , $\Lambda \in SO_c(m, n)$, if and only if it possesses the bi-gyration decomposition

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \quad (147)$$

$$\in SO_c(m, n) = SO(m) \times \mathbb{R}_c^{n \times m} \times SO(n),$$

parametrized by the velocity matrix $V \in \mathbb{R}_c^{n \times m}$, and the two orientation parameters $O_m \in SO(m)$ and $O_n \in SO(n)$.

A slightly different decomposition, called the *polar decomposition* of Lorentz groups of any signature, is provided by the following theorem.

Theorem 16. (Lorentz Transformation Polar Decomposition ([19] Theorem 5.21)). A matrix $\Lambda \in \mathbb{R}^{(m+n) \times (m+n)}$, $m, n \in \mathbb{N}$, is the matrix representation of a Lorentz transformation of signature (m, n) , $\Lambda \in SO_c(m, n)$, if and only if it possesses the polar decomposition

$$\Lambda = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \quad (148)$$

$$= \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in SO_c(m, n),$$

for any $V \in \mathbb{R}_c^{n \times m}$, $O_m \in SO(m)$ and $O_n \in SO(n)$.

It follows from (147), or (148) that a Lorentz transformation Λ of signature (m, n) without rotations is what we call a Lorentz bi-boost, $B_c(V)$, of signature (m, n) ,

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in SO_c(m, n), \quad (149)$$

parametrized by the velocity matrix $V \in \mathbb{R}_c^{n \times m}$. Like (101), the inverse of $B_c(V)$ is given by

$$B_c^{-1}(V) = B_c(-V). \quad (150)$$

The matrix $B_c(V)$ is a bi-boost (as opposed to a boost) in the sense that it admits in (147) and in (148) the bi-rotation $(O_m, O_n) \in SO(m) \times SO(n)$.

Here $B_c(V) \in \mathbb{R}^{(m+n) \times (m+n)}$ is an $(m+n) \times (m+n)$ block matrix consisting of the four blocks $\Gamma_V^R \in \mathbb{R}^{m \times m}$, $\Gamma_V^L \in \mathbb{R}^{n \times n}$, $\Gamma_V^L V \in \mathbb{R}^{n \times m}$, and $c^{-2} \Gamma_V^R V^t \in \mathbb{R}^{m \times n}$.

An elegant, straightforward demonstration that $B_c(V) \in \text{SO}_c(m, n)$, as stated in (149), based on the commuting relations in (146), is presented in ([19] Section 5.8).

In the special case when the signature is $(m, n) = (1, n)$, the Lorentz bi-boost $B_c(V)$ in (149) descends to the common Lorentz transformation of Einstein's special theory of relativity, as shown in (151), where the parameter $V \in \mathbb{R}_c^{n \times 1} = \mathbb{R}_c^n$ represents relativistically admissible n -dimensional velocities.

Indeed, when $m = 1$ the bi-boost $B_c(V) \in \text{SO}(m, n)$ in (149) can be manipulated by means of (146) and by means of (127) and (128), obtaining the following chain of equations.

$$\begin{aligned} B_c(V) &= \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ V \Gamma_V^R & \Gamma_V^L \end{pmatrix} \\ &= \begin{pmatrix} \gamma_V & \frac{1}{c^2} \gamma_V V^t \\ \gamma_V V & I_n + \frac{1}{c^2} \frac{\gamma_V^2}{1 + \gamma_V} V V^t \end{pmatrix} \in \text{SO}_c(1, n), \quad (m = 1) \end{aligned} \quad (151)$$

where $V \in \mathbb{R}_c^{n \times 1} \subset \mathbb{R}^{n \times 1} = \mathbb{R}^n$ is a column vector in the ball $\mathbb{R}_c^{n \times 1} = \mathbb{R}_c^n$ of \mathbb{R}^n and where γ_V is given by (127).

The extreme right-hand side of (151) turns out to be the standard special relativistic $(n+1) \times (n+1)$ matrix representation of the Lorentz group in one time dimension and n space dimensions [28] ([20] p. 254) ([21] p. 447), presented in (100). Accordingly, it follows from (151) that in the special case when $m = 1$ the Lorentz group of signature (m, n) specializes to the Lorentz group of special relativity theory, where $(m, n) = (1, n)$.

The bi-rotation decomposition (147) presents the generic Lorentz transformation of signature (m, n) as a bi-boost along with a left rotation $O_n \in \text{SO}(n)$ and a right rotation $O_m \in \text{SO}(m)$ acting on $V \in \mathbb{R}_c^{n \times m}$. Collectively, the pair (O_n, O_m) of a left and a right rotation, taking V into $O_n V O_m$, is called a *bi-rotation*.

In the limit of large c , the Lorentz bi-boost $B_c(V)$ of signature (m, n) tends to its Galilean counterpart $B_\infty(V)$, called the Galilei bi-boost of signature (m, n) . Indeed, by means of (125) we have

$$\lim_{c \rightarrow \infty} B_c(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} =: B_\infty(V) \in \text{SO}_\infty(m, n), \quad (152)$$

$V \in \mathbb{R}_c^{n \times m} \xrightarrow{c \rightarrow \infty} \mathbb{R}^{n \times m}$, where $\text{SO}_\infty(m, n)$ is the group of all Galilei transformations of signature (m, n) . The Galilei bi-boost of signature $(1, 3)$ is the common Galilei boost of classical mechanics. The physical interpretation of the Galilei bi-boost of signature $(1, 3)$ is intuitively clear. We will see in Section 20 that the physical interpretation of the Galilei bi-boost of signature $(m, 3)$ for any $m > 1$ is intuitively clear as well. The physical interpretation of the Galilei bi-boost of signature $(m, 3)$, in turn, leads to a physical interpretation of the Lorentz bi-boost of signature $(m, 3)$, by means of the additive decomposition (155) of the Lorentz bi-boost, which we present in Section 18.

18. Additive Decomposition of the Lorentz Bi-Boost

Theorem 17. (Additive Decomposition of the Lorentz Bi-boost ([19] Theorem 5.83)). *Let*

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in \text{SO}_c(m, n), \quad (153)$$

$V \in \mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$, be the Lorentz bi-boost of signature (m, n) , $m, n \in \mathbb{N}$, and let

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} \in \mathbb{R}^{(m+n) \times (m+n)}, \quad (154)$$

$V \in \mathbb{R}^{n \times m}$, be the Galilei bi-boost of signature (m, n) .

Then, $B_c(V)$ and $B_\infty(V)$ are related to each other by the additive decomposition of the Lorentz bi-boost,

$$B_c(V) = B_\infty(V) + \frac{1}{c^2} \begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m + \Gamma_V^R} V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t V & \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t \end{pmatrix}, \quad (155)$$

for all $V \in \mathbb{R}_c^{n \times m}$.

In (155) we continue using the convenient matrix division notation, as in (126).

The additive decomposition (155) of the Lorentz bi-boost $B_c(V)$ forms the natural generalization of the additive decomposition (104) of the Lorentz boost to the Lorentz bi-boost, that is, a generalization from signature $(1, n)$ to any signature (m, n) , $m, n \in \mathbb{N}$.

The additive decomposition (155) follows immediately from (153) and (126), and it provides a correspondence between Lorentz and Galilei bi-boosts. Thus, the set of all Lorentz bi-boosts $B_c(V)$, $V \in \mathbb{R}_c^{n \times m}$, of signature (m, n) is associated in (155) with the set of all Galilei bi-boosts $B_\infty(V)$, $V \in \mathbb{R}^{n \times m}$, of same signature (m, n) , and conversely, the set of all Galilei bi-boosts of signature (m, n) is associated in (155) with the set of all Lorentz bi-boosts of same signature (m, n) , $m, n \in \mathbb{N}$. This association is important, enabling us to interpret Lorentz transformations of signature (m, n) in terms of the intuitively clear interpretation of corresponding Galilei transformations of same signature (m, n) that we will study in Section 20. It suggests the following formal definition.

Definition 9. (Additive Decomposition of the Lorentz Bi-boost ([19] Definition 5.84)). *Let*

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2} \Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix}, \quad (156)$$

$V \in \mathbb{R}_c^{n \times m}$, and

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix}, \quad (157)$$

$V \in \mathbb{R}^{n \times m}$, be the Lorentz bi-boost of signature (m, n) and its corresponding Galilei bi-boost of same signature (m, n) , $m, n \in \mathbb{N}$. Furthermore, let

$$E(V) = \begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m + \Gamma_V^R} V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t V & \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t \end{pmatrix}, \quad (158)$$

$V \in \mathbb{R}_c^{n \times m}$, so that, by Theorem 17, we have the Lorentz bi-boost additive decomposition

$$B_c(V) = B_\infty(V) + \frac{1}{c^2}E(V) \quad (159)$$

for all $V \in \mathbb{R}_c^{n \times m}$.

Following the additive decomposition (159) we say that a Lorentz bi-boost $B_c(V)$ of signature (m, n) has two components. These are (i) the Galilean component, a Galilei bi-boost $B_\infty(V)$ of signature (m, n) ; and (ii) a relativistic entanglement component, $c^{-2}E(V)$.

Owing to the presence of the factor c^{-2} in the relativistic entanglement component $c^{-2}E(V)$ of the additive decomposition (159), non-Galilean, relativistic effects are directly noticeable only at very high speeds.

The physical interpretation of the Lorentz bi-boost of signature $(m, 3)$ is well-known when $m = 1$, being the Lorentz boost of Einstein's special theory of relativity. The relativistic entanglement component of the Lorentz bi-boost $B_c(V)$ of signature $(1, 3)$ is responsible for the relativistic intertwining of space and time, as well as for other relativistic effects such as time dilation, length contraction, Thomas precession and quantum mechanical energy levels. The shifting of energy levels that results from quantum entanglement, leading to Lorentz symmetry violation, is studied in [9].

The physical interpretation of the Lorentz bi-boost of signature $(1, 3)$ leads to Einstein's special theory of relativity. In order to pave the road to extending the physical interpretation of the Lorentz bi-boost $B_c(V)$ of signature $(m, 3)$ from $m = 1$ to $m \geq 1$, we present in Sections 19 and 20 the intuitively clear physical interpretation of the Galilei bi-boost $B_\infty(V)$ of signature $(m, 3)$, $m \geq 1$.

19. Application of the Galilei Bi-Boost of Signature (1,3)

In this section about the Galilei bi-boost of signature $(1, 3)$, we make way on the road to the presentation of the Galilei bi-boost of signature $(2, 3)$, and hence, of signature $(m, 3)$, $m > 1$, in Section 20.

A Galilei bi-boost of signature $(1, 3)$ is the common Galilei boost of classical mechanics. Let $B_\infty(V) = B_\infty(\mathbf{v})$ be the Galilei bi-boost of signature $(m, n) = (1, 3)$, parametrized by the velocity parameter $V = (\mathbf{v})$,

$$V = (\mathbf{v}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 = \mathbb{R}^{3 \times 1}. \quad (160)$$

In order to conform to the formalism of bi-boosts of signature (m, n) , $m, n > 1$, in Section 20, we view V in (160) as a 3×1 matrix in $\mathbb{R}^{3 \times 1}$, and \mathbf{v} as a vector in \mathbb{R}^3 , while noting that $\mathbb{R}^{3 \times 1} = \mathbb{R}^3$. Accordingly, $V = (\mathbf{v})$ is a matrix V the single column of which is \mathbf{v} , so that \mathbf{v} is a velocity vector and V is a velocity matrix.

Let

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^{4 \times 1} \quad (161)$$

be a 4×1 matrix that represents the time-space coordinates of a particle with position $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. The point (t, \mathbf{x}) is said to be a particle of signature $(m, n) = (1, 3)$ with position $\mathbf{x} \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. A particle of signature (m, n) is also called an (m, n) -particle, in short.

The application of the Galilei bi-boost $B_\infty(V)$ of signature $(m, n) = (1, 3)$ to a $(1, 3)$ -particle (t, \mathbf{x}) in $m + n = 1 + 3$ time-space dimensions is described in the following chain of equations.

$$\begin{aligned} \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} &:= B_\infty(V) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ v_1 t + x_1 \\ v_2 t + x_2 \\ v_3 t + x_3 \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix}. \end{aligned} \quad (162)$$

Here, $B_\infty(V)$ in (162) is given by (157) with $m = 1$ and $n = 3$.

Accordingly, the Galilei bi-boost $B_\infty(V)$ of signature $(1, 3)$ is a Galilei boost that keeps the time invariant, $t' = t$, and boosts the position $\mathbf{x} \in \mathbb{R}^3$ of the particle $(t, \mathbf{x})^t$ into the position $\mathbf{x}' = \mathbf{x} + \mathbf{v}t \in \mathbb{R}^3$, $\mathbf{v} \in \mathbb{R}^3$, of the boosted particle $(t, \mathbf{x} + \mathbf{v}t)^t$, at time t .

20. Application of the Galilei Bi-boost of any Signature

We are now in the position to consider the Galilei bi-boost of signature (m, n) for all $m, n \in \mathbb{N}$, while paying special attention to the case when $(m, n) = (2, 3)$ as an illustrative example.

Let $B_\infty(V) = B_\infty(\mathbf{v}_1, \mathbf{v}_2)$ be the Galilei bi-boost of signature $(2, 3)$, parametrized by the velocity matrix $V = (\mathbf{v}_1 \ \mathbf{v}_2)$,

$$V = (\mathbf{v}_1 \ \mathbf{v}_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{pmatrix} \in \mathbb{R}^{3 \times 2}, \quad (163)$$

of two velocity vectors $\mathbf{v}_k = (v_{1k}, v_{2k}, v_{3k})^t \in \mathbb{R}^3$, $k = 1, 2$. These two velocity vectors form the two columns of the velocity matrix V , in analogy with (160).

Furthermore, let

$$\begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \in \mathbb{R}^{5 \times 2} \quad (164)$$

be a 5×2 matrix that represents a $(2, 3)$ -particle consisting of the time-space coordinates of two subparticles, (t_k, \mathbf{x}_k) , $k = 1, 2$, with positions $\mathbf{x}_k = (x_{1k}, x_{2k}, x_{3k})^t \in \mathbb{R}^3$, at time $t_k \in \mathbb{R}$, respectively. Here

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \quad (165)$$

$t_1, t_2 > 0$, is a 2×2 diagonal matrix that represents the times t_1 and t_2 when two subparticles are observed at positions \mathbf{x}_1 and \mathbf{x}_2 in \mathbb{R}^3 , respectively; and

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = (\mathbf{x}_1 \ \mathbf{x}_2) \in \mathbb{R}^{3 \times 2} \quad (166)$$

is a 3×2 matrix the columns of which represent the positions $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ of two subparticles at times $t_1, t_2 \in \mathbb{R}$, respectively.

Accordingly, the point $\begin{pmatrix} T \\ X \end{pmatrix} = (T, X) \in \mathbb{R}^{5 \times 2}$ represents a (2,3)-particle, which is a multi-particle system consisting of two subparticles (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) with positions \mathbf{x}_1 and \mathbf{x}_2 in \mathbb{R}^3 at times t_1 and t_2 , respectively. Here we use the *displayed notation*, $\begin{pmatrix} T \\ X \end{pmatrix}$, and the *inline notation*, (T, X) , interchangeably.

The collective application of the Galilei bi-boost $B_\infty(V)$ of signature (2,3) to the pair of subparticles (T, X) in $m + n = 2 + 3$ time-space dimensions yields

$$\begin{pmatrix} T' \\ X' \end{pmatrix} := B_\infty(V) \begin{pmatrix} T \\ X \end{pmatrix}, \tag{167}$$

which is described in the following chain of equations,

$$\begin{aligned} \begin{pmatrix} t'_1 & 0 \\ 0 & t'_2 \\ \mathbf{x}'_1 & \mathbf{x}'_2 \end{pmatrix} &:= B_\infty(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ v_{11} & v_{12} & 1 & 0 & 0 \\ v_{21} & v_{22} & 0 & 1 & 0 \\ v_{31} & v_{32} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \\ &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ v_{11}t_1 + x_{11} & v_{12}t_2 + x_{12} \\ v_{21}t_1 + x_{21} & v_{22}t_2 + x_{22} \\ v_{31}t_1 + x_{31} & v_{32}t_2 + x_{32} \end{pmatrix} \\ &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix}. \end{aligned} \tag{168}$$

Here, $B_\infty(V)$ in (167) and (168) is given by (157) with $m = 2$ and $n = 3$.

The chain of equations (168) describes the application of a Galilei bi-boost $B_\infty(V)$ of signature (2,3) to collectively bi-boost two subparticles, (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) , into the two bi-boosted subparticles, $(t_1, \mathbf{x}_1 + \mathbf{v}_1 t_1)$ and $(t_2, \mathbf{x}_2 + \mathbf{v}_2 t_2)$, by 3-dimensional velocity vectors $\mathbf{v}_1 = (v_{11}, v_{21}, v_{31})^t$ and $\mathbf{v}_2 = (v_{12}, v_{22}, v_{32})^t$ in \mathbb{R}^3 . It is important to note that the two collectively bi-boosted subparticles are not *entangled* in the sense that the boost of each boosted subparticle is independent of the boost of the other boosted subparticle. Interestingly, this observation fails when we replace Galilei bi-boosts of signature $(m, 3)$, $m \geq 2$, by corresponding Lorentz bi-boosts of same signature $(m, 3)$, as we will see in the sequel.

Each of the two particles (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) possesses a one-dimensional time, $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$, respectively. Accordingly, the system consisting of the two particles possesses the two-dimensional time, $(t_1, t_2) \in \mathbb{R}^2$. Each of the two particles possesses its own clock, so that the two-dimensional time of the system is measured by two clocks. In general, a multi-particle system consisting of m particles possesses an m -dimensional time, measured by m clocks, $m \in \mathbb{N}$.

The extension of (163)–(168) from signature (2,3) to signature (m, n) , $m, n \in \mathbb{N}$, is now obvious. The Galilei bi-boost $B_\infty(V)$ of signature (m, n) is parametrized by a velocity matrix $V \in \mathbb{R}^{n \times m}$ of order $n \times m$ that consists of m columns, $V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_m)$, which represent the m velocity vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ relative to some inertial frame. Furthermore, when $B_\infty(V)$ is applied to collectively bi-boost m particles in \mathbb{R}^n (i) it keeps invariant each of the times $t_k, k = 1, \dots, m$ of the m particles (t_k, \mathbf{x}_k) , that is, $t'_k = t_k$, and (ii) it bi-boosts their positions $\mathbf{x}_k \in \mathbb{R}^n$ into the bi-boosted positions $\mathbf{x}_k + \mathbf{v}_k t_k \in \mathbb{R}^n$ at times t_k , respectively. The m collectively bi-boosted particles are not entangled in the sense that (i) the boost of each boosted particle is independent of the boosts and times of the other

boosted particles and (ii) the time of each boosted particle is independent of the times and boosts of the other boosted particles.

A Galilei bi-boost of signature $(m, 3)$, applied collectively to the m subparticles of an $(m, 3)$ -particle is thus equivalent to m Galilei boosts applied individually to each subparticle. Hence, a Galilei bi-boost of signature (m, n) , $m, n \geq 2$, can be viewed as a Galilei multi-boost acting on multi-particle systems. While Galilei multi-boosts involve no entanglement, we will see that Lorentz multi-boosts accommodate entanglement of space and time coordinates of multi-particle systems.

The chain of equations (168) for the action of Galilei bi-boosts of signature $(2, 3)$ and its obvious extension to the action of Galilei bi-boosts of any signature (m, n) , $m, n \in \mathbb{N}$, demonstrate that the extension of the common Galilei boost of signature $(1, 3)$ to Galilei bi-boosts of any signature (m, n) is quite natural and intuitively clear. The additive decomposition (155) provides a correspondence between Galilei bi-boosts, $B_\infty(V)$, $V \in \mathbb{R}^{n \times m}$, of signature (m, n) , and Lorentz bi-boosts, $B_c(V)$, $V \in \mathbb{R}_c^{n \times m}$, of same signature (m, n) . This correspondence indicates that the extension of the common Lorentz boost of signature $(1, 3)$ to Lorentz bi-boosts of any signature (m, n) is quite natural as well. Yet, unlike Lorentz bi-boosts, Galilei bi-boosts of signature (m, n) are intuitively clear. We, therefore, explore the Lorentz bi-boosts of any signature (m, n) in Section 21.

21. Application of the Lorentz Bi-Boost of any Signature

We are now in the position to explore the interpretation of the Lorentz bi-boost of signature (m, n) for all $m, n \in \mathbb{N}$, paying special attention to the case when $(m, n) = (2, 3)$ as an illustrative example.

The collective application of the Lorentz bi-boost $B_c(V)$ of signature $(2, 3)$ to the pair of particles (T, X) in $2 + 3$ time-space dimensions is obtained by replacing $B_\infty(V)$ by $B_c(V)$ in (168), where $B_\infty(V)$ and $B_c(V)$ are related by the Lorentz bi-boost additive decomposition (159). Accordingly, in full analogy with (168) we obtain the chain of equations

$$\begin{aligned} \begin{pmatrix} T' \\ X' \end{pmatrix} &= \begin{pmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \\ \mathbf{x}'_1 & \mathbf{x}'_2 \end{pmatrix} := B_c(V) \begin{pmatrix} T \\ X \end{pmatrix} = B_c(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \\ &= \{B_\infty(V) + \frac{1}{c^2}E(V)\} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \\ &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix} + \frac{1}{c^2}E(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}, \end{aligned} \tag{169}$$

where T and X are given by (165) and (166), $V \in \mathbb{R}_c^{3 \times 2} \subset \mathbb{R}^{3 \times 2}$ is given by (163) and $E(V)$ is given by (158) with $(m, n) = (2, 3)$. Owing to the presence of the factor $1/c^2$ in (169), the interaction terms that appear in the entangled time components t'_{21} and t'_{12} in (169) are directly noticeable only at very high speeds.

The relativistic squared *bi-norm* (111) of each subparticle of the $(2, 3)$ -particle (T, X) remains invariant under the application in (169) of the bi-boost $B_c(V) \in \text{SO}_c(2, 3)$, that is,

$$\begin{aligned} (t'_{11})^2 + (t'_{21})^2 - c^{-2}(\mathbf{x}'_1)^2 &= t_1^2 - c^{-2}\mathbf{x}_1^2 \\ (t'_{12})^2 + (t'_{22})^2 - c^{-2}(\mathbf{x}'_2)^2 &= t_2^2 - c^{-2}\mathbf{x}_2^2, \end{aligned} \tag{170}$$

where we use the notation $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$ for vectors $\mathbf{x} \in \mathbb{R}^n$.

Moreover, the relativistic *bi-inner product* (110) of the two subparticles $(\mathbf{t}_k, \mathbf{x}_k)$, $k = 1, 2$, of the $(2, 3)$ -particle (T, X) remains invariant under a bi-boost application as well.

$$t'_{11}t'_{12} + t'_{21}t'_{22} - c^{-2}\mathbf{x}'_1 \cdot \mathbf{x}'_2 = t_1 0 + 0t_2 - c^{-2}\mathbf{x}_1 \cdot \mathbf{x}_2 = -c^{-2}\mathbf{x}_1 \cdot \mathbf{x}_2, \tag{171}$$

as implied from ([19] Theorem 6.15).

Remark 2. ([59] Remark 8). *The invariance in (171) results from the theorem in ([19] Theorem 6.15) for $(T, X) \in \mathbb{R}^{(m+n) \times m}$, $m, n \in \mathbb{N}$, in the special case when $(m, n) = (2, 3)$. Hence, it is important to note the following immediate generalization of the theorem. The two theorems in [19] (Theorem 6.14) and [19] (Theorem 6.15) are stated for $(T, X) \in \mathbb{R}^{(m+n) \times m}$ for all $m, n \in \mathbb{N}$, where $T \in \mathbb{R}^{m \times m}$ and $X \in \mathbb{R}^{n \times m}$. However, these two theorems and their proofs can be unified under a generalization of the theorem in [19] (Theorem 6.15) to the case when $(T, X) \in \mathbb{R}^{(m+n) \times k}$ for all $m, n, k \in \mathbb{N}$, so that $T \in \mathbb{R}^{m \times k}$ and $X \in \mathbb{R}^{n \times k}$. Then, each invariance in (170) corresponds to $(m, n, k) = (2, 3, 1)$, and the invariance in (171) corresponds to $(m, n, k) = (2, 3, 2)$.*

According to (169), the collective application of a Lorentz bi-boost $B_c(V)$ of signature $(m, n) = (2, 3)$ to the constituents of a system of two 3-dimensional subparticles generates classical effects and bi-relativistic effects.

The classical effects boost the two subparticles of the system individually by velocity vectors \mathbf{v}_1 and \mathbf{v}_2 , as we see from the first term on the extreme right-hand side of (169), which is the application of the Galilean component of the Lorentz bi-boost of signature $(2, 3)$.

The bi-relativistic effects involve collective entanglement of the times and positions of the system constituents, as we see from the second term on the extreme right-hand side of (169), which is the application of the relativistic entanglement component of the Lorentz bi-boost of signature $(2, 3)$.

It is evidenced from (170) that owing to the presence of entanglement, $(2, 3)$ -particles involve Lorentz symmetry violation. Indeed, the symmetry group of the $(2 + 3)$ -dimensional spacetime of a $(2, 3)$ -particle is the Lorentz group of signature $(2, 3)$, rather than the common, special relativistic Lorentz group of signature $(1, 3)$. Accordingly, the spacetime of a $(2, 3)$ -particle obeys the $(2, 3)$ -Lorentz symmetry, and as such, violates the common $(1, 3)$ -Lorentz symmetry.

The extension of (169) to the application of Lorentz bi-boosts of any signature (m, n) , $m, n \in \mathbb{N}$, to multi-particle systems of m n -dimensional subparticles is now clear. It suggests that Lorentz bi-boosts of signature (m, n) form the symmetry group by which multi-particle systems of m n -dimensional moving entangled particles can be understood, just as the Lorentz group of signature $(1, 3)$ forms the symmetry group by which Einstein’s special relativity can be understood.

22. Lorentz Bi-boost Composition Law

The special relativistic Lorentz bi-boost composition law for signature $(1, n)$ is presented in (103). In full analogy, the Lorentz bi-boost composition law for any signature (m, n) , $n, m \in \mathbb{N}$, is given by ([19] Theorem 6.7)

$$\begin{aligned} B_c(V_1)B_c(V_2) &= B_c(V_1 \oplus_E V_2) \begin{pmatrix} \text{rgyr}[V_1, V_2] & 0_{m,n} \\ 0_{n,m} & \text{lgyr}[V_1, V_2] \end{pmatrix} \\ &= \begin{pmatrix} \text{rgyr}[V_1, V_2] & 0_{m,n} \\ 0_{n,m} & \text{lgyr}[V_1, V_2] \end{pmatrix} B_c(V_2 \oplus_E V_1) \end{aligned} \tag{172}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n \times m}$. Here

$$\oplus_E = \oplus_{E,(1,n),c} \tag{173}$$

is the Einstein addition of signature (m, n) in the ball $\mathbb{R}_c^{n \times m}$, given by each of the following mutually equivalent equations (Equations 6.54 [19]):

$$\begin{aligned} V_1 \oplus_{E,(m,n),c} V_2 &= \sqrt{I_n - c^{-2} V_1 V_1^t} (I_n + c^{-2} V_2 V_2^t)^{-1} (V_1 + V_2) \sqrt{I_m - c^{-2} V_1^t V_1}^{-1} \\ V_1 \oplus_{E,(m,n),c} V_2 &= \sqrt{I_n - c^{-2} V_1 V_1^t}^{-1} (V_1 + V_2) (I_m + c^{-2} V_1^t V_2)^{-1} \sqrt{I_m - c^{-2} V_1^t V_1}, \end{aligned} \quad (174)$$

$V_1, V_2 \in \mathbb{R}_c^{n \times m}$. In bi-gyrolanguage, the Einstein addition of signature (m, n) is said to be a bi-gyroaddition. In the special case when $m = 1$ Einstein bi-gyroaddition (174) descends to the Einstein gyroaddition (2) of special relativity, as shown in ([19] Section 5.17). Not unexpectedly, Einstein bi-gyroaddition (174) in $\mathbb{R}_c^{n \times m}$ obeys the so called bi-gyrotriangle inequality ([19] Theorem 5.108),

$$\|V_1 \oplus_E V_2\| \leq \|V_1\| \oplus_E \|V_2\|, \quad (175)$$

$V_1, V_2 \in \mathbb{R}_c^{n \times m}$, which descends to the gyrotriangle inequality (14) when $m = 1$.

The Lorentz bi-boost composition law (172) involves two kinds of gyrations generated by $V_1, V_2 \in \mathbb{R}_c^{n \times m}$. These are left gyrations $\text{lgyr}[V_1, V_2] \in \text{SO}(m)$ and right gyrations $\text{rgyr}[V_1, V_2] \in \text{SO}(n)$, given by (Equation 5.329 [19]),

$$\begin{aligned} \text{lgyr}[V_2, V_1] &= \sqrt{I_n - c^{-2} V_2 V_2^t} (I_n + c^{-2} V_1 V_1^t)^{-1} \sqrt{I_n - c^{-2} V_1 V_1^t} \sqrt{I_n + c^{-2} E_{21} E_{21}^t} \\ \text{rgyr}[V_2, V_1] &= \sqrt{I_m + c^{-2} E_{21}^t E_{21}} \sqrt{I_m - c^{-2} V_2^t V_2} (I_m + c^{-2} V_1^t V_2)^{-1} \sqrt{I_m - c^{-2} V_1^t V_1}, \end{aligned} \quad (176)$$

where

$$E_{21} = \sqrt{I_n - c^{-2} V_1 V_1^t}^{-1} (V_1 + V_2) \sqrt{I_m - c^{-2} V_2^t V_2}^{-1}. \quad (177)$$

Furthermore, the extended version of the gyrator equation (17) from signature $(1, n)$ to signature (m, n) for any $m, n \in \mathbb{N}$ is implicit in ([19] Theorem 5.75).

$$\text{lgyr}[V_1, V_2] V_3 \text{rgyr}[V_2, V_1] = \ominus_E (V_1 \oplus_E V_2) \oplus_E \{V_1 \oplus_E (V_2 \oplus_E V_3)\} \quad (178)$$

for all $V_1, V_2, V_3 \in \mathbb{R}_c^{n \times m}$. Note that the left gyration in (178) is generated by V_1 and V_2 , while the right gyration in (178) is generated by V_2 and V_1 .

When $m = 1$, right gyrations are trivial and left gyrations descend to the gyrations of special relativity, where they are also known as Thomas rotations.

The polar decomposition (172) of bi-boost products determines the Einstein addition \oplus_E in $\mathbb{R}_c^{n \times m}$, given by (174). Similarly, the identity

$$B_c(r \otimes V) = (B_c(V))^r \quad (179)$$

for all $r \in \mathbb{R}$ and $V \in \mathbb{R}_c^{n \times m}$, determines $r \otimes V \in \mathbb{R}_c^{n \times m}$ in terms of r and V . The resulting Einstein scalar multiplication $r \otimes V$ is presented in Theorem 5.86 of [19]. The triple $(\mathbb{R}_c^{n \times m}, \oplus_E, \otimes)$ forms a bi-gyrovector space of signature (m, n) , $m, n \in \mathbb{N}$, studied in [19] along with its bi-hyperbolic geometry. It is shown in [19] that bi-hyperbolic geometry enables us to study bi-relativistic entanglement geometrically, but this will not be reviewed here. Bi-hyperbolic geometry is hyperbolic geometry of signature (m, n) , $m, n \in \mathbb{N}$. When $(m, n) = (1, n)$ bi-hyperbolic geometry descends to the hyperbolic geometry of Lobachevsky and Bolyai, studied analytically in [20,21,31,54,56–58], which involves no entanglement. When $m \geq 2$, analytic bi-hyperbolic geometry of signature (m, n) involves entanglement, studied in [19], where it is illustrated graphically for signature $(m, n) = (3, 2)$.

As suggested by (178), a left and a right gyration, lgyr and rgyr , determine a gyration, $\text{gyr}[V_1, V_2] : \mathbb{R}_c^{n \times m} \rightarrow \mathbb{R}_c^{n \times m}$ by the equation ([19] Equation 5.526),

$$\text{gyr}[V_1, V_2]V_3 := \text{lgyr}[V_1, V_2]V_3\text{rgyr}[V_2, V_1], \tag{180}$$

for all $V_1, V_2, V_3 \in \mathbb{R}_c^{n \times m}$.

Then, an extension of (18) from signature $(1, n)$ to any signature (m, n) , $m, n \in \mathbb{N}$, emerges, giving rise to the following elegant bi-gyrogroup identities of any signature (m, n) ([19] Chap. 5):

$$\begin{aligned} V_1 \oplus_E V_2 &= \text{gyr}[V_1, V_2](V_2 \oplus_E V_1) && \text{Gyrocommutative Law} \\ V_1 \oplus_E (V_2 \oplus_E V_3) &= (V_1 \oplus_E V_2) \oplus_E \text{gyr}[V_1, V_2]V_3 && \text{Left Gyroassociative Law} \\ (V_1 \oplus_E V_2) \oplus_E V_3 &= V_1 \oplus_E (V_2 \oplus_E \text{gyr}[V_2, V_1]V_3) && \text{Right Gyroassociative Law} \\ \text{gyr}[V_1 \oplus_E V_2, V_2] &= \text{gyr}[V_1, V_2] && \text{Left Reduction Property} \\ \text{gyr}[V_1, V_2 \oplus_E V_1] &= \text{gyr}[V_1, V_2] && \text{Right Reduction Property} \\ \text{gyr}[\ominus_E V_1, \ominus_E V_2] &= \text{gyr}[V_1, V_2] && \text{Gyration Even Property} \\ (\text{gyr}[V_1, V_2])^{-1} &= \text{gyr}[V_2, V_1] && \text{Gyration Inversion Law} \end{aligned} \tag{181}$$

for all $V_1, V_2, V_3 \in \mathbb{R}_c^{n \times m}$, $m, n \in \mathbb{N}$. Accordingly, the pair $(\mathbb{R}_c^{n \times m}, \oplus_E)$ is a gyrocommutative gyrogroup, called an Einstein bi-gyrogroup of signature (m, n) .

23. A Supporting Property

A velocity matrix $V \in \mathbb{R}_c^{n \times m}$, $m \geq 2$, whose $m \geq 2$ columns are equal to each other, is called an *equi-column velocity matrix*. Equi-column velocity matrices are necessarily employed when one wishes to translate the spacetime coordinates between inertial observers who observe the same multi-particle system. As an illustrative example, we consider the simple case of $m = 3$. The general case of $m \geq 2$ will, then, be clear. Accordingly, let (T, X) be the multi-particle system of three constituent particles (that may be called *subparticles*) given by

$$\begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}, \tag{182}$$

as observed by a moving observer S , who moves with velocity \mathbf{v} relative to an observer S' at rest relative to some laboratory. The velocity $\mathbf{v} \in \mathbb{R}^2$ is such that the equi-column velocity matrix V is *bi-relativistically admissible*, that is,

$$V = (\mathbf{v} \mathbf{v} \mathbf{v}) \in \mathbb{R}_c^{n \times m}, \tag{183}$$

where $(m, n) = (3, 2)$.

Then, the coordinates of the multi-particle system (182) as observed by observer S' are given by

$$\begin{pmatrix} T' \\ X' \end{pmatrix} = B_c(V) \begin{pmatrix} T \\ X \end{pmatrix}. \tag{184}$$

We say that the multi-particle system (182) is *integrated* (as opposed to *disintegrated*) if its constituent particles have equal positions at equal times, that is, if

$$\begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \\ x_1 & x_1 & x_1 \\ x_2 & x_2 & x_2 \end{pmatrix}. \tag{185}$$

We may consider the integrated system (T, X) in (185) as a single particle capable of being disintegrated into three subparticles. To disintegrate the particle (T, X) we replace V in (183) by $V_{123} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ are distinct velocity vectors. The bi-boost $B_c(V_{123})$ takes the integrated $(3, 2)$ -particle (T, X) in (185) into a disintegrated multi-particle system $(T', X') = B_c(V_{123})(T, X)$ of three entangled subparticles that move with distinct relative velocities. In this case we say that the bi-boost $B_c(V_{123})$ disintegrates the integrated particle (T, X) , resulting in the disintegrated multi-particle system (T', X') of entangled subparticles.

A multi-particle system is *partially integrated* if several of its subsystems are integrated. Clearly, if an inertial observer observes that a given multi-particle system is partially integrated in some sense, then all inertial observers observe that the system is partially integrated in the same sense. Hence, we naturally raise the following property (Property 1) of Lorentz bi-boosts that are parametrized by equi-column velocity matrices.

Property 1. *The property of a multi-particle system to be partially integrated is observer-independent.*

Formally, Property 1 asserts that any bi-boost $B_c(V)$ parametrized by an equi-column velocity matrix $V \in \mathbb{R}_c^{n \times m}$ preserves partially integrated structures of multi-particle systems.

Clearly, Property 1 must hold true if the Lorentz bi-boost $B_c(V)$ of signature (m, n) possesses physical significance as the symmetry group of multi-particle systems.

Property 1 implies, for instance, that the system (T, X) in (184) is integrated if and only if the bi-boosted system (T', X') in (184) is integrated.

Property 1 is corroborated numerically in ([19] Chap. 6) by several examples. However, we do not have a proof of Property 1. Property 1, if it holds true, supports the claim that the Lorentz group $SO_c(m, n)$ is the symmetry group of the time and space coordinates of any multi-particle system of m n -dimensional particles, for any $m, n \in \mathbb{N}$.

It is, therefore, hoped that this review article will stimulate the search for experimental evidence that establishes the physical significance of Lorentz groups $SO_c(m, 3)$ for all $m \geq 2$, just as Einstein's special relativity establishes the physical significance of the common Lorentz group $SO_c(1, 3)$.

24. Conclusions

Einstein addition of relativistically admissible velocities, presented in Section 2, is neither commutative nor associative, and as such, it is not a group operation. However, it turns out in Section 3 that, being a gyrocommutative gyrogroup operation, Einstein addition possesses a rich algebraic structure called a gyrocommutative gyrogroup. In particular, it turns out that the departure of Einstein addition from commutativity and associativity is strictly controlled by gyrations, which are automorphisms studied in Section 4. In Sections 5–12, we therefore pave the road from Einstein addition to gyrogroups, and review elements of gyrogroup theory in order to demonstrate the power and elegance that gyrogroup theory shares with group theory.

In particular, we pay attention to the elegant gyro-identities (18) that Einstein addition and its gyrations obey. These remarkable gyro-identities survive unimpaired in (181) in their transition from Einstein addition (2) of relativistically admissible velocity vectors in the ball \mathbb{R}_c^n to Einstein addition (174) of bi-relativistically admissible velocity matrices in the ball $\mathbb{R}_c^{n \times m}$.

Our plan to extend the Lorentz boost from signature $(1, 3)$ in special relativity to signature (m, n) for all $m, n \in \mathbb{N}$, is guided by analogies with the Galilei boost, the analogous extension of which is clear and intuitive. Therefore, in Section 13 we present both Galilei and Lorentz boosts, paying special attention to the additive decomposition (105) that expresses the Lorentz boost as the sum of two components: (i) a Galilei boost and (ii) a relativistic component, which is directly noticeable only in high speeds. We indicate in Section 13 that Galilei and Lorentz boosts, which act on particles, are to be extended to Galilei and Lorentz multi-boosts, which act on multi-particle systems.

The additive decomposition (105) is generalized in (155) from signature (1,3) to signature (m, n) for all $m, n \in \mathbb{N}$. The resulting Galilei bi-boost of signature (m, n) has a clear and intuitive interpretation as a multi-boost that acts collectively on a multi-particle system. As such, it induces our interpretation of the resulting Lorentz bi-boost of signature (m, n) as a multi-boost that acts collectively on a multi-particle system as well. Contrastingly, Galilei multi-boosts are intuitively clear and involve no entanglement, whereas Lorentz multi-boosts are counterintuitive and involve entanglement. Hence, the association of Lorentz groups to Galilei groups of any signature (m, n) , $m, n \in \mathbb{N}$ that the additive decomposition in (105) and in (155) offers is significantly important. It allows us to interpret a Lorentz bi-boost as a Lorentz multi-boost, just as a Galilei bi-boost has a clear interpretation as a Galilei multi-boost.

Having at hand the theory of Galilei and Lorentz boosts of signature $(1, n)$, $n \in \mathbb{N}$, in Sections 14–22 we extend the theory to multi-boosts, called bi-boosts, of any signature (m, n) , $m, n \in \mathbb{N}$. We have, thus, placed the theory of Galilei and Lorentz bi-boosts of signature (m, n) under the same umbrella for all $m, n \in \mathbb{N}$. Guided by the interpretation of Galilei bi-boosts, the resulting unified theory suggests that the Lorentz bi-boost of signature (m, n) , $m, n \geq 2$, is a plausible candidate for the symmetry group of a multi-particle system of m n -dimensional entangled particles.

Finally, in Section 23 we note that if Lorentz bi-boosts have physical significance, they must imply that the property of a multi-particle system to be partially integrated is observer-independent. Accordingly, we raise a property asserting that this implication holds true. As yet we have no proof of the property, but several numerical examples in ([19] Chap. 6) corroborate it.

As such, the Lorentz bi-boost of signature $(m, 3)$, $m \geq 2$, is indeed the relativistic symmetry group of a multi-particle system of m 3-dimensional entangled particles that are observed in quantum mechanics.

Hence, it is hoped that this review article will stimulate the search for experimental evidence that (i) establishes the physical significance of the Lorentz group $SO_c(m, 3)$, $m \geq 2$, as the symmetry group of the spacetime coordinates of a multi-particle system of entangled particles, just as Einstein's special relativity (ii) establishes the physical significance of the common Lorentz group $SO_c(1, 3)$ as the symmetry group of the spacetime coordinates of a particle.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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