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The Fifth Coefficient of Bazilevič Functions

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Abstract: Let $f \in \mathcal{A}$, the class of normalized analytic functions defined in the unit disk $\mathbb{D}$, and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. This paper presents a new approach to finding bounds for $|a_n|$. As an application, we find the sharp bound for $|a_5|$ for the class $B_1(\alpha)$ of Bazilevič functions when $\alpha \geq 1$.

Keywords: univalent functions, Bazilevič functions, coefficient problems

MSC: 30C45; 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Then for $z \in \mathbb{D}$, $f \in \mathcal{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Denote by $\mathcal{S}$, the subset of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$.

We remark at the outset that in a great number of the more familiar subclasses of $\mathcal{S}$, sharp bounds have been found for the coefficients $|a_n|$, when $2 \leq n \leq 4$, but bounds when $n = 5$ and beyond are much more difficult to obtain. (See, e.g., [1]).

Denote by $\mathcal{S}^*$, the class of starlike functions defined as follows.

Definition 1. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*$ if, and only if, for $z \in \mathbb{D}$,*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

An application of the method introduced in this paper to estimate the fifth coefficient of functions in $\mathcal{A}$, concerns the $\mathcal{B}_1(\alpha)$ Bazilevič functions defined as follows.

Definition 2. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{B}_1(\alpha)$ if, and only if, for $\alpha \geq 0$, and $z \in \mathbb{D}$,*

$$\operatorname{Re}\left\{ f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \right\} > 0. \tag{2}$$

We note that $\mathcal{B}_1(0) = \mathcal{S}^*$, and each of the above classes are necessarily subclasses of $\mathcal{S}$. Apart from $\alpha = 0$, where $|a_n| \leq n$ for $n \geq 2$, we also note that sharp bounds for $|a_n|$ are known for $f \in \mathcal{B}_1(\alpha)$

when $\alpha \geq 0$ only when $2 \leq n \leq 4$, [2], and only partial solutions are known for $|a_n|$ when $n \geq 5$ [3,4] for $\alpha \geq 1$.

It was conjectured in [4], that when $\alpha \geq 1$, the sharp bound for $|a_n|$ when $n \geq 2$ is given by

$$|a_n| \leq \frac{2}{n-1+\alpha},$$

and a partial solution to this problem in the case $n = 5$ was given in [3].

In this paper, we illustrate our method by giving a complete solution to finding the sharp bound for $|a_5|$ when $f \in \mathcal{B}_1(\alpha)$ for $\alpha \geq 1$.

2. Auxiliary Results

Denote by $\mathcal{P}$, the class of analytic functions p with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

Lemma 1 ([5]). *If the functions*

$$1 + \sum_{n=1}^{\infty} b_n z^n \quad \textit{and} \quad 1 + \sum_{n=1}^{\infty} c_n z^n$$

belong to $\mathcal{P}$, then the same is true of the function

$$1 + \frac{1}{2}\sum_{n=1}^{\infty} b_n c_n z^n.$$

Lemma 2 ([6]). *Let $h(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$ and $1 + G(z) = 1 + d_1 z + d_2 z^2 + \cdots$ be functions in $\mathcal{P}$, $\epsilon_0 = 1$, and*

$$\epsilon_n = \frac{1}{2^n}\left[1 + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\beta_k\right], \quad n \in \mathbb{N}. \tag{4}$$

If A_n $(n \in \mathbb{N})$ is defined by

$$\sum_{n=1}^{\infty}(-1)^{n+1}\epsilon_{n-1}G^n(z) = \sum_{n=1}^{\infty} A_n z^n,$$

then $|A_n| \leq 2$.

We first outline the method of proof.

Let $p \in \mathcal{P}$ be in the form (3), and

$$\Psi = p_4 + B_1 p_1^4 + B_2 p_1^2 p_2 + B_3 p_1 p_3 + B_4 p_2^2 \tag{5}$$

with $B_i \in \mathbb{C}$, $i \in \{1,2,3,4\}$. Assume that there exists $q \in \mathcal{P}$ of the form $q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then by Lemma 1 the function

$$1 + \frac{1}{2}(p(z)-1) * (q(z)-1) = 1 + \sum_{n=1}^{\infty}\frac{1}{2}b_n p_n z^n$$

also belongs to $\mathcal{P}$. Let

$$1 + G(z) = 1 + \frac{1}{2}(p(z)-1) * (q(z)-1) = 1 + \sum_{n=1}^{\infty} v_n z^n.$$

Then $1 + G(z) \in \mathcal{P}$, and $v_n = b_n p_n/2$, $n \in \mathbb{N}$.

Now assume that $h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n \in \mathcal{P}$. Then by Lemma 2 we obtain $|A_4| \le 2$, where

$$A_4 = \frac{1}{2}\epsilon_0 b_4 p_4 - \frac{1}{4}\epsilon_1 b_2^2 p_2^2 - \frac{1}{2}\epsilon_1 b_1 b_3 p_1 p_3 + \frac{3}{8}\epsilon_2 b_1^2 b_2 p_1^2 p_2 - \frac{1}{16}\epsilon_3 b_1^4 p_1^4. \tag{6}$$

Here, ϵ_i, $i \in \{1,2,3,4\}$, are given by

$$\begin{aligned}\epsilon_0 &= 1,\\ \epsilon_1 &= \frac{1}{2}\left(1 + \frac{1}{2}u_1\right),\\ \epsilon_2 &= \frac{1}{4}\left(1 + u_1 + \frac{1}{2}u_2\right),\\ \epsilon_3 &= \frac{1}{8}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right).\end{aligned}$$

Hence we have the following.

(A) *Let $p \in \mathcal{P}$ be in the form (3). If there exist $q, h \in \mathcal{P}$ such that q and h are represented by*

$$q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n,$$

respectively, with

$$\begin{cases} b_4 = 2 \\ B_1 = -\frac{1}{128}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right) b_1^4 \\ B_2 = \frac{3}{32}\left(1 + u_1 + \frac{1}{2}u_2\right) b_1^2 b_2 \\ B_3 = -\frac{1}{4}\left(1 + \frac{1}{2}u_1\right) b_1 b_3 \\ B_4 = -\frac{1}{8}\left(1 + \frac{1}{2}u_1\right) b_2^2, \end{cases}$$

then $|\Psi| = |A_4| \le 2$, where Ψ and A_4 are given by (5) and (6), respectively.

We now recall a recent result of Cho et al. [7], where they obtained the following parametric formulas for the initial coefficients of Carathéodory functions (see also [8]). We recall the Möbius transformation $\psi_\zeta : \mathbb{D} \to \mathbb{D}$, $\zeta \in \mathbb{D}$, defined by

$$\psi_\zeta(z) = \frac{z - \zeta}{1 - \overline{\zeta} z} \tag{7}$$

and let

$$L(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{8}$$

Lemma 3 ([7])**.** *If $p \in \mathcal{P}$ is of the form (3), then*

$$p_1 = 2\zeta_1, \tag{9}$$

$$p_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{10}$$

and

$$p_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \tag{11}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1,2,3\}$. For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with p_1 as in (9), namely,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p = L \circ \omega \in \mathcal{P}$ with p_1 and p_2 as in (9)-(10), where

$$\omega(z) = z\psi_{-\zeta_1}(\zeta_2 z), \quad z \in \mathbb{D},$$

i.e.,

$$p(z) = \frac{1 + (\overline{\zeta}_1\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\overline{\zeta}_1\zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}.$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there is a unique function $p = L \circ \omega \in \mathcal{P}$ with p_1, p_2 and p_3 as in (9)-(11), where

$$\omega(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D},$$

i.e.,

$$p(z) = \frac{1 + (\overline{\zeta}_2\zeta_3 + \overline{\zeta}_1\zeta_2 + \zeta_1)z + (\overline{\zeta}_1\zeta_3 + \zeta_1\overline{\zeta}_2\zeta_3 + \zeta_2)z^2 + \zeta_3 z^3}{1 + (\overline{\zeta}_2\zeta_3 + \overline{\zeta}_1\zeta_2 - \zeta_1)z + (\overline{\zeta}_1\zeta_3 - \zeta_1\overline{\zeta}_2\zeta_3 - \zeta_2)z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}.$$

Conversely, if $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \overline{\mathbb{D}}$ are given, then we can construct a (unique) function $p \in \mathcal{P}$ of the form (3) so that p_i, $i \in \{1, 2, 3\}$, satisfying the identities in (9)–(11). For this, we define

$$\omega(z) = \omega_{\zeta_1,\zeta_2,\zeta_3}(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D}, \tag{12}$$

where ψ_ζ is the function defined as in (7). Then $\omega \in \mathcal{B}$. Moreover, if we define $p(z) = (1 + \omega(z))/(1 - \omega(z))$, $z \in \mathbb{D}$, then p is represented by (3), where p_1, p_2 and p_3 satisfy the identities in (9)–(11) (see the proof of ([7], Lemma 2.4)).

Assume that the function $q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{P}$ is constructed by $\xi_1, \xi_2 \in \mathbb{D}$, $\xi_3 \in \overline{\mathbb{D}}$, and the function $h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n \in \mathcal{P}$ is constructed by $\zeta_1, \zeta_2 \in \mathbb{D}$, $\zeta_3 \in \overline{\mathbb{D}}$. Namely, $q = L \circ \omega_1$ and $h = L \circ \omega_2$, where L is the function defined by (8), $\omega_1(z) = z\psi_{-\xi_1}(z\psi_{-\xi_2}(\xi_3 z))$ and $\omega_2(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z))$. Then by combining the above argument, we conclude (B) below.

(B) *Let $p \in \mathcal{P}$ be in the form (3). If there exist ζ_1, ζ_2, ξ_1, $\xi_2 \in \mathbb{D}$, ζ_3, $\xi_3 \in \overline{\mathbb{D}}$ satisfying the following conditions*

$$\begin{cases} b_1 = 2\xi_1 \\ b_2 = 2\xi_1^2 + 2(1 - |\xi_1|^2)\xi_2 \\ b_3 = 2\xi_1^3 + 4(1 - |\xi_1|^2)\xi_1\xi_2 - 2(1 - |\xi_1|^2)\overline{\xi}_1\xi_2^2 + 2(1 - |\xi_1|^2)(1 - |\xi_2|^2)\xi_3 \\ u_1 = 2\zeta_1 \\ u_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \\ u_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta}_1\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \\ b_4 = 2 \\ B_1 = -\frac{1}{128}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right) b_1^4 \\ B_2 = \frac{3}{32}\left(1 + u_1 + \frac{1}{2}u_2\right) b_1^2 b_2 \\ B_3 = -\frac{1}{4}\left(1 + \frac{1}{2}u_1\right) b_1 b_3 \\ B_4 = -\frac{1}{8}\left(1 + \frac{1}{2}u_1\right) b_2^2, \end{cases}$$

then $|\Psi| = |A_4| \leq 2$, where Ψ and A_4 are given by (5) and (6), respectively.

Since the system of equations in (B) has many solutions, we now place some restrictions on the parameters sufficient for our purpose.

We fix

$$q(z) = q_\tau(z) = \frac{1 + 2\tau z + z^2}{1 - z^2}.$$

Then if $\tau \in [-1, 1]$, $q \in \mathcal{P}$ and is given by $q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, with

$$b_1 = b_3 = 2\tau, \quad \text{and} \quad b_2 = b_4 = 2.$$

We also assume that ζ_i, $i \in \{1, 2, 3\}$ take real values. Then the identities for u_i, $i \in \{1, 2, 3\}$ become

$$\begin{cases} u_1 = 2\zeta_1 \\ u_2 = 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2 \\ u_3 = 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - \zeta_2^2)\zeta_3. \end{cases}$$

Thus we are able to conclude the following.
(C) *Let $p \in \mathcal{P}$ be in the form (3). If there exist ζ_1, $\zeta_2 \in (-1, 1)$ and ζ_3, $\tau \in [-1, 1]$ such that*

$$\begin{cases} u_1 = 2\zeta_1 \\ u_2 = 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2 \\ u_3 = 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - \zeta_2^2)\zeta_3 \\ b_4 = 2 \\ B_1 = -\frac{1}{8}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right)\tau^4 \\ B_2 = \frac{3}{4}\left(1 + u_1 + \frac{1}{2}u_2\right)\tau^2 \\ B_3 = -\left(1 + \frac{1}{2}u_1\right)\tau^2 \\ B_4 = -\frac{1}{2}\left(1 + \frac{1}{2}u_1\right), \end{cases}$$

then $|\Psi| = |A_4| \leq 2$, where Ψ and A_4 are given by (5) and (6), respectively.

3. The Fifth Coefficient of $\mathcal{B}_1(\alpha)$ Bazilevič Functions

We now use the above method to find the sharp bound for $|a_5|$ when $\alpha \geq 1$.

Theorem 1. *Let $f \in \mathcal{B}_1(\alpha)$, and be given by (1), then $|a_5| \leq 2/(4 + \alpha)$, provided $\alpha \geq 1$. The inequality is sharp, with extreme function $f \in \mathcal{A}$ defined by*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha - 1} = \frac{1 + z^5}{1 - z^5}. \tag{13}$$

Proof. From (2) we can write

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha - 1} = p(z), \tag{14}$$

for some function $p \in \mathcal{P}$ of the form (3). Putting the series (1) and (3) into (14) by equating the coefficients we get

$$p_1 = (1 + \alpha)a_2, \quad p_2 = \frac{1}{2}(2 + \alpha)[(-1 + \alpha)a_2^2 + 2a_3], \tag{15}$$

$$p_3 = \frac{1}{6}(3 + \alpha)[(2 - 3\alpha + \alpha^2)a_2^3 + 6(-1 + \alpha)a_2a_3 + 6a_4] \tag{16}$$

and

$$p_4 = \frac{1}{24}(4+\alpha)[(-6+11\alpha-6\alpha^2+\alpha^3)a_2^4+12(3-3\alpha+\alpha^2)a_2^2a_3 \\ +24(-1+\alpha)a_2a_4+12((-1+\alpha)a_3^2+2a_5)]. \tag{17}$$

From the equalities (15)–(17), a_5 can be written as

$$a_5 = \frac{1}{4+\alpha}\cdot\Psi, \tag{18}$$

where

$$\Psi = p_4+(4+\alpha)(1-\alpha)(\mu_1p_1p_3+\mu_2p_2^2+\mu_3p_1^2p_2+\mu_4p_1^4),$$

with

$$\mu_1 = \frac{1}{(1+\alpha)(3+\alpha)}, \quad \mu_2 = \frac{1}{2(2+\alpha)^2},$$

$$\mu_3 = \frac{1-2\alpha}{2(1+\alpha)^2(2+\alpha)}, \quad \text{and} \quad \mu_4 = \frac{(3\alpha-1)(2\alpha-1)}{24(1+\alpha)^4}.$$

Thus it is enough to show that $|\Psi| \leq 2$.

When $\alpha = 1$, it is clear that $|\Psi| \leq 2$ holds trivially, and so we can assume that $\alpha > 1$.

Put

$$\zeta_1 = \frac{\alpha-5}{(1+\alpha)^2},$$

$$\zeta_2 = \frac{18-2\alpha-\alpha^2+\alpha^3}{3(2+\alpha)(6+\alpha+\alpha^2)},$$

and

$$\zeta_3 = \frac{108-2880\alpha-1472\alpha^2+1985\alpha^3+2110\alpha^4+1060\alpha^5+326\alpha^6+43\alpha^7}{(1+\alpha)^3(972+828\alpha+428\alpha^2+180\alpha^3+48\alpha^4+8\alpha^5)},$$

then $|\zeta_i| < 1$ $(i = 1, 2, 3)$ when $\alpha > 1$.

Now let $\omega \in \mathbb{B}$ be defined by (12). Thus, the function k defined by $k = L \circ \omega$, where L is given by (8), belongs to $\mathcal{P}$, with $k(0) = 1$.

Setting

$$k(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n,$$

we obtain

$$\beta_1 = \frac{2(-5+\alpha)}{(1+\alpha)^2},$$

$$\beta_2 = \frac{2(78-\alpha-7\alpha^2+\alpha^3+\alpha^4)}{3(1+\alpha)^2(2+\alpha)},$$

and

$$\beta_3 = \frac{2(-222-21\alpha-374\alpha^2-228\alpha^3+148\alpha^4+105\alpha^5+16\alpha^6)}{3(1+\alpha)^6(2+\alpha)}.$$

With ϵ_n $(n \in \mathbb{N})$ as in (4), we obtain

$$\epsilon_1 = \frac{-4+3\alpha+\alpha^2}{2(1+\alpha)^2}, \tag{19}$$

$$\epsilon_2 = \frac{12-29\alpha+4\alpha^2+11\alpha^3+2\alpha^4}{6(1+\alpha)^3(2+\alpha)}, \tag{20}$$

and

$$\epsilon_3 = \frac{(3+\alpha)^2(-4+23\alpha-38\alpha^2+13\alpha^3+6\alpha^4)}{24(1+\alpha)^6}. \tag{21}$$

Now define q by

$$q(z) = \frac{1+2\tau z+z^2}{1-z^2} = 1+\sum_{n=1}^{\infty} c_n z^n$$

with $\tau := \sqrt{(1+\alpha)/(3+\alpha)}$. Then $0 < \tau < 1$ for $\alpha > 1$, which implies that $q \in \mathcal{P}$, with

$$c_1 = c_3 = 2\tau, \quad \text{and} \quad c_2 = c_4 = 2. \tag{22}$$

Since $p \in \mathcal{P}$, by Lemma 1 the function

$$h(z) := 1+\frac{1}{2}\sum_{n=1}^{\infty} c_n p_n z^n$$

also belongs to $\mathcal{P}$. Therefore from Lemma 2 with $d_n = c_n p_n/2$ we have $|A_4| \le 2$, where

$$A_4 = \frac{1}{2}c_4p_4 - \frac{1}{4}\epsilon_1(c_2^2p_2^2+2c_1c_3p_1p_3)+\frac{3}{8}\epsilon_2c_1^2c_2p_1^2p_2-\frac{1}{16}\epsilon_3c_1^4p_1^4.$$

Finally, (19)–(22) shows that $A_4 = \Psi$, and so $|\Psi| \le 2$. Thus it follows from (18) that the inequality $|a_5| \le 2/(4+\alpha)$ holds.

Let $f \in \mathcal{B}_1(\alpha)$ be the function defined by (13). Then, by equating coefficients in (13), we get $a_2 = a_3 = a_4 = 0$ and $a_5 = 2/(4+\alpha)$, which shows that this result is sharp. This completes the proof of Theorem 1. □

4. Conclusions

There are several instances in the literature where only partial solutions are known for the bounds for $|a_5|$ for functions in subclasses of $\mathcal{S}$ (again, see [1]). Applying the method introduced in this paper may well provide improved, or complete solutions to some of these.

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