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Implicit Extragradient-Like Method for Fixed Point Problems and Variational Inclusion Problems in a Banach Space

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Abstract: In a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$, one use VIP to indicate a variational inclusion problem involving two accretive mappings and CFPP to denote the common fixed-point problem of an infinite family of strict pseudocontractions of order q. In this paper, we introduce a composite extragradient implicit method for solving a general symmetric system of variational inclusions (GSVI) with certain VI and CFPP. We then investigate its convergence analysis under some weak conditions. Finally, we consider the celebrated LASSO problem in Hilbert spaces.

Keywords: banach space; implicit method; system of variational inclusions; fixed point problem

1. Introduction-Preliminaries

Throughout this article, one always supposes that H is a real infinite dimensional Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let the nonempty subset $C \subset H$ be convex and closed, and let the mapping P_C be the nearest point (metric) projection of H onto C. Let $A : C \to H$ be a nonself mapping. The celebrated variational inequality problem (VIP) is to find an element $x^* \in C$ s.t. $\langle y - x^*, Ax^* \rangle \geq 0$, for all $y \in C$. One denotes by $\mathrm{VI}(C, A)$ the set of all solutions of the VIP. The VIP acts as a unified framework for lots of real industrial and applied problems, such as, machine learning, transportation, image processing and economics; see, e.g., [1–7]. One knows that Korpelevich's extragradient algorithm [8] is now one of the most popular algorithms to numerically solve the VIP. It reads as follows:

$$\begin{cases} x_{j+1} = P_C(x_j - \ell A y_j) \quad \forall j \geq 0, \\ y_j = P_C(x_j - \ell A x_j), \end{cases}$$

with $\ell \in (0, \frac{1}{L})$, where L is the Lipschitz constant of the mapping A. If its solution set is not empty, it was obtained that sequence $\{x_j\}$ is weakly convergent. The price for the weak convergence is that A must be a Lipschitz and monotone mapping. To date, now, the extragradient method and relaxed extragradient methods have received much attention and has been studied extensively; see, e.g., [9–13]. Next, one assumes that B is monotone set-valued mapping defined on H and A is a monotone single-valued mapping defined on H, respectively. The so-called variational inclusion problem is to find an element $x^* \in H$ s.t. $0 \in (A + B)x^*$ and it has recently been studied by many authors based on splitting-based approximation methods; see, e.g., [14–17]. This model provides a unified framework for a lot of theoretical and practical problems and was investigated via different methods [18–22]. To the best of the authors' knowledge, there are no few associated results obtained in Banach spaces. Next, one will turn our attention to Banach spaces.

Next, E^* will be used to present be the dual space of Banach space E. Let $\emptyset \neq C \subset E$ be a convex and closed set. Given a nonlinear mapping T on C, one uses the symbol $\mathrm{Fix}(T)$ to denote the set of all the fixed points of T. Recall that the mapping T is said to be a Lipschitz mapping if and only if, $\forall x, y \in C$, $\|Tx - Ty\| \leq L\|x - y\|$. If the Lipschitz constant L is just one, one say that T is a nonexpansive mapping whose complementary mappings are monotone.

Recall that the duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\}.$$

For a particular case, one uses J to stand for J_2, which is commonly called the normalized duality. Recall that a mapping T defined on C is said to be a strict pseudocontraction of order q if, $\forall x, y \in C$, there is $j_q(x-y) \in J_q(x-y)$ such that the following inequality holds $\langle Tx - Ty, j_q(x-y)\rangle \leq \|x - y\|^q - \varsigma\|(I-T)x - (I-T)y\|^q$ for some $\varsigma > 0$.

The convexity modulus of space E, δ_E, which maps the interval $(0, 2]$ to the interval $[0, 1]$, is defined as follows

$$\delta_E(\epsilon) = \inf\{\frac{2 - \|x + x'\|}{2} : x', x \in E,\ \epsilon \leq \|x' - x\|,\ \|x'\| = 1 = \|x\|\}.$$

The smoothness modulus of space E, ρ_E, which maps the interval $[0, \infty)$ to the interval $[0, \infty)$, is defined as follows

$$\rho_E(\tau) = \sup\{\frac{\|\tau x' + x\| + \|\tau x' - x\| - 2}{2} : x', x \in E,\ \|x'\| = \|x\| = 1\}.$$

Recall that a space E is said to be uniformly convex if $\delta_E(e) > 0$, $\forall e \in (0, 2]$. Recall that a space E is said to be uniformly smooth if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Further it is said to be q-uniformly ($q > 1$) smooth if $\exists c > 0$ s.t. $c \geq s^{-q}\rho_E(s)$, $\forall s > 0$. In q-uniformly ($q > 1$) smooth spaces, one has the following celebrated inequality, for some $j_q(x+y) \in J_q(x+y)$,

$$\|x + y\|^q - \|x\|^q \leq q\langle y, j_q(x+y)\rangle \quad \forall x, y \in E. \tag{1}$$

One says that an operator $\Pi : C \to D$, where D is convex and closed subset of C, is said to be a sunny mapping if, $(1-\xi)\Pi(x) + \xi x \in C$ for $\xi \geq 0$ and $x \in C$, $\Pi(x) = \Pi[\Pi(x) + \xi(x - \Pi(x))]$. If Π is both nonexpansive and sunny, then $\langle \bar{x} - x', J(\Pi(\bar{x}) - \Pi(x'))\rangle \geq \|\Pi(\bar{x}) - \Pi(x')\|^2$, $\forall \bar{x}, x' \in C$.

In the setting of q-uniformly smooth spaces, we recall that an operator B is said to be accretive if, for some $j_q(x' - \bar{x}) \in J_q(x' - \bar{x})$, $\langle u - v, j_q(x' - \bar{x})\rangle \geq 0$, for all $u \in Bx'$, $v \in B\bar{x}$. An accretive operator B is said to be inverse-strongly of order q if, for each $x', \bar{x} \in C$, $\exists j_q(x' - \bar{x}) \in J_q(x' - \bar{x})$ s.t. $\langle u - v, j_q(x' - \bar{x})\rangle \geq \alpha\|u - v\|^q$, for all $u \in Bx'$, $v \in B\bar{x}$ for some $\alpha > 0$. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. Furthermore, one can define a mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)\ \ \forall \lambda > 0$. From [9], we have $\text{Fix}(T_\lambda) = (A + B)^{-1}0\ \forall \lambda > 0$ and $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ for $0 < \lambda \leq r$ and $y \in C$. From [23], one has $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x)$ for all $\lambda, \mu > 0$, $x \in E$ and $\text{Fix}(J_\lambda^B) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$.

Let $A_1, A_2 : C \to E$ and $B_1, B_2 : C \to 2^E$ be nonlinear mappings with $B_k x \neq \emptyset\ \forall x \in C, k = 1, 2$. Consider the symmetrical system of variational inclusions, which consists of finding the pair (x^*, y^*) in $C \times C$ s.t.

$$\begin{cases} 0 \in \zeta_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \zeta_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{2}$$

where ζ_k is a positive constant for $k = 1, 2$. Ceng, Postolache and Yao [24] obtain the fact that problem (2) is equivalent to a fixed point problem.

Based on the equivalent relation, Ceng et al. [24] suggested a composite viscosity implicit rule for solving the GSVI (2) as follows:

$$\begin{cases} y_j = J_{\zeta_2}^{B_2}(x_j - \zeta_2 A_2 x_j), \\ x_j = \alpha_j f(x_{j-1}) + \delta_j x_{j-1} + \beta_j V x_{j-1} + \gamma_j[\mu S x_j + (1-\mu)J_{\zeta_1}^{B_1}(y_j - \zeta_1 A_1 y_j)] \quad \forall j \geq 1, \end{cases}$$

where $\mu \in (0,1)$, $S := (1-\alpha)I + \alpha T$ with $0 < \alpha < \min\{1, \frac{2\lambda}{\kappa_2}\}$, and the sequences $\{\alpha_j\}, \{\delta_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ are such that (i) $\alpha_j + \delta_j + \beta_j + \gamma_j = 1\ \forall j \geq 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = 0$; (iii) $\lim_{j\to\infty} \gamma_j = 1$; (iv) $\sum_{j=0}^{\infty} \alpha_j = \infty$. They proved that $\{x_j\}$ converges strongly to a point of $\mathrm{Fix}(G) \cap \mathrm{Fix}(T)$, which solves a certain VIP.

In this article, we introduce and investigate an implicitly composite solution method to solve the GSVI (2) with certain VIP and CFPP constraints. We then analyze convergence of the suggested method in the setting of real Hilbert spaces under some mild conditions. An application is also considered.

From now on, one always uses $\kappa_q > 0$ to denote the smoothness coefficient; see [25,26]. One also lists some essential lemmas for the strong convergence theorem next.

Lemma 1 ([27]). *Let E be q-uniformly smooth with $q \in (1,2]$, and $\emptyset \neq C \subset E$ a closed convex set. Let $T : C \to C$ be a ς-strict pseudocontraction of order q. Given $\alpha \in (0,1)$. Define a single-valued nonlinear mapping $T_\alpha : C \to C$ by $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then T_α is nonexpansive with* $\mathrm{Fix}(T_\alpha) = \mathrm{Fix}(T)$ *provided $0 < \alpha \leq (\frac{\varsigma q}{\kappa_q})^{\frac{1}{q-1}}$.*

Lemma 2 ([28]). *Let E be q-uniformly smooth with $q \in (1,2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q + \|(I-\lambda A)x - (I-\lambda A)y\|^q \leq \|x-y\|^q, \quad \forall x, y \in C.$$

Lemma 3 ([26,28]). *Let $q \in (1,2]$ be a fixed real number and let E be a q-uniformly smooth Banach space. Then $\|x+y\|^q \leq q\langle y, J_q(x)\rangle + \|x\|^q + \kappa_q\|y\|^q, \forall x, y \in E$. Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators. Let $A_i : C \to E$ $(i = 1,2)$ be σ_i-inverse-strongly accretive mapping of order q for each $i = 1,2$. Define an operator $G : C \to C$ by*

$$G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2).$$

If $0 \leq \zeta_i \leq (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ $(i = 1,2)$, then G is a nonexpansive mapping.

The following lemmas was proved in [29], which is a simple extension of the inequality established in [26].

Lemma 4. *In a uniformly convex real Banach space E, there is a convex, strictly increasing, continuous function g, which maps $[0,+\infty)$ to $[0,+\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x' + \mu\bar{x} + \nu\tilde{x}\|^q \leq \lambda\|x'\|^q + \mu\|\bar{x}\|^q + \nu\|\tilde{x}\|^q - W_q g(\|\bar{x} - \tilde{x}\|),$$

where $q > 1$ is any real number and W_q is a real function associated with μ and ν, for all $x', \bar{x}, \tilde{x} \in \{x \in E : \|x\| \leq r\}$ (r is some real number) and $\lambda, \mu, \nu \in [0,1]$ such that $\lambda + \nu + \mu = 1$.

Lemma 5 ([26]). *If E be a uniformly convex Banach space, then there exist a convex, strictly increasing, continuous function h, which maps $[0,+\infty)$ to $[0,+\infty)$ with $h(0) = 0$ such that*

$$h(\|x-y\|) \leq \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q,$$

where x and y are in some bounded subset of E and $j_q(y) \in J_q(y)$.

Lemma 6 ([30]). *Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on C such that*

$$\sum_{n=1}^{\infty} \sup_{x\in C} \|S_n x - S_{n-1}x\| < \infty.$$

Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be the self-mapping on C defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup_{x\in C} \|S_n x - Sx\| = 0$.

Lemma 7 ([31]). *Let E be a strictly convex Banach space. Let T_n be a nonexpansive mapping defined on a convex and closed subset C of E for each $n \geq 1$. Let $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ be nonempty. Let $\{\lambda_n\}$ be a positive sequence such that $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ $\forall x \in C$ is well defined, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 8 ([32]). *Let E be a smooth Banach space. Let $T : C \to C$, where C is convex and closed set in E, be a self-nonexpansive mapping. Suppose that λ is constant in the interval $(0, 1)$. Then $\{z^\lambda\}$, where $z^\lambda = (1-\lambda)Tz^\lambda + \lambda u$, converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves $\langle u - x^*, J(x^* - x)\rangle \geq 0$ for all $x \in \mathrm{Fix}(T)$.*

Lemma 9 ([33]). *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \leq (1 - s_n)a_n + s_n\nu_n$ $\forall n \geq 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \leq 0$ or $\sum_{n=0}^{\infty} |s_n\nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 10 ([34]). *Let $\{\Gamma_n\}$ be a real sequence such that there exists a real sequence $\{\Gamma_{n_i}\}$, which is a subsequence of $\{\Gamma_n\}$ such that $\Gamma_{n_i+1} > 1\Gamma_{n_i}$ for each integer $i \geq 1$. Let $\{\tau(n)\}_{n\geq n_0}$ be a integer Define the sequence defined by $\tau(n) = \max\{k \leq n : \Gamma_{k+1} > \Gamma_k\}$, where the integer $n_0 \geq 1$ is chosen in such a way that $\{k \leq n_0 : \Gamma_{k+1}\} > \Gamma_k \neq \emptyset$. (i) $\Gamma_{\tau(n)+1} \geq \Gamma_{\tau(n)}$ and $\Gamma_{\tau(n)+1} \geq \Gamma_n$ for each $n \geq n_0$; (ii) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ and $\tau(n) \to \infty$.*

2. Results

Throughout this section, suppose that C is a convex closed set in a Banach space E, which is both uniformly convex and q-uniformly smooth with $q \in (1, 2]$. Let both $B_1 : C \to 2^E$ and $B_2 : C \to 2^E$ be m-accretive operators. Let $A_k : C \to E$ be single-valued σ_k-inverse strongly accretive mapping for each $k = 1, 2$. Further, one assumes that $G : C \to C$ is a self mapping defined as $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ with constants $\zeta_1, \zeta_2 > 0$. Let S_n be a ς-uniformly strictly pseudocontractive mapping for each $n \geq 1$. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Assume that the feasibility set $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap (A + B)^{-1}0 \cap \mathrm{Fix}(G)$ is nonempty.

Algorithm 1: Composite extragradient implicit method for the GSVI (2) with VIP and CFPP constraints.

Initial Step. Given $\xi \in (0, 1)$, $\alpha \in (0, \min\{1, (\frac{\varsigma q}{\kappa_q})^{\frac{1}{q-1}}\})$. Let $x_0 \in C$ be an arbitrary initial.
Iteration Steps. Compute x_{n+1} from the current x_n as follows:
Step 1. Calculate

$$\begin{cases} w_n = s_n((1-\xi)x_n + \xi G w_n) + (1 - s_n)((1-\alpha)x_n + \alpha S_n x_n), \\ u_n = J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n); \end{cases}$$

Step 2. Calculate $y_n = J_{\lambda_n}^{B}(u_n - \lambda_n A u_n)$;
Step 3. Calculate $z_n = J_{\lambda_n}^{B}(u_n - \lambda_n A y_n + r_n(y_n - u_n))$;
Step 4. Calculate $x_{n+1} = \alpha_n u + \beta_n u_n + \gamma_n((1-\alpha)z_n + \alpha S_n z_n)$, where u is a fixed element in C, $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\{\lambda_n\} \subset (0, \infty)$ and $\alpha_n + \beta_n + \gamma_n = 1$.
Set $n := n + 1$ and go to Step 1.

Lemma 11. *Let the vector sequence $\{x_n\}$ be constructed by Algorithm 1. One has that the sequence $\{x_n\}$ is bounded.*

Proof. Putting $v_n = J^{B_2}_{\zeta_2}(w_n - \zeta_2 A_2 w_n)$ and $S_{\alpha,n} := (1-\alpha)I + \alpha S_n \ \forall n \geq 0$, we know from Lemma 1 that each $S_{\alpha,n} : C \to C$ is nonexpansive with $\mathrm{Fix}(S_{\alpha,n}) = \mathrm{Fix}(S_n)$. Let $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0$. Then we observe that $p = Gp = S_n p = J^B_{\lambda_n}((1-r_n)p + r_n(p - \frac{\lambda_n}{r_n}Ap))$. By use of Lemmas 2 and 3, we deduce that $I - \zeta_1 A_1$, $I - \zeta_2 A_2$ and $G := J^{B_1}_{\zeta_1}(I - \zeta_1 A_1) J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)$ are nonexpansive mappings. It is clear that there is only one element $w_n \in C$ satisfying

$$w_n = s_n((1-\xi)x_n + \xi G w_n) + (1-s_n)(\alpha S_n x_n + (1-\alpha)x_n).$$

Since G and $S_{\alpha,n}$ are both nonexpansive mappings, we get

$$\begin{aligned}\|w_n - p\| &\leq s_n((1-\xi)\|x_n - p\| + \xi\|Gw_n - p\|) + (1-s_n)\|S_{\alpha,n}x_n - p\| \\ &\leq s_n(\xi\|w_n - p\| + (1-\xi)\|x_n - p\|) + (1-s_n)\|x_n - p\|,\end{aligned}$$

and hence $\|w_n - p\| \leq \|x_n - p\| \ \forall n \geq 0$. Using the nonexpansivity of G again, we deduce from $u_n = Gw_n$ that $\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\|$. By use of Lemmas 2 and 4, we have

$$\begin{aligned}\|y_n - p\|^q &\leq \|(I - \lambda_n A)p - (I - \lambda_n A)u_n\|^q \\ &\leq \|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q,\end{aligned}$$

which leads to $\|y_n - p\| \leq \|u_n - p\|$. On the other hand,

$$\begin{aligned}\|z_n - p\|^q &= \|J^B_{\lambda_n}((1-r_n)u_n + r_n(y_n - \tfrac{\lambda_n}{r_n}Ay_n)) - J^B_{\lambda_n}((1-r_n)p + r_n(p - \tfrac{\lambda_n}{r_n}Ap))\|^q \\ &\leq r_n\|(I - \tfrac{\lambda_n}{r_n}A)y_n - (I - \tfrac{\lambda_n}{r_n}A)p\|^q + (1-r_n)\|u_n - p\|^q \\ &\leq r_n[\|y_n - p\|^q - \tfrac{\lambda_n}{r_n}(\sigma q - \tfrac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n - Ap\|^q] + (1-r_n)\|u_n - p\|^q \\ &\leq \lambda_n(\tfrac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}} - \sigma q)\|Ay_n - Ap\|^q + r_n\lambda_n(\kappa_q \lambda_n^{q-1} - \sigma q)\|Au_n - Ap\|^q + \|u_n - p\|^q.\end{aligned}$$

This ensures that $\|z_n - p\| \leq \|u_n - p\|$. So it follows from (3.2) that

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\|p - u\| + \beta_n\|u_n - p\| + \gamma_n\|S_{\alpha,n}z_n - p\| \\ &\leq \alpha_n\|p - u\| + (1-\alpha_n)\|x_n - p\|,\end{aligned}$$

which leads to $\|x_n - p\| \leq \max\{\|p - u\|, \|p - x_0\|\}$. □

Theorem 1. *Suppose that $\{x_n\}$ is the vector sequence generated/defined by Iterative Algorithm 1. Assume that the parameter sequences satisfy $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$; $0 < a \leq \beta_n \leq b < 1$ and $0 < c \leq s_n \leq d < 1$; $0 < r \leq r_n < 1$ and $0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < (\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$; $0 < \zeta_i < (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$, $i = 1, 2$. Suppose $\sum_{n=0}^{\infty} \sup_{x \in D} \|S_{n+1}x - S_n x\| < \infty$, where D is a bounded set in set C. Define the mapping S by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. Then the sequence $\{x_n\}$ converges to $x^* \in \Omega$ strongly. The solution also uniquely solves $\langle x^* - u, J(x^* - \bar{x})\rangle \leq 0$ for all $\bar{x} \in \Omega$.*

Proof. First, we set $v_n = J^{B_2}_{\zeta_2}(w_n - \zeta_2 A_2 w_n)$, $x^* \in \Omega$ and $y^* = J^{B_2}_{\zeta_2}(x^* - \zeta_2 A_2 x^*)$. Since $u_n = J^{B_1}_{\zeta_1}(I - \zeta_1 A_1)v_n$ and $v_n = J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)w_n$, we have $u_n = Gw_n$. Using Lemma 2 yields that

$$\|v_n - y^*\|^q \leq \zeta_2(\kappa_q \zeta_2^{q-1} - \sigma_2 q)\|A_2 w_n - A_2 x^*\|^q + \|w_n - x^*\|^q$$

and

$$\|u_n - x^*\|^q \leq \zeta_1(\kappa_q \zeta_1^{q-1} - \sigma_1 q)\|A_1 v_n - A_1 y^*\|^q + \|v_n - y^*\|^q.$$

Combining the last two inequalities, we have

$$\|u_n - x^*\|^q \leq \zeta_2(\kappa_q \zeta_2^{q-1} - \sigma_2 q)\|A_2 w_n - A_2 x^*\|^q + \zeta_1(\kappa_q \zeta_1^{q-1} - \sigma_1 q)\|A_1 v_n - A_1 y^*\|^q + \|w_n - x^*\|^q.$$

Using (1) and Lemma 4, we obtain that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^q\\
&\le q\alpha_n\langle u-x^*, J_q(x_{n+1}-x^*)\rangle + \|\beta_n(u_n-x^*)+\gamma_n(S_{\alpha,n}z_n-x^*)\|^q\\
&\le \beta_n\|x^*-u_n\|^q+\gamma_n[\|u_n-x^*\|^q - r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\
&\quad -\lambda_n(\sigma q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q] - W_q g(\|u_n-S_{\alpha,n}z_n\|)+q\alpha_n\langle u-x^*, J_q(x_{n+1}-x^*)\rangle\\
&\le (1-\alpha_n)\|x_n-x^*\|^q-\gamma_n[\zeta_2(\sigma_2 q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q\\
&\quad +\zeta_1(\sigma_1 q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q + r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\
&\quad +\lambda_n(\sigma q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ax^*-Ay_n\|^q] - W_q g(\|u_n-S_{\alpha,n}z_n\|)+\alpha_n q\langle u-x^*, J_q(x_{n+1}-x^*)\rangle.
\end{aligned}
\tag{3}
$$

Put

$$
\begin{aligned}
\Gamma_n &= \|x_n-x^*\|^q,\\
\eta_n &= \gamma_n[\zeta_2(\sigma_2 q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q+\zeta_1(\sigma_1 q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q\\
&\quad -\lambda_n r_n(\kappa_q\lambda_n^{q-1}-q\sigma)\|Au_n-Ax^*\|^q+(\sigma q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\lambda_n\|Ay_n-Ax^*\|^q]\\
&\quad -W_q g(\|u_n-S_{\alpha,n}z_n\|)\\
\delta_n &= \langle u-x^*, J_q(x_{n+1}-x^*)\rangle q\alpha_n.
\end{aligned}
$$

Then (3) can be rewritten as the following formula:

$$\Gamma_{n+1}+\eta_n \le (1-\alpha_n)\Gamma_n+\delta_n. \tag{4}$$

□

We next give two possible cases.

Case 1. We assume that there is an integer $n_0 \ge 1$ with the restriction that $\{\Gamma_n\}$ is non-increasing. From (4), we get $\eta_n \le \Gamma_n - \alpha_n\Gamma_n - \Gamma_{n+1}+\delta_n$. From $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\delta_n = 0$, one sees that $\lim_{n\to\infty}\eta_n = 0$. It is easy to see from Lemma 4 that $\lim_{n\to\infty} g(\|u_n-S_{\alpha,n}z_n\|)=0$,

$$\lim_{n\to\infty}\|A_2w_n-A_2x^*\| = \lim_{n\to\infty}\|A_1v_n-A_1y^*\| = 0 \tag{5}$$

and

$$\lim_{n\to\infty}\|Au_n-Ax^*\| = \lim_{n\to\infty}\|Ay_n-Ax^*\| = 0. \tag{6}$$

From the fact that g is a strictly increasing, continuous and convex function with $g(0)=0$, one has

$$\lim_{n\to\infty}\|u_n-S_{\alpha,n}z_n\| = 0. \tag{7}$$

By use of Lemma 5, we get

$$
\begin{aligned}
\|v_n-y^*\|^q &\le \langle w_n-\zeta_2A_2w_n-(x^*-\zeta_2A_2x^*), J_q(v_n-y^*)\rangle\\
&\le \tfrac{1}{q}[\|w_n-x^*\|^q+(q-1)\|v_n-y^*\|^q-\tilde{h}_1(\|w_n-x^*-v_n+y^*\|)]\\
&\quad +\zeta_2\langle A_2x^*-A_2w_n, J_q(v_n-y^*)\rangle,
\end{aligned}
$$

where $\tilde{h}_1$ is a convex, strictly increasing, continuous function as in Lemma 5. This hence entails

$$\|v_n-y^*\|^q \le \|w_n-x^*\|^q-\tilde{h}_1(\|w_n-v_n-x^*+y^*\|)+q\zeta_2\|A_2x^*-A_2w_n\|\|v_n-y^*\|^{q-1}.$$

In a similar way, one concludes

$$\begin{aligned}\|u_n - x^*\|^q &\le \langle v_n - \zeta_1 A_1 v_n - (y^* - \zeta_1 A_1 y^*), J_q(u_n - x^*)\rangle \\ &\le \tfrac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\ &\quad + \zeta_1\langle A_1 y^* - A_1 v_n, J_q(u_n - x^*)\rangle,\end{aligned}$$

which hence entails

$$\begin{aligned}&\|u_n - x^*\|^q \\ &\le \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) + q\zeta_1\|A_1 y^* - A_1 v_n\|\|u_n - x^*\|^{q-1} \\ &\le \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2 x^* - A_2 w_n\|\|v_n - y^*\|^{q-1} \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1 y^* - A_1 v_n\|\|u_n - x^*\|^{q-1}.\end{aligned} \tag{8}$$

Note that

$$\begin{aligned}q\|y_n - x^*\|^q &\le q\langle(u_n - \lambda_n A u_n) - x^* + \lambda_n A x^*), J_q(y_n - x^*)\rangle \\ &\le \|(u_n - \lambda_n A u_n) - x^* + \lambda_n A x^*)\|^q - h_1(\|u_n - y_n - \lambda_n(Au_n - Ax^*)\|) \\ &\quad + (q-1)\|y_n - x^*\|^q,\end{aligned}$$

which leads us to

$$\|y_n - x^*\|^q \le \|u_n - x^*\|^q - h_1(\|u_n - y_n - \lambda_n(Au_n - Ax^*)\|).$$

From (8), one has

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \\ &\le \gamma_n\{(1-r_n)\|u_n - x^*\|^q + r_n[\|u_n - x^*\|^q + \alpha_n\|x^* - u\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)]\} + \beta_n\|u_n - x^*\|^q \\ &\le \beta_n\|x_n - x^*\|^q + \alpha_n\|x^* - u\|^q + \gamma_n\{\|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1 y^* - A_1 v_n\|\|u_n - x^*\|^{q-1} \\ &\quad + q\zeta_2\|A_2 x^* - A_2 w_n\|\|v_n - y^*\|^{q-1} - r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\le \|x_n - x^*\|^q + \alpha_n\|x^* - u\|^q - \gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\quad + q\zeta_1\|A_1 y^* - A_1 v_n\|\|u_n - x^*\|^{q-1} + q\zeta_2\|A_2 x^* - A_2 w_n\|\|v_n - y^*\|^{q-1},\end{aligned}$$

which immediately yields that

$$\begin{aligned}&\gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\le \alpha_n\|u - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\zeta_1\|A_1 y^* - A_1 v_n\|\|u_n - x^*\|^{q-1} \\ &\quad + q\zeta_2\|A_2 x^* - A_2 w_n\|\|v_n - y^*\|^{q-1}.\end{aligned}$$

Since $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$ and the fact that $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions, one concludes that $\lim_{n\to\infty}\|w_n - v_n - x^* + y^*\| = \lim_{n\to\infty}\|v_n - u_n + x^* - y^*\| = \lim_{n\to\infty}\|u_n - y_n\|$. This immediately implies that

$$\lim_{n\to\infty}\|w_n - u_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{9}$$

Furthermore, borrowing $w_n = s_n((1-\xi)x_n + \xi G w_n) + (1-s_n)S_{\alpha,n}x_n$ defined as the first step in the iteration procedure, we obtain that

$$\begin{aligned}
&\|w_n - x^*\|^q = \langle s_n((1-\xi)x_n + \xi G w_n) + (1-s_n)S_{\alpha,n}x_n - x^*, J_q(w_n - x^*)\rangle \\
&= s_n[(1-\xi)\langle x_n - x^*, J_q(w_n - x^*)\rangle + \xi\langle Gw_n - x^*, J_q(w_n - x^*)\rangle] \\
&\quad + (1-s_n)\langle S_{\alpha,n}x_n - x^*, J_q(w_n - x^*)\rangle \\
&\le s_n[(1-\xi)\langle x_n - x^*, J_q(w_n - x^*)\rangle + \xi\|w_n - x^*\|^q] + (1-s_n)\langle S_{\alpha,n}x_n - x^*, J_q(w_n - x^*)\rangle,
\end{aligned}$$

which yields

$$\begin{aligned}
\|w_n - x^*\|^q \quad &\le \tfrac{1}{1-s_n\xi}[s_n(1-\xi)\langle x_n - x^*, J_q(w_n - x^*)\rangle + (1-s_n)\langle S_{\alpha,n}x_n - x^*, J_q(w_n - x^*)\rangle] \\
&\le \tfrac{1}{q}[\|x_n - x^*\|^q + (q-1)\|w_n - x^*\|^q] - \tfrac{1}{1-s_n\xi}[\tfrac{s_n(1-\xi)}{q}h_3(\|x_n - w_n\|) \\
&\quad + \tfrac{1-s_n}{q}\tilde{h}_3(\|S_{\alpha,n}x_n - w_n\|)].
\end{aligned}$$

This further implies that

$$\begin{aligned}
&\|u_n - x^*\|^q \le \|w_n - x^*\|^q \\
&\qquad \le \|x_n - x^*\|^q - \frac{1}{1-s_n\xi}[s_n(1-\xi)h_3(\|x_n - w_n\|) + (1-s_n)\tilde{h}_3(\|S_{\alpha,n}x_n - w_n\|)].
\end{aligned} \tag{10}$$

In a similar way, one further concludes

$$\begin{aligned}
\|z_n - x^*\|^q \quad &\le \|(x^* - \lambda_n Ax^*) - (u_n - \lambda_n Ay_n + r_n(y_n - u_n))\|^q \\
&\quad - h_2(\|u_n + (r_n y_n - r_n u_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - z_n\|) \\
&\le \|u_n - x^*\|^q - h_2(\|u_n + (r_n y_n - r_n u_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - z_n\|).
\end{aligned}$$

Using (10) leads us to

$$\begin{aligned}
\|x_{n+1} - x^*\|^q \quad &\le \gamma_n[\|u_n - x^*\|^q + \beta_n\|u_n - x^*\|^q \\
&\quad - h_2(\|u_n + (r_n y_n - r_n u_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - z_n\|)] + \alpha_n\|x^* - u\|^q \\
&\le \|x_n - x^*\|^q - \gamma_n\{\tfrac{1}{1-s_n\xi}[s_n(1-\xi)h_3(\|x_n - w_n\|) \\
&\quad + (1-s_n)\tilde{h}_3(\|S_{\alpha,n}x_n - w_n\|)] + h_2(\|u_n + (r_n y_n - r_n u_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - z_n\|)\} \\
&\quad + \alpha_n\|x^* - u\|^q.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\gamma_n\{\tfrac{1}{1-s_n\xi}[s_n(1-\xi)h_3(\|x_n - w_n\|) + (1-s_n)\tilde{h}_3(\|S_{\alpha,n}x_n - w_n\|)] \\
&\quad + h_2(\|u_n + (r_n y_n - r_n u_n) - (\lambda_n Ay_n - \lambda_n Ax^*) - z_n\|)\} \\
&\le \Gamma_n + \alpha_n\|u - x^*\|^q - \Gamma_{n+1}.
\end{aligned}$$

Note that $h_2(0) = h_3(0) = \tilde{h}_3(0) = 0$ and the fact that h_2, h_3 and $\tilde{h}_3$ are strictly increasing, continuous and convex functions. From (6) and (9) we have

$$\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|S_{\alpha,n}x_n - w_n\| = \lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{11}$$

By using (9) and (11), one concludes $\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|x_n - z_n\| = 0$. It follows that

$$\begin{aligned}
\|x_n - Gx_n\| \quad &\le \|x_n - u_n\| + \|u_n - Gx_n\| \\
&\le \|x_n - u_n\| + \|w_n - x_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{12}$$

Thanks to (11), we get $\lim_{n\to\infty}\|S_{\alpha,n}x_n - x_n\| = 0$, which, together with $S_{\alpha,n}x_n - x_n = \alpha(S_n x_n - x_n)$, leads to

$$\|S_n x_n - x_n\| = \frac{1}{\alpha}\|S_{\alpha,n}x_n - x_n\| \to 0 \quad (n\to\infty). \tag{13}$$

From the boundedness of $\{x_n\}$ and setting $D = \overline{\text{conv}}\{x_n : n \geq 0\}$, we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$. Lemma 6 yields that $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$. So, $\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0$. Further, from (13), we have

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{14}$$

Letting $S_\alpha x := (1-\alpha)x + \alpha Sx\ \forall x \in C$, we deduce from Lemma 1 that $S_\alpha : C \to C$ is a nonexpansive mapping. It is easy to see from (14) that $\lim_{n\to\infty} \|x_n - S_\alpha x_n\| = 0$. For each $n \geq 0$, set $T_{\lambda_n} := J^B_{\lambda_n}(I - \lambda_n A)$. It follows that

$$\begin{aligned} \|x_n - T_{\lambda_n} x_n\| &\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n} u_n\| + \|T_{\lambda_n} u_n - T_{\lambda_n} x_n\| \\ &\leq 2\|x_n - u_n\| + \|u_n - y_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

In light of $0 < \lambda \leq \lambda_n$ for all $n \geq 0$, we obtain

$$\|T_\lambda x_n - x_n\| \leq 2\|T_{\lambda_n} x_n - x_n\| \to 0 \quad (n \to \infty). \tag{15}$$

We define a mapping $\Psi : C \to C$ by $\Psi x := \theta_1 S_\alpha x + \theta_2 Gx + (1 - \theta_1 - \theta_2) T_\lambda x\ \forall x \in C$ with $\theta_1 + \theta_2 < 1$ for constants $\theta_1, \theta_2 \in (0,1)$. Lemma 7 guarantees that Ψ is nonexpansive and

$$\text{Fix}(\Psi) = \text{Fix}(S_\alpha) \cap \text{Fix}(G) \cap \text{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{Fix}(G) \cap (A+B)^{-1}0 \ \ (=: \Omega).$$

Taking into account that

$$\|\Psi x_n - x_n\| \leq \theta_1 \|S_\alpha x_n - x_n\| + \theta_2 \|Gx_n - x_n\| + (1 - \theta_1 - \theta_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (12) and (15) that

$$\lim_{n\to\infty} \|\Psi x_n - x_n\| = 0. \tag{16}$$

Let $z^\lambda = \lambda u + (1-\lambda)\Psi z^\lambda$, $\forall \lambda \in (0,1)$. Lemma 8 guarantees that $\{z^\lambda\}$ converges to a point $x^* \in \text{Fix}(\Psi) = \Omega$ in norm, and x^* further solves the VIP: $\langle x^* - u, J(x^* - p)\rangle \leq 0, \forall p \in \Omega$. From (1), we have

$$\begin{aligned} \|z^\lambda - x_n\|^q &\leq \lambda q \|z^\lambda - x_n\|^q + (1-\lambda)^q (\|\Psi z^\lambda - \Psi x_n\| + \|\Psi x_n - x_n\|)^q + \lambda q \langle u - z^\lambda, J_q(z^\lambda - x_n)\rangle \\ &\leq \lambda q \|z^\lambda - x_n\|^q + (1-\lambda)^q (\|\Psi x_n - x_n\| + \|z^\lambda - x_n\|)^q + \lambda q \langle u - z^\lambda, J_q(z^\lambda - x_n)\rangle. \end{aligned}$$

Further, from (16), one has

$$\limsup_{n\to\infty} \langle u - z_t, J_q(x_n - z_t)\rangle \leq M \frac{(qt-1) + (1-t)^q}{qt},$$

where M is a constant such that $\|z_t - x_n\|^q \leq M$ for all $n \geq 0$ and $t \in (0,1)$. From the properties of J_q and the fact that $z_t \to x^*$ as $t \to 0$, one gets $\lim_{t\to 0} \|J_q(x_n - x^*) - J_q(x_n - z_t)\| = 0$. A simple calculation indicates that

$$\limsup_{n\to\infty} \langle u - x^*, J_q(x_n - x^*)\rangle \leq 0 \tag{17}$$

and then

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \|u - x_n\| + \beta_n \|u_n - x_n\| + \gamma_n (\|S_{\alpha,n} z_n - u_n\| + \|u_n - x_n\|) \\ &\leq \alpha_n \|u - x_n\| + \|u_n - x_n\| + \|S_{\alpha,n} z_n - u_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

Using (17), we have $\limsup_{n\to\infty} \langle u - x^*, J_q(x_{n+1} - x^*)\rangle \leq 0$. An application of Lemma 9 yields that $\Gamma_n \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$.

Case 2. We assume that there is $\{\Gamma_{k_i}\} \subset \{\Gamma_k\}$ s.t. $\Gamma_{k_i} < \Gamma_{k_i+1}$ $\forall i \in N$, where N is the set of all positive integers. We now give a new mapping $\tau : N \to N$ by $\tau(k) := \max\{i \le k : \Gamma_i < \Gamma_{i+1}\}$. Using Lemma 10, one concludes

$$\Gamma_{\tau(k)+1} \ge \Gamma_{\tau(k)} \quad \text{and} \quad \Gamma_{\tau(k)+1} \ge \Gamma_k.$$

Putting $\Gamma_k = \|x_k - x^*\|^q$ $\forall k \in \mathbf{N}$ and using the same reasoning as in Case 1 we can obtain

$$\lim_{k\to\infty} \|x_{\tau(k)} - x_{\tau(k)+1}\| = 0 \tag{18}$$

$$\limsup_{k\to\infty} \langle u - x^*, J_q(x_{\tau(k)+1} - x^*)\rangle \le 0. \tag{19}$$

In view of $\alpha_{\tau(k)} > 0$ and $\Gamma_{\tau(k)+1} \ge \Gamma_{\tau(k)}$, we conclude that

$$\frac{q}{1-\delta}\langle u - x^*, J_q(x_{\tau(k)+1} - x^*)\rangle \ge \|x_{\tau(k)} - x^*\|^q.$$

Consequently, $\lim_{k\to\infty} \|x_{\tau(k)} - x^*\|^q = 0$. Using Lemma 3, we have that

$$\begin{aligned}
&\|x_{\tau(k)+1} - x^*\|^q - \|x^* - x_{\tau(k)}\|^q \\
&\le q\langle x_{\tau(k)+1} - x_{\tau(k)}, J_q(x_{\tau(k)} - x^*)\rangle + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q \\
&\le q\|x_{\tau(k)} - x^*\|^{q-1}\|x_{\tau(k)+1} - x_{\tau(k)}\| + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q \to 0 \quad (k \to \infty).
\end{aligned}$$

Thanks to $\Gamma_k \le \Gamma_{\tau(k)+1}$, we get

$$\|x_k - x^*\|^q \le \|x_{\tau(k)} - x^*\|^q + q\|x_{\tau(k)+1} - x_{\tau(k)}\|\|x_{\tau(k)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q.$$

It is easy to see from (18) that $x_k \to x^*$ as $k \to \infty$. This completes the proof.

It is well known that $\kappa_2 = 1$ in Hilbert spaces. From Theorem 1, we derive the following conclusion.

Corollary 1. *Let $\emptyset \ne C \subset H$ be a closed convex set. Let $\{S_n\}_{n=0}^\infty$ be a family of ς-uniformly strict pseudocontraction mappings defined on C. Suppose that $B_1, B_2 : C \to 2^H$ are both maximal monotone operators and $A_k : C \to H$ is σ_k-inverse-strongly monotone mapping for $k = 1, 2$. Define the mapping $G : C \to C$ by $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ for constants $\zeta_1, \zeta_2 > 0$. Let $A : C \to H$ and $B : C \to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator, respectively. For any given $x_0 \in C$, $\xi \in (0,1)$ and $\alpha \in (0, \min\{1, 2\varsigma\})$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases}
w_n = s_n((1-\xi)x_n + \xi G w_n) + (1-s_n)((1-\alpha)x_n + \alpha S_n x_n), \\
u_n = J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\
y_n = J_{\lambda_n}^{B}(u_n - \lambda_n A u_n), \\
z_n = J_{\lambda_n}^{B}(u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\
x_{n+1} = \alpha_n u + \beta_n u_n + \gamma_n((1-\alpha)z_n + \alpha S_n z_n) \quad \forall n \ge 0,
\end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1]$ with the additional restrictions $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0,\infty)$ are such that $\sum_{n=0}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ as $n \to \infty$; $0 < a \le \beta_n \le b < 1$ and $0 < c \le s_n \le d < 1$; $0 < r \le r_n < 1$ and $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < 2\sigma$; $0 < \zeta_k < 2\sigma_k$ for $k = 1, 2$. Assume that $\sum_{n=0}^\infty \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$, where D is a bounded subset of C. Define a self mapping S by $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$, and further assume that $\bigcap_{n=0}^\infty \mathrm{Fix}(S_n) = \mathrm{Fix}(S)$. If $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \ne \emptyset$, then $\lim_{n\to\infty} \|x_n - x^*\| = 0$, where $x^* \in \Omega$ uniquely solves $\langle x^* - u, p - x^*\rangle \ge 0$ $\forall p \in \Omega$.

Next, we recall the least absolute shrinkage and selection operator (LASSO) [35], which can be formulated as a convex constrained optimization problem:

$$\min_{y\in H}\frac{1}{2}\|Ty-b\|_2^2 \quad \text{subject to } \|y\|_1\le s, \tag{20}$$

where T is a bounded operator on H, b is a fixed vector in H, and $s>0$ is a real number. In this section, Λ is employed to denote the set of solutions of LASSO (20). LASSO, which acts as a unified model for a number of real problems, has been investigated in different settings. Ones know that a solution to (20) is a minimizer to the following minimization problem: $\min_{y\in H} g(y)+h(y)$, where $g(y):=\frac{1}{2}\|Ty-b\|_2^2$, $h(y):=\lambda\|y\|_1$. It is known that $\nabla g(y)=T^*(Ty-b)$ is $\frac{1}{\|T^*T\|}$-inverse-strongly monotone. Hence, we have that z solves the LASSO iff z solves the problem, which consists of finding $z\in H$ s.t.

$$\begin{aligned} 0\in\partial h(z)+\nabla g(z) &\Leftrightarrow z-\lambda\nabla g(z)\in z+\lambda\partial h(z)\\ &\Leftrightarrow z=\mathrm{prox}_h(z-\lambda\nabla g(z)), \end{aligned}$$

where $\lambda>0$ is real, and $\mathrm{prox}_h(y)$ is the proximal of $h(y):=\lambda\|y\|_1$ defined as follows

$$\mathrm{prox}_h(y)=\mathrm{argmin}_{u\in H}\{\lambda\|u\|_1+\frac{1}{2}\|u-y\|_2^2\} \quad \forall y\in H.$$

This is separable in indices. So, $y\in H$, for $i=1,2,\ldots,n$, $\mathrm{prox}_h(y)=\mathrm{prox}_{\lambda\|\cdot\|_1}(y)=(\mathrm{prox}_{\lambda|\cdot|}(y_1),\mathrm{prox}_{\lambda|\cdot|}(y_2),\ldots,\mathrm{prox}_{\lambda|\cdot|}(y_n))$, with $\mathrm{prox}_{\lambda|\cdot|}(y_i)=\mathrm{sgn}(y_i)\max\{|y_i|-\lambda,0\}$.

By putting $C=H$, $A=\nabla g$, $B=\partial h$ and $\sigma=\frac{1}{\|T^*T\|}$ in Corollary 1, we obtain the following result immediately.

Corollary 2. *Let A_k,B_k $(k=1,2)$ and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 1 with $C=H$. Assume that $\Omega:=\bigcap_{n=0}^\infty \mathrm{Fix}(S_n)\cap\mathrm{Fix}(G)\cap\Lambda\neq\emptyset$. For any given $x_0\in H$, $\xi\in(0,1)$ and $\alpha\in(0,\min\{1,2\varsigma\})$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n=s_n((1-\xi)x_n+\xi Gw_n)+(1-s_n)((1-\alpha)x_n+\alpha S_nx_n),\\ u_n=J_{\zeta_1}^{B_1}(I-\zeta_1A_1)J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n),\\ y_n=\mathrm{prox}_h(u_n-\lambda_nT^*(Tu_n-b)),\\ z_n=\mathrm{prox}_h(u_n-\lambda_nT^*(Ty_n-b)+(r_ny_n-r_nu_n)),\\ x_{n+1}=\alpha_nu+(1-\alpha)\gamma_nz_n+\alpha\gamma_nS_nz_n+\beta_nu_n \quad \forall n\ge 0, \end{cases}$$

where the sequences $\{r_n\},\{s_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1]$, $\alpha_n+\beta_n+\gamma_n=1$ and $\{\lambda_n\}\subset(0,\infty)$ are such that the conditions presented in Corollary 3.1 hold where $\sigma=\frac{1}{\|T^*T\|}$. Then $x_n\to x^*\in\Omega$ as $n\to\infty$.

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