symmetry

MDPI

Article

A New Integral Transform: ARA Transform and Its Properties and Applications

Rania Saadeh, Ahmad Qazza * and Aliaa Burqan

Department of Mathematics, Zarqa University, Zarqa 13132, Jordan; rsaadeh@zu.edu.jo (R.S.); aliaaburqan@zu.edu.jo (A.B.)
* Correspondence: aqazza@zu.edu.jo

Received: 24 February 2020; Accepted: 19 May 2020; Published: 2 June 2020

Abstract: In this paper, we introduce a new type of integral transforms, called the ARA integral transform that is defined as: $\mathcal{G}_n[g(t)](s) = G(n,s) = s\int_0^\infty t^{n-1}e^{-st}g(t)dt,\ s > 0$. We prove some properties of ARA transform and give some examples. Also, some applications of the ARA transform are given.

Keywords: ARA transform; Fourier transform; Laplace transform; Sumudu transform; Elzaki transform; Mellin transform; Natural transform; Yang transform; Shehu transform; Novel transform; Ordinary differential equation; Integral equation

1. Introduction

The integral transforms play a vital role in finding solutions to initial value problems and initial boundary value problems. An integral transform T [1] has the form

$$T[g(t)](u) = \int_{t_1}^{t_2} g(t)K(t,u)dt, \tag{1}$$

where the input function of the transform is $g(t)$ and the output is $T[g(t)](u)$, and the function $K(t,u)$ is a kernel function. Moreover, the inverse transform related to the inverse kernel function is given by:

$$g(t) = \int_{u_1}^{u_2} T[g(t)](u)\ K^{-1}(u,t)du. \tag{2}$$

The integral transform was introduced by the French mathematician and physicist P.S. Laplace [2,3] in 1780. In 1822, J. Fourier [4] introduced the Fourier transform. Laplace and Fourier transforms form the foundation of operational analysis, a branch of mathematics that has very powerful applications, not only in applied mathematics but also in other branches of science like physics, engineering, astronomy, etc.

In recent years, mathematicians have been interested in developing and establishing new integral transforms. In 1993, Watugula [5] introduced the Sumudu transform. The natural transform was introduced by Khan and Khan [6] in 2008. In 2011, the Elzaki transform [7] was devised by Elzaki. Atangana and Kiliçman [8] in 2013, introduced the Novel transform. In 2015, Srivastava, Luo and Raina [9] introduced the M-transform. In 2016, many transforms were introduced, like the ZZ transform by Zafar [10], Ramadan Group (RG) transform [11], a polynomial transform by Barnes [12], also, a new integral transform was presented by Yang [13]. In the year 2017, other transforms were introduced, such as the Aboodh transform [14] and Rangaig transform [15], while the Shehu transform [16] was established in 2019, by Shehu and Weidong.

In this paper, we proclaim a new integral transform called the ARA integral transform.

This transform is a powerful and versatile generalization that unifies some variants of the classical Laplace transform, namely, the Sumudu transform, the Elzaki transform, the Natural transform, the Yang transform, and the Shehu transform.

In Section 2, we state our definition of the ARA transform and some related theorems. In Section 3, we provide the properties of the ARA transform, and in the last section, we give some applications.

2. Definitions and Theorems

Definition 1. *The ARA integral transform of order n of the continuous function $g(t)$ on the interval $(0, \infty)$ is defined as:*

$$\mathcal{G}_n[g(t)](s) = G(n,s) = s\int_0^\infty t^{n-1}e^{-st}g(t)dt,\ s>0 \tag{3}$$

Definition 2. *The inverse of the ARA transform is given by*

$$\begin{aligned} g(t) &= \mathcal{G}_{n+1}^{-1}[\mathcal{G}_{n+1}[g(t)]] \\ &= \tfrac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Bigg((-1)^n\bigg(\tfrac{1}{s\Gamma(n-1)}\int_0^s (s-x)^{n-1}G(n \\ &\quad +1,x)dx + \sum_{k=0}^{n-1}\tfrac{s^k}{k!}\tfrac{\partial^k G(0)}{\partial s^k}\bigg)\Bigg)ds, \end{aligned} \tag{4}$$

where

$$G(s) = \int_0^\infty e^{-st}\, g(t)\, dt,$$

is $(n-1)$ times differentiable.

In fact, from the definition of ARA transform of a function $g(t)$, we have

$$G(n+1,s) = \mathcal{G}_{n+1}[g(t)](s) = s\int_0^\infty t^n e^{-st}g(t)dt = (-1)^n\, s\,\frac{d^nG(s)}{ds^n}\ ,$$

thus

$$\frac{1}{s\,\Gamma(n-1)}\int_0^s (s-t)^{n-1}G(n+1,t)dt = (-1)^n\left(G(s) - \sum_{k=0}^{n-1}\frac{s^k}{k!}\frac{\partial^k G(0)}{\partial s^k}\right),$$

and so,

$$\frac{(-1)^n}{s\,\Gamma(n-1)}\int_0^s (s-t)^{n-1}G(n+1,t)dt + \sum_{k=0}^{n-1}\frac{s^k}{k!}\frac{\partial^k G(0)}{\partial s^k} = G(s).$$

It follows that:

$$\begin{aligned} &\tfrac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Bigg((-1)^n\bigg(\tfrac{1}{s\Gamma(n-1)}\int_0^s (s-x)^{n-1}G(n+1,x)dx \\ &+\sum_{k=0}^{n-1}\tfrac{s^k}{k!}\tfrac{\partial^k G(0)}{\partial s^k}\bigg)\Bigg)ds = \tfrac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big((-1)^n G(s)\Big)ds. \end{aligned}$$

$$\mathcal{G}_{n+1}^{-1}[\mathcal{G}_{n+1}[g(t)]] = \frac{(-1)^{2n}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}G(s)ds = g(t).$$

Theorem 1. *The sufficient condition for the existence of ARA transform. If the function $g(t)$ is piecewise continuous in every finite interval $0 \le t \le \alpha$ and satisfies:*

$$\left|t^{n-1}g(t)\right| \le Ke^{\beta t}, \tag{5}$$

then ARA transform exists for all $s > \beta$.

Proof of Theorem 1. We have

$$s\int_0^\infty t^{n-1}e^{-st}g(t)dt = s\int_0^\alpha t^{n-1}e^{-st}g(t)dt + s\int_\alpha^\infty t^{n-1}e^{-st}g(t)dt,$$

since the function $g(t)$ is piecewise continuous then the first integral on the right side exists. Also, the second integral on the right side converges because:

$$\begin{aligned}\left|s\int_\alpha^\infty t^{n-1}e^{-st}g(t)dt\right| &\le s\int_\alpha^\infty e^{-st}\left|t^{n-1}g(t)\right|dt \le s\int_\alpha^\infty e^{-st}ke^{\beta t}dt \\ = sk\int_\alpha^\infty e^{\beta t-st}dt &= \lim_{b\to\infty} -sk\,\frac{e^{-t(s-\beta)}}{s-\beta}\bigg|_\alpha^b = \frac{sk}{s-\beta}e^{-\alpha(s-\beta)},\end{aligned}$$

and this improper integral is convergent for all $s > \beta$. Thus, $\mathcal{G}_{n+1}[g(t)](s)$ exists. □

3. Dualities between ARA Transform and Some Integral Transform:

Duality between ARA and Laplace Transforms [17]:

i:

$$\mathcal{G}_0[f(t)](s) = s\int_0^\infty t^{-1}e^{-st}f(t)dt = s\,\mathcal{L}\left[\frac{f(t)}{t}\right] = s\int_s^\infty F(u)du, \tag{6}$$

where $F(u) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt$

ii:

$$\begin{aligned}\mathcal{G}_1[f(t)](s) = s\int_0^\infty t^{1-1}e^{-st}f(t)dt = s\int_0^\infty e^{-st}f(t)dt = s\,\mathcal{L}[f(t)] \\ = s\,F(s).\end{aligned} \tag{7}$$

iii:

$$\mathcal{L}^{-1}[\mathcal{G}_n[f(t)](s)] = t^{n-2}(2H(t)-1)((n-1)f(t) + tf'(t)) \tag{8}$$

where

$$H(t) = \int_{-\infty}^t \delta(s)ds,$$

is a Heaviside function (integral of Dirac delta).

There are some functions in which the Laplace transform does not exist.

Proof: Relations i and ii are obvious. Here, we prove relation iii.

$$\begin{aligned}
\mathcal{L}^{-1}[\mathcal{G}_n[f(t)](s)] &= \tfrac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\mathcal{G}_n[f(t)](s)ds \\
&= \tfrac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\left(s\int_0^{\infty} e^{-st}t^{n-1}f(t)dt\right)ds \\
&= \tfrac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}s\,\mathcal{L}\Big[t^{n-1}f(t)\Big]ds = \mathcal{L}^{-1}\Big[s\,\mathcal{L}\Big[t^{n-1}f(t)\Big]\Big] \\
&= \mathcal{L}^{-1}[s]*\mathcal{L}^{-1}\Big[\mathcal{L}\Big[t^{n-1}f(t)\Big]\Big] = \delta'(t)*t^{n-1}f(t) \\
&= \int_0^t \delta'(t-\tau)\tau^{n-1}f(\tau)d\tau \\
&= t^{-2+n}(-1+2H(t))((-1+n)f(t)+tf'(t)).
\end{aligned}$$

and for relation iv, the Laplace transform for the function $\frac{e^{-t}}{t}$ does not exist, while:

$$\mathcal{G}_2\left[\frac{e^{-t}}{t}\right](s) = s\int_0^{\infty} t^{2-1}e^{-st}\frac{e^{-t}}{t}dt = \frac{s}{s+1}.$$

The duality between ARA and Laplace Carson transforms [18]

$$\mathcal{L}_*[g(t)] = \mathcal{G}_1[g(t)](s)$$

$$\mathcal{G}_n[g(t)](s) = \mathcal{L}_*\Big[t^{n-1}g(t)\Big]$$

where

$$\mathcal{L}_*[g(t)] = s\int_0^{\infty} e^{-st}g(t)dt.$$

Duality between ARA and Aboodh Transforms [19]

$$A[g(t)] = \frac{1}{s^2}\mathcal{G}_1[g(t)](s)$$

$$\mathcal{G}_n[g(t)](s) = s^2 A\Big[t^{n-1}\,g(t)\Big]$$

where

$$A[g(t)] = \frac{1}{s}\int_0^{\infty} g(t)e^{-st}dt.$$

Duality between ARA and Mohand Transforms [20]:

$$M[g(t)] = s\mathcal{G}_1[g(t)](s)$$

$$\mathcal{G}_n[g(t)](s) = \frac{1}{s}M\Big[t^{n-1}\,g(t)\Big]$$

where

$$M[g(t)] = s^2\int_0^{\infty} g(t)e^{-st}dt.$$

□

4. Properties of ARA Transform

In this section, we establish some properties of the ARA transform, which enable us to calculate further transform of functions in applications.

Property 1. *(Linearity property) Let $u(t)$ and$v(t)$ be two functions in which ARA transform exists, then*

$$\mathcal{G}_n[\alpha u(t) + \beta v(t)](s) = \alpha \mathcal{G}_n[u(t)](s) + \beta \mathcal{G}_n[v(t)](s) \tag{9}$$

where α and β are nonzero constants.

Proof of Property 1:

$$\begin{aligned} \mathcal{G}_n[\alpha u(t) + \beta v(t)](s) &= s \int_0^\infty t^{n-1} e^{-st} (\alpha u(t) + \beta v(t))\, dt \\ &= \alpha s \int_0^\infty t^{n-1} e^{-st} u(t) dt + \beta s \int_0^\infty t^{n-1} e^{-st} v(t)\, dt \\ &= \alpha \mathcal{G}_n[u(t)](s) + \beta \mathcal{G}_n[v(t)](s). \end{aligned}$$

□

Property 2. *Change of scale property*

$$\mathcal{G}_n[g(at)](s) = \frac{1}{a^{n-1}} \mathcal{G}_n[g(t)]\left(\frac{s}{a}\right) = \frac{1}{a^{n-1}} G\left(n, \frac{s}{a}\right). \tag{10}$$

Proof of Property 2: Using the definition of ARA transform for $g(at)$, we get

$$\mathcal{G}_n[g(at)](s) = s \int_0^\infty t^{n-1} e^{-st} g(at) dt \tag{11}$$

and a substitution of $u = at$ in Equation (5) yields:

$$\begin{aligned} \mathcal{G}_n[g(at)](s) &= \tfrac{s}{a^n} \int_0^\infty u^{n-1} e^{-\frac{s}{a}u} g(u) du = \tfrac{1}{a^{n-1}} \tfrac{s}{a} \int_0^\infty u^{n-1} e^{-\frac{s}{a}u} g(u) du \\ &= \tfrac{1}{a^{n-1}} G\left(n, \tfrac{s}{a}\right) \end{aligned}$$

□

Property 3. *Shifting in $s-$ Domain*

$$\mathcal{G}_n\left[e^{-ct} g(t)\right](s) = \frac{s}{s+c} G(n, s+c). \tag{12}$$

Proof of Property 3:

$$\begin{aligned} \mathcal{G}_n\left[e^{-ct} g(t)\right](s) &= s \int_0^\infty t^{n-1} e^{-st} e^{-ct} g(t) dt = s \int_0^\infty t^{n-1} e^{-(s+c)t} g(t) dt \\ &= \tfrac{s}{s+c} (s+c) \int_0^\infty t^{n-1} e^{-(s+c)t} g(t) dt \\ &= \tfrac{s}{s+c} G(n, s+c). \end{aligned}$$

□

Property 4. *Shifting in n− Domain*

$$\mathcal{G}_n[t^m g(t)](s) = \mathcal{G}_{n+m}[g(t)] = G(n+m,s). \tag{13}$$

Proof of Property 4:

$$\begin{aligned}\mathcal{G}_n[t^m g(t)](s) &= s\int_0^\infty t^{n-1}e^{-st}t^m g(t)dt = s\int_0^\infty t^{m+n-1}e^{-st}g(t)dt \\ &= \mathcal{G}_{n+m}[g(t)](s) = G(n+m,s).\end{aligned}$$

Also, $\mathcal{G}_n\left[\frac{g(t)}{t^m}\right] = \mathcal{G}_{n-m}[g(t)] = G(n-m,s)$. □

Property 5. *Shifting on t− domain*

$$\mathcal{G}_n[u_c(t)g(t-c)](s) = e^{-cs}\mathcal{G}_1\left[g(v)(v+c)^{n-1}\right] \tag{14}$$

Proof of Property 5:

$$\begin{aligned}\mathcal{G}_n[u_c(t)g(t-c)](s) &= s\int_c^\infty t^{n-1}e^{-st}\ u_c(t)\ g(t-c)dt \\ &= s\int_c^\infty t^{n-1}e^{-st}\ \ g(t-c)dt,\end{aligned}$$

letting $t-c=v$ and substituting in the above equation we get

$$\begin{aligned}s\int_0^\infty (v+c)^{n-1}e^{-s(v+c)}\ \ g(v)dv &= se^{-sc}\int_0^\infty e^{-sv}\ \ g(v)(v+c)^{n-1}dv \\ &= e^{-sc}\mathcal{G}_1\left[g(v)(v+c)^{n-1}\right].\end{aligned}$$

□

Property 6. *ARA transform for derivatives*

$$\mathcal{G}_n\left[f^{(m)}(t)\right](s) = (-1)^{n-1}s\frac{d^{n-1}}{ds^{n-1}}\left(\frac{\mathcal{G}_1\left[f^{(m)}(t)\right](s)}{s}\right) \tag{15}$$

Proof of Property 6:

$$\begin{aligned}\mathcal{G}_n\left[f^{(m)}(t)\right](s) &= s\int_0^\infty t^{n-1}e^{-st}f^{(m)}(t)dt = s\int_0^\infty\left(t^{n-1}f^{(m)}(t)\right)e^{-st}dt \\ &= \mathcal{G}_1\left[t^{n-1}f^{(m)}(t)\right](s) \\ &= (-1)^{n-1}s\frac{d^{n-1}}{ds^{n-1}}\left(\frac{\mathcal{G}_1\left[f^{(m)}(t)\right](s)}{s}\right).\end{aligned}$$

Moreover:

$$\mathcal{G}_n\left[f^{(n)}(t)\right](s) = (-1)^{n-1} s \frac{d^{n-1}}{ds^{n-1}} \left(s^{n-1} \mathcal{G}_1[f(t)](s) - \sum_{j=1}^{n} s^{n-j} f^{(j-1)}(0) \right) \tag{16}$$

Since

$$\begin{aligned} \mathcal{G}_n\left[f^{(n)}(t)\right](s) &= (-1)^{n-1} s \tfrac{d^{n-1}}{ds^{n-1}} \mathcal{L}\left[f^{(n)}(t)\right] \\ &= (-1)^{n-1} s \tfrac{d^{n-1}}{ds^{n-1}} \Big(s^{n-1} \mathcal{G}_1[f(t)](s) - s^{n-1} f(0) - \cdots \\ &\quad - f^{(n-1)}(0) \Big) \\ &= (-1)^{n-1} s \tfrac{d^{n-1}}{ds^{n-1}} \Big(s^{n-1} \mathcal{G}_1[f(t)](s) \\ &\quad - \sum_{j=1}^{n} s^{n-j} f^{(j-1)}(0) \Big). \end{aligned}$$

Using the properties of the convolution of Laplace transform, we get the following property. □

Property 7. *Convolution*

$$\mathcal{G}_n[f(t) * h(t)](s) = (-1)^{n-1} s \sum_{j=0}^{n-1} c_j^{n-1} F^{(j)}(s) \cdot H^{(n-1-j)}(s) \tag{17}$$

Proof of Property 7:

$$\begin{aligned} \mathcal{G}_n[f(t) * g(t)](s) &= s \int_0^\infty (f(t) * g(t)) t^{n-1} e^{-st} dt \\ &= \mathcal{G}_1\left[t^{n-1} f(t) * g(t)\right] = (-1)^{n-1} s \tfrac{d^{n-1}}{ds^{n-1}} \tfrac{\mathcal{G}_1[f(t) * g(t)]}{s} \\ &= (-1)^{n-1} s \sum_{j=0}^{n-1} c_j^{n-1} F^{(j)}(s) \cdot H^{(n-1-j)}(s), \end{aligned}$$

where c_n^k is the binomial coefficient. □

Now, we introduce some practical examples for finding ARA transform for some functions.
Example 4.1:

$$\begin{aligned} \mathcal{G}_n[1](s) &= s \int_0^\infty t^{n-1} e^{-st} dt = \Gamma(n) \left(\tfrac{1}{s}\right)^n s \int_0^\infty \tfrac{t^{n-1} e^{-st}}{\Gamma(n) \left(\frac{1}{s}\right)^n} dt = \Gamma(n) \left(\tfrac{1}{s}\right)^n s \\ &= \tfrac{\Gamma(n)}{s^{n-1}} = \tfrac{(n-1)!}{s^{n-1}}. \end{aligned}$$

Example 4.2:

$$\begin{aligned} \mathcal{G}_n[t](s) &= s \int_0^\infty t^{n-1} e^{-st} t \, dt = s \int_0^\infty t^n e^{-st} \, dt \\ &= \Gamma(n+1) \left(\tfrac{1}{s}\right)^{n+1} s \int_0^\infty \tfrac{t^n e^{-st}}{\Gamma(n+1) \left(\frac{1}{s}\right)^{n+1}} \, dt \\ &= \left(\tfrac{1}{s}\right)^{n+1} \Gamma(n+1) s = \tfrac{\Gamma(n+1)}{s^n}. \end{aligned}$$

Example 4.3:

$$\begin{aligned} \mathcal{G}_n[t^m](s) &= s \int_0^\infty t^{n-1} e^{-st} t^m \, dt = s \int_0^\infty t^{m+n-1} e^{-st} \, dt \\ &= \Gamma(m+n) \left(\tfrac{1}{s}\right)^{m+n} s \int_0^\infty \tfrac{t^{m+n-1} e^{-st}}{\Gamma(m+n) \left(\frac{1}{s}\right)^{m+n}} \, dt \\ &= \left(\tfrac{1}{s}\right)^{m+n} \Gamma(m+n) s = \tfrac{\Gamma(m+n)}{s^{m+n-1}}. \end{aligned}$$

Example 4.4:

$$\mathcal{G}_n\Big[e^{at}\Big](s) = \quad s\int\limits_0^\infty t^{n-1}e^{-t(s-a)}\,dt = \Gamma(n)\Big(\tfrac{1}{s-a}\Big)^n s\int\limits_0^\infty \frac{t^{n-1}e^{-t(s-a)}}{\Gamma(n)\big(\frac{1}{s-a}\big)^n}\,dt$$
$$= \tfrac{s}{(s-a)^n}\Gamma(n),\ \text{for all}\ \ s > a.$$

Example 4.5:

$$\begin{aligned}\mathcal{G}_n\Big[t^m e^{at}\Big](s) &= s\int\limits_0^\infty t^{m+n-1}e^{-t(s-a)}\,dt\\ &= \Gamma(m+n)\Big(\tfrac{1}{s-a}\Big)^{m+n} s\int\limits_0^\infty \frac{t^{m+n-1}e^{-t(s-a)}}{\Gamma(m+n)\big(\frac{1}{s-a}\big)^{m+n}}\,dt\\ &= \tfrac{s}{(s-a)^{m+n}}\Gamma(m+n).\end{aligned}$$

Example 4.6:

$$\begin{aligned}\mathcal{G}_n[\sin(at)](s) &= \mathcal{G}_n\Big[\tfrac{e^{iat}-e^{-iat}}{2i}\Big] = \tfrac{1}{2i}\Big(\mathcal{G}_n\Big[e^{iat}\Big] - \mathcal{G}_n\Big[-e^{-iat}\Big]\Big)\\ &= \tfrac{s}{2i}\Gamma(n)\bigg(\tfrac{1}{(s-ia)^n} - \tfrac{1}{(s+ia)^n}\bigg)\\ &= \tfrac{s}{2i}\Gamma(n)\Bigg(\frac{2i}{(s^2+a^2)^{\frac{n}{2}}}\sin\Big(n\tan^{-1}\Big(\tfrac{a}{s}\Big)\Big)\Bigg)\\ &= \Big(1+\tfrac{a^2}{s^2}\Big)^{-\frac{\mathrm{n}}{2}}\mathrm{s}^{1-\mathrm{n}}\Gamma(n)\sin\Big(n\tan^{-1}\Big(\tfrac{a}{s}\Big)\Big).\end{aligned}$$

Example 4.7:

$$\mathcal{G}_n[\cos(at)](s) = \left(1+\frac{a^2}{s^2}\right)^{-\frac{\mathrm{n}}{2}}\mathrm{s}^{1-\mathrm{n}}\Gamma(n)\cos\!\left(n\tan^{-1}\left(\frac{a}{s}\right)\right).$$

With similar arguments to example 4.6, we obtain the result.

Example 4.8:

$$\begin{aligned}\mathcal{G}_n[\sinh(at)] &= \mathcal{G}_n\Big[\tfrac{e^{at}-e^{-at}}{2}\Big] = \tfrac{1}{2}\Big(\mathcal{G}_n\Big[e^{at}\Big] - \mathcal{G}_n\Big[-e^{-at}\Big]\Big)\\ &= \tfrac{1}{2}\bigg(\tfrac{s}{(s-a)^n}\Gamma(n) - \tfrac{s}{(s+a)^n}\Gamma(n)\bigg)\\ &= \tfrac{s}{2}\Gamma(n)\bigg(\tfrac{1}{(s-a)^n} - \tfrac{1}{(s+a)^n}\bigg)\\ &= \tfrac{s}{2}\Gamma(n)\tfrac{1}{s^n}\Bigg(\tfrac{1}{\big(1-\frac{a}{s}\big)^n} - \tfrac{1}{\big(1+\frac{a}{s}\big)^n}\Bigg).\end{aligned}$$

Example 4.9:

$$\begin{aligned}\mathcal{G}_n[\cosh(at)](s) &= \mathcal{G}_n\Big[\tfrac{e^{at}+e^{-at}}{2}\Big] = \tfrac{1}{2}\Big(\mathcal{G}_n\Big[e^{at}\Big] + \mathcal{G}_n\Big[-e^{-at}\Big]\Big)\\ &= \tfrac{1}{2}\bigg(\tfrac{s}{(s-a)^n}\Gamma(n) + \tfrac{s}{(s+a)^n}\Gamma(n)\bigg)\\ &= \tfrac{s}{2}\Gamma(n)\bigg(\tfrac{1}{(s-a)^n} + \tfrac{1}{(s+a)^n}\bigg)\\ &= \tfrac{s}{2}\Gamma(n)\tfrac{1}{s^n}\Bigg(\tfrac{1}{\big(1-\frac{a}{s}\big)^n} + \tfrac{1}{\big(1+\frac{a}{s}\big)^n}\Bigg)\end{aligned}$$

Example 4.10:

$$\mathcal{G}_n\Big[u_3(t)e^{t-3}\Big](s) = e^{-3s}\mathcal{G}_1\Big[g(v)(v+3)^{n-1}\Big].$$

For $n = 1$

$$\mathcal{G}_1\Big[u_3(t)e^{t-3}\Big](s) = e^{-3s}\mathcal{G}_1\Big[e^v(v+3)^{1-1}\Big] = \frac{s}{s-1}\,e^{-3s}.$$

For $n = 2$

$$\mathcal{G}_2\Big[u_3(t)e^{t-3}\Big](s) = e^{-3s}\mathcal{G}_1\Big[e^v(v+3)^{2-1}\Big] = e^{-3s}\mathcal{G}_1[v\,e^v + 3\,e^v]$$
$$= s\,e^{-3s}\Big(\tfrac{1}{(s-1)^2} + \tfrac{3}{s-1}\Big).$$

We present a list of ARA transform of some special functions and General properties of the ARA transform in Table A1 (Appendix A).

5. Applications of the ARA Transform

In this section, we give some applications of ordinary differential equations, in which the efficiency and high accuracy of ARA transform are illustrated.

Example 5.1:

Consider the initial value problem:

$$y'(t) + y(t) = 0, \quad y(0) = 1. \tag{18}$$

Solution:

Applying ARA transform $\mathcal{G}_1$ on both side of Equation (18)

$$\mathcal{G}_1[y'(t)](s) + \mathcal{G}_1[y(t)](s) = 0$$

$$s\mathcal{G}_1[y(t)](s) - sy(0) + \mathcal{G}_1[y(t)](s) = 0$$

$$\mathcal{G}_1[y(t)](s) = \frac{s}{s+1}.$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ we get:

$$y(t) = e^{-t}.$$

Example 5.2:

Consider the initial value problem:

$$y'(t) - y(t) = e^{2t}, \quad y(0) = 1. \tag{19}$$

Solution:

Applying ARA transform $\mathcal{G}_1$ on both side of Equation (19)

$$\mathcal{G}_1[y'(t)](s) - \mathcal{G}_1[y(t)](s) = \mathcal{G}_1\Big[e^{2t}\Big](s)$$

$$s\mathcal{G}_1[y(t)](s) - sy(0) - \mathcal{G}_1[y(t)](s) = \frac{s}{s-2}$$

$$\mathcal{G}_1[y(t)](s) = \frac{1}{s-1}\Big(\frac{s}{s-2} + s\Big)$$

$$\mathcal{G}_1[y(t)](s) = \frac{s}{(s-2)(s-1)} + \frac{s}{s-1} = \frac{s}{s-2}.$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ we get:

$$y(t) = e^{2t}.$$

Example 5.3:

Consider the initial value problem:

$$y''(t) + y(t) = 0, \quad y(0) = 1, \; y'(0) = 1. \tag{20}$$

Solution:
First, we solve the initial value problem (20) applying $\mathcal{G}_1$:

$$\mathcal{G}_1[y''(t)](s)+\mathcal{G}_1[y(t)](s)=0$$

$$s^2\mathcal{G}_1[y(t)](s)-s^2y(0)-sy'(0)+\mathcal{G}_1[y(t)](s)=0$$
$$s^2\mathcal{G}_1[y(t)](s)-s^2-s+\mathcal{G}_1[y(t)](s)=0$$
$$\left(s^2+1\right)\mathcal{G}_1[y(t)](s)=s^2+s$$
$$\mathcal{G}_1[y(t)](s)=\frac{s^2}{s^2+1}+\frac{s}{s^2+1}.$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$, we get

$$y(t)=\cos(t)+\sin(t).$$

Now, we solve the initial value problem (20) applying $\mathcal{G}_2$:

$$\mathcal{G}_2[y''(t)]+\mathcal{G}_2[y(t)]=0$$

$$sy(0)-s^2\mathcal{G}_1'[y(t)](s)-s\mathcal{G}_1[y(t)](s)+\frac{\mathcal{G}_1[y(t)](s)}{s}-\mathcal{G}_1'[y(t)](s)=0$$
$$s^2-s^3\mathcal{G}_1'[y(t)](s)-s^2\mathcal{G}_1[y(t)](s)+\mathcal{G}_1[y(t)](s)-s\mathcal{G}_1'[y(t)](s)=0$$
$$-\left(s^3+s\right)\mathcal{G}_1'[y(t)](s)-\left(s^2-1\right)\mathcal{G}_1[y(t)](s)=-s^2$$
$$\mathcal{G}_1'[y(t)](s)+\frac{s^2-1}{s^3+s}\mathcal{G}_1[y(t)](s)=\frac{s^2}{s^3+s}$$
$$\frac{d}{ds}\left(\mathcal{G}_1[y(t)](s)\frac{s^2+1}{s}\right)=\frac{s^2}{s^3+s}\frac{s^2+1}{s}=1$$
$$\mathcal{G}_1[y(t)](s)=\frac{s(s+c)}{s^2+1}.$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ and using the second initial condition $y'(0)=1$, we get:

$$y(t)=\cos t+\sin t.$$

Example 5.4:

$$y''-y'+6y=8e^{2t},\ y(0)=0,\ y'(0)=-3. \tag{21}$$

Solution:
Applying ARA transform $\mathcal{G}_2$ on both side of Equation (21)

$$\mathcal{G}_2[y''(t)]-\mathcal{G}_2[y'(t)]+6\mathcal{G}_2[y(t)]=8\mathcal{G}_2\left[e^{2t}\right]$$

$$\begin{gathered}sy(0)-s^2\mathcal{G}_1'[y(t)](s)-s\mathcal{G}_1[y(t)](s)+s\mathcal{G}_1'[y(t)](s)\\ +6\left(\tfrac{\mathcal{G}_1[y(t)](s)}{s}-\mathcal{G}_1'[y(t)](s)\right)=\tfrac{8s}{(s-2)^2}\end{gathered}$$

$$\begin{gathered}-s^3\mathcal{G}_1'[y(t)](s)-s^2\mathcal{G}_1[y(t)](s)+s^2\mathcal{G}_1'[y(t)](s)+6\mathcal{G}_1[y(t)](s)\\ -6s\mathcal{G}_1'[y(t)](s)=\tfrac{8s^2}{(s-2)^2}\end{gathered}$$

$$-\left(s^3-s^2+6s\right)\mathcal{G}_1'[y(t)](s)-\left(s^2-6\right)\mathcal{G}_1[y(t)](s)=\frac{8s^2}{(s-2)^2}$$

$$\mathcal{G}_1'[y(t)](s)+\frac{s^2-6}{s^3-s^2+6s}\mathcal{G}_1[y(t)](s)=\frac{-8s^2}{(s-2)^2(s^3-s^2+6s)}$$

$$\frac{d}{ds}\left(\mathcal{G}_1[y(t)](s)\frac{s^2-s+6}{s}\right)=\frac{-8}{(s-2)^2}$$

$$\begin{aligned}\mathcal{G}_1[y(t)](s)&=\left(8(s-2)^{-1}+c\right)\left(\tfrac{s}{s^2-s+6}\right)\\&=\tfrac{8}{(s-2)(s^2-s+6)}+\tfrac{cs}{s^2-s+6}\\&=\tfrac{1}{s-2}-\tfrac{s+1}{s^2-s+6}+\tfrac{cs}{s^2-s+6}.\end{aligned}$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ and using second initial condition $y'(0)=-3$, we get:

$$y(t)=e^{2t}-e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{23}}{2}t\right)-\frac{9}{\sqrt{23}}e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{23}}{2}t\right).$$

Example 5.5:

$$y''+2y'+5y=e^{-t}\sin t,\ y(0)=0,\ y'(0)=1. \tag{22}$$

Solution:
Applying ARA transform $\mathcal{G}_2$ on both side of the differential Equation (22):

$$\mathcal{G}_2[y''(t)]+2\mathcal{G}_2[y'(t)]+5\mathcal{G}_2[y(t)]=\mathcal{G}_2\left[e^{-t}\sin t\right]$$

$$\begin{gathered}sy(0)-s^2\mathcal{G}_1'[y(t)](s)-s\mathcal{G}_1[y(t)](s)+2\left(-s\mathcal{G}_1'[y(t)](s)\right)\\+5\left(\tfrac{\mathcal{G}_1[y(t)](s)}{s}-\mathcal{G}_1'[y(t)](s)\right)\\=\left(1+\tfrac{1}{(-1-s)^2}\right)^{-2/2}s(1\\+s)^{-2}\Gamma(2)\sin\left(2\tan^{-1}\left(\tfrac{1}{1+s}\right)\right),\end{gathered}$$

$$\begin{gathered}-s^3\mathcal{G}_1'[y(t)](s)-s^2\mathcal{G}_1[y(t)](s)-2s^2\mathcal{G}_1'[y(t)](s)+5\mathcal{G}_1[y(t)](s)\\-5s\mathcal{G}_1'[y(t)](s)\\=s\left(\tfrac{(s+1)^2+1}{(s+1)^2}\right)^{-1}\tfrac{s}{(1+s)^2}2\sin\left(\tan^{-1}\left(\tfrac{1}{1+s}\right)\right)\cos\left(\tan^{-1}\left(\tfrac{1}{1+s}\right)\right)\\=\tfrac{2s^2}{1+(s+1)^2}\ \tfrac{s+1}{1+(s+1)^2}=\tfrac{2s^2(s+1)}{\left(1+(s+1)^2\right)^2}\end{gathered}$$

$$-\left(s^3+2s^2+5s\right)\mathcal{G}_1'[y(t)](s)-\left(s^2-5\right)\mathcal{G}_1[y(t)](s)=\frac{2s^2(s+1)}{\left(1+(s+1)^2\right)^2}$$

$$\begin{gathered}\mathcal{G}_1'[y(t)](s)+\tfrac{s^2-5}{s^3+2s^2+5s}\mathcal{G}_1[y(t)](s)\\=\tfrac{-2s^2(s+1)}{(s^3+2s^2+5s)\left(1+(s+1)^2\right)^2}\end{gathered}$$

$$\begin{gathered}\tfrac{d}{ds}\left(\mathcal{G}_1[y(t)](s)\tfrac{s^2+2s+5}{s}\right)\\=\tfrac{-2s^2(s+1)}{(s^3+2s^2+5s)\left(1+(s+1)^2\right)^2}\ \tfrac{s^2+2s+5}{s}\end{gathered}$$

$$\mathcal{G}_1[y(t)](s)=\frac{1}{1+(s+1)^2}\ \frac{s}{s^2+2s+5}=s\frac{1}{1+(s+1)^2}\ \frac{1}{4+(s+1)^2}.$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ and using the second initial condition $y'(0)=1$ we get:

$$y(t)=\frac{1}{3}e^{-t}\sin t+\frac{2}{3}e^{-t}\sin 2t.$$

Example 5.6:
Consider the initial boundary value problem

$$u_t = u_{xx} \tag{23}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(2\pi x).$$

Solution:
Applying ARA transform $\mathcal{G}_1$ on both side of the Equation (23)

$$\mathcal{G}_1[u_t] = \mathcal{G}_1[u_{xx}]$$

$$s\mathcal{G}_1[u(x,t)](s) - su(x,0) = \frac{d^2}{dx^2}(\mathcal{G}_1[u(x,t)](s))$$

$$\mathcal{G}_1''[u(x,t)](s) - s\mathcal{G}_1[u(x,t)](s) = -s\sin(2\pi x). \tag{24}$$

The general solution of Equation (24) can be written as

$$\mathcal{G}_1[u(x,t)](s) = \mathcal{G}_{1c}[u(x,t)](s) + \mathcal{G}_{1p}[u(x,t)](s),$$

where $\mathcal{G}_{1c}[u(x,t)](s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$, and the solution of the nonhomogeneous part is given by

$$\mathcal{G}_{1p}[u(x,t)](s) = \alpha_1 \sin(2\pi x) + \alpha_2 \cos(2\pi x),$$

after simple calculations, we get $\alpha_2 = 0$.
Hence:

$$\mathcal{G}_1[u(x,t)](s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{s}{4\pi^2 + s}\sin(2\pi x).$$

Using boundary conditions, we get $c_1 = c_2 = 0$:

$$\mathcal{G}_1[u(x,t)](s) = \frac{s}{4\pi^2 + s}\sin(2\pi x).$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$:

$$u(x,t) = e^{-4\pi^2 t}\sin(2\pi x).$$

Example 5.7
Consider the initial boundary value problem:

$$u_{tt} = u_{xx} + \sin(\pi x), \tag{25}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = u_t(x,0) = 0.$$

Solution:
Applying ARA transform $\mathcal{G}_1$ on both sides of the Equation (25)

$$\mathcal{G}_1[u_{tt}] = \mathcal{G}_1[u_{xx}] + \mathcal{G}_1[\sin(\pi x)]$$

$$s^2\mathcal{G}_1[u(x,t)](s) - s^2u(x,0) - su_t(x,0)$$
$$= \tfrac{d^2}{dx^2}(\mathcal{G}_1[u(x,t)](s)) + \sin(\pi x)$$

$$\frac{d^2}{dx^2}(\mathcal{G}_1[u(x,t)](s)) - s^2\mathcal{G}_1[u(x,t)](s) = -\sin(\pi x). \tag{26}$$

The general solution of Equation (26) can be written as

$$\mathcal{G}_1[u(x,t)](s) = \mathcal{G}_{1c}[u(x,t)](s) + \mathcal{G}_{1p}[u(x,t)](s),$$

where $\mathcal{G}_{1c}[u(x,t)](s) = c_1 e^{sx} + c_2 e^{-sx}$, and the solution of the nonhomogeneous part is given by:

$$\mathcal{G}_{1p}[u(x,t)](s) = A\sin(\pi x) + B\cos(\pi x).$$

To find A and B, we substituting $\mathcal{G}_{1p}$ in Equation (26) to get:

$$\mathcal{G}_{1p}[u(x,t)](s) = \frac{1}{\pi^2 + s^2}\sin(\pi x)$$

$$\mathcal{G}_1[u(x,t)](s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{1}{\pi^2 + s^2}\sin(\pi x).$$

Using boundary conditions, we get:

$$\mathcal{G}_1[u(x,t)](s) = \frac{1}{\pi^2 + s^2}\sin(\pi x).$$

Taking the inverse ARA transform $\mathcal{G}_1^{-1}$ we get the solution:

$$u(x,t) = \frac{\sin(\pi x)}{\pi^2}(1 - \cos(\pi t)).$$

6. Conclusions

In this paper, we introduced a new integral operator transform called the ARA transform. We presented its existence and inverse transform. We presented some properties and their application in the solving of ordinary and partial differential equations that arise in some branches of science like physics, engineering, etc.

Author Contributions: Data curation, R.S., A.Q. and A.B.; Formal analysis, R.S., A.Q. and A.B.; Investigation, R.S., A.Q. and A.B.; Methodology, R.S. and A.Q.; Project administration, A.Q.; Resources, R.S.; Writing—original draft, A.Q.; Writing—review & editing, R.S. and A.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors express their gratitude to the dear unknown referees and the editor for their helpful suggestions, which improved the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. Here we present a list of ARA transform of some special functions and General properties of the ARA transform.

$g(t)$	$G(n,s)$
1	$s^{1-n}\Gamma(n)$
t	$s^{-n}\Gamma(1+n)$
$\sqrt{t}$	$s^{\frac{1}{2}-n}\Gamma\left(\frac{1}{2}+n\right)$
$\sqrt{\frac{\pi}{t}}$	$\sqrt{\pi}s^{\frac{3}{2}-n}\Gamma\left(-\frac{1}{2}+n\right)$

Table A1. *Cont.*

$g(t)$	$G(n,s)$
t^2	$s^{-1-n}\Gamma(2+n)$
t^m	$s^{1-m-n}\Gamma(m+n)$
$t^{m-1/2}$	$s^{\frac{3}{2}-m-n}\Gamma\left(-\frac{1}{2}+m+n\right)$
$e^{\alpha t}$	$s(s-\alpha)^{-n}\Gamma(n)$
$e^{-\alpha t}$	$s(s+\alpha)^{-n}\Gamma(n)$
$te^{\alpha t}$	$s(s-\alpha)^{-1-n}\Gamma(1+n)$
$t^m e^{\alpha t}$	$s(s-\alpha)^{-m-n}\Gamma(m+n)$
$\sin(at)$	$\frac{s}{2i}\Gamma(n)\left(\frac{1}{(s-ia)^n}-\frac{1}{(s+ia)^n}\right)=\left(1+\frac{a^2}{s^2}\right)^{-\frac{n}{2}}s^{1-n}\Gamma(n)\sin\left(n\tan^{-1}\left(\frac{a}{s}\right)\right)$
$\cos(at)$	$\frac{s}{2i}\Gamma(n)\left(\frac{1}{(s-ia)^n}+\frac{1}{(s+ia)^n}\right)=\left(1+\frac{a^2}{s^2}\right)^{-\frac{n}{2}}s^{1-n}\Gamma(n)\cos\left(n\tan^{-1}\left(\frac{a}{s}\right)\right)$
$t\sin(at)$	$\frac{\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n+2)\sin\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)}{1+n}$
$t\cos(at)$	$\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n+1)\cos\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)$
$\sin(at)-a\,t\cos(at)$	$\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n)\left(-a\,n\cos\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)+\sqrt{1+\frac{a^2}{s^2}}s\,\sin\left(n\tan^{-1}\left(\frac{a}{s}\right)\right)\right)$
$\sin(at)+a\,t\cos(at)$	$\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n)\left(a\,n\cos\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)+\sqrt{1+\frac{a^2}{s^2}}s\,\sin\left(n\tan^{-1}\left(\frac{a}{s}\right)\right)\right)$
$\cos(at)-a\,t\sin(at)$	$\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n)\left(s\cos\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)-a(-1+n)\,\sin\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)\right)$
$\cos(at)+a\,t\sin(at)$	$\left(1+\frac{a^2}{s^2}\right)^{\frac{1}{2}(-1-n)}s^{-n}\Gamma(n)\left(s\,\cos\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)+a(1+n)\sin\left((1+n)\tan^{-1}\left(\frac{a}{s}\right)\right)\right)$
$\sin(at+b)$	$\left(1+\frac{a^2}{s^2}\right)^{-n/2}s^{1-n}\,\Gamma(n)\sin\left(b+n\,\tan^{-1}\left(\frac{a}{s}\right)\right)$
$\cos(at+b)$	$\left(1+\frac{a^2}{s^2}\right)^{-n/2}s^{1-n}\,\Gamma(n)\cos\left(b+n\,\tan^{-1}\left(\frac{a}{s}\right)\right)$
$e^{at}\sin(bt)$	$\left(1+\frac{b^2}{(a-s)^2}\right)^{-n/2}s(-a+s)^{-n}\Gamma(n)\sin\left(n\tan^{-1}\left(\frac{b}{-a+s}\right)\right)$
$e^{at}\cos(bt)$	$\left(1+\frac{b^2}{(a-s)^2}\right)^{-n/2}s(-a+s)^{-n}\Gamma(n)\cos\left(n\tan^{-1}\left(\frac{b}{a-s}\right)\right)$
$\sinh(at)$	$\frac{1}{2}s\left(-\frac{a^2}{s}+s\right)^{-n}\Gamma(n)\left(-\left(1-\frac{a}{s}\right)^n+\left(\frac{a+s}{s}\right)^n\right)$
$\cosh(at)$	$\frac{1}{2}s\left(-a^2+s^2\right)^{-n}\Gamma(n)\left((s-\lvert a\rvert)^n+(s+\lvert a\rvert)^n\right)$
$e^{at}\sinh(at)$	$\frac{1}{2}\left(1-\frac{b^2}{(a-s)^2}\right)^{-n}s(-a+s)^{-n}\Gamma(n)\left(-\left(1+\frac{b}{a-s}\right)^n+\left(1+\frac{b}{-a+s}\right)^n\right)$
$e^{at}\cosh(at)$	$\frac{1}{2}s\left(-a^2+s^2\right)^{-n}\Gamma(n)\left(\left((1+\frac{\sqrt{b^2}}{a-s}\right)^{-n}+\left(1+\frac{\sqrt{b^2}}{-a+s}\right)^{-n}\right)$
$\mathcal{G}_2[y(t)](s)$	$-s\frac{d}{ds}\left(\frac{\mathcal{G}_1[y(t)](s)}{s}\right)$
$\mathcal{G}_3[y(t)](s)$	$s\frac{d^2}{ds^2}\left(\frac{\mathcal{G}_1[y(t)](s)}{s}\right)$
$\mathcal{G}_n[y(t)](s)$	$s(-1)^{n-1}\frac{d^{n-1}}{ds^{n-1}}\left(\frac{\mathcal{G}_1[y(t)](s)}{s}\right)$
$\mathcal{G}_1[t\,y(t)](s)$	$-s\frac{d}{ds}\left(\frac{\mathcal{G}_1[y(t)](s)}{s}\right)$
$\mathcal{G}_1[y'(t)](s)$	$s\mathcal{G}_1[y](s)-sy(0)$
$\mathcal{G}_1[y''](s)$	$s^2\mathcal{G}_1[y(t)](s)-s^2y(0)-sy'(0)$
$\mathcal{G}_2[y(t)](s)$	$\frac{\mathcal{G}_1[y(t)](s)}{s}-\mathcal{G}_1'[y(t)](s)$
$\mathcal{G}_2[y'(t)](s)$	$-s\frac{d}{ds}(\mathcal{G}_1[y(t)](s))$
$\mathcal{G}_2[y''(t)](s)$	$sy(0)-s^2\frac{d}{ds}(\mathcal{G}_1[y(t)](s))-s\mathcal{G}_1[y(t)](s)$
$\mathcal{G}_n[\alpha u(t)+\beta v(t)](s)$	$\alpha\mathcal{G}_n[u(t)](s)+\beta\mathcal{G}_n[v(t)](s)$
$\mathcal{G}_n[g(at)](s)$	$\frac{1}{a^{n-1}}G\left(n,\frac{s}{a}\right)$
$\mathcal{G}_n\left[e^{-ct}g(t)\right](s)$	$\frac{s}{s+c}G(n,s+c)$
$\mathcal{G}_n[t^m g(t)](s)$	$G(n+m,s)$
$\mathcal{G}_n\left[\frac{g(t)}{t^m}\right]$	$G(n-m,s)$
$\mathcal{G}_n\left[f^{(m)}(t)\right](s)$	$(-1)^{n-1}s\frac{d^{n-1}}{ds^{n-1}}\left(\frac{\mathcal{G}_1\left[f^{(m)}(t)\right](s)}{s}\right)$
$\mathcal{G}_n\left[f^{(n)}(t)\right](s)$	$(-1)^{n-1}s\frac{d^{n-1}}{ds^{n-1}}\left(s^{n-1}\mathcal{G}_1[f(t)](s)-\sum_{j=1}^{n}s^{n-j}f^{(j-1)}(0)\right)$
$\mathcal{G}_n[f(t)*h(t)](s)$	$(-1)^{n-1}s\sum_{j=0}^{n-1}c_j^{n-1}F^{(j)}(s)\cdot H^{(n-1-j)}(s)$
$\mathcal{G}_n[u_c(t)g(t-c)](s)$	$e^{-cs}\mathcal{G}_1\left[g(u)(u+c)^{n-1}\right]$

References

1. Debnath, L.; Bhatta, D. *Integral Transforms and Their Applications*, 2nd ed.; C.R.C. Press: London, UK, 2007.
2. Widder, D.V. *The Laplace Transform*; Princeton University Press: London, UK, 1946.

3. Spiegel, M.R. *Theory and Problems of Laplace Transforms*; Schaums Outline Series; McGraw-Hill: New York, NY, USA, 1965.
4. Bochner, S.; Chandrasekharan, K. *Fourier Transforms*; Princeton University Press: London, UK, 1949.
5. Watugula, G.K. Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *Int. J. Math. Edu. Sci. Technol.* **1993**, *24*, 35–43. [CrossRef]
6. Khan, Z.H.; Khan, W.A. Natural transform-properties and applications. *NUST J. Eng. Sci.* **2008**, *1*, 127–133.
7. Elzaki, T.M. The new integral transform "Elzaki transform". *Glob. J. Pure Appl. Math.* **2011**, *7*, 57–64.
8. Atangana, A.; Kiliçman, A. A novel integral operator transform and its application to some FODE and FPDE with some kind of singularities. *Math. Probl. Eng.* **2013**. [CrossRef]
9. Srivastava, H.; Luo, M.; Raina, R. A new integral transform and its applications. *Acta Math. Sci.* **2015**, *35B*, 1386–1400. [CrossRef]
10. Zafar, Z. ZZ transform method. *Int. J. Adv. Eng. Glob. Technol.* **2016**, *4*, 1605–1611.
11. Ramadan, M.; Raslan, K.; El-Danaf, T.; Hadhoud, A. A new general integral transform: Some properties and remarks. *J. Math. Comput. Sci.* **2016**, *6*, 103–109.
12. Barnes, B. Polynomial integral transform for solving differential Equations. *Eur. J. Pure Appl. Math.* **2016**, *9*, 140–151.
13. Yang, X.J. A new integral transform with an application in heat transf. *Therm. Sci.* **2016**, *20*, 677–681. [CrossRef]
14. Aboodh, K.; Abdullahi, I.; Nuruddeen, R. On the Aboodh transform connections with some famous integral transforms. *Int. J. Eng. Inform. Syst.* **2017**, *1*, 143–151.
15. Rangaig, N.; Minor, N.; Penonal, G.; Filipinas, J.; Convicto, V. On Another Type of Transform Called Rangaig Transform. *Int. J. Partial. Diff. Equ. Appl.* **2017**, *5*, 42–48. [CrossRef]
16. Maitama, S.; Zhao, W. New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. *Int. J. Anal. Appl.* **2019**, *17*, 167–190.
17. Aggarwal, S.; Chaudhary, R. A comparative study of Mohand and Laplace transforms. *J. Emerg. Technol. Innov. Res.* **2019**, *6*, 230–240.
18. Aggarwal, S.; Gupta, A.R.; Sharma, S.D.; Chauhan, R.; Sharma, N. Mahgoub transform (Laplace-Carson transform) of error function. *Int. J. Latest Technol. Eng. Manag. Appl. Sci.* **2019**, *8*, 92–98.
19. Aggarwal, S.; Singh, G.P. Aboodh transform of error function. *Univers. Rev.* **2019**, *10*, 137–150.
20. Aggarwal, S.; Gupta, A.R.; Kumar, D. Mohand transform of error function. *Int. J. Res. Advent Technol.* **2019**, *7*, 224–231. [CrossRef]