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# Solvability and Bifurcation of Solutions of Nonlinear Equations with Fredholm Operator

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**Abstract:** The necessary and sufficient conditions of existence of the nonlinear operator equations' branches of solutions in the neighbourhood of branching points are derived. The approach is based on the reduction of the nonlinear operator equations to finite-dimensional problems. Methods of nonlinear functional analysis, integral equations, spectral theory based on index of Kronecker-Poincaré, Morse-Conley index, power geometry and other methods are employed. Proposed methodology enables justification of the theorems on existence of bifurcation points and bifurcation sets in the nonstandard models. Formulated theorems are constructive. For a certain smoothness of the nonlinear operator, the asymptotic behaviour of the solutions is analysed in the neighbourhood of the branch points and uniformly converging iterative schemes with a choice of the uniformization parameter enables the comprehensive analysis of the problems details. General theorems and effectiveness of the proposed methods are illustrated on the nonlinear integral equations.

**Keywords:** branch points; bifurcation points; Fredholm operator; uniformization; asymptotics; iterations; regularization

## 1. Introduction

The progress in the methods of analysis development and the new nonlinear problems of applied mathematics enable the nonlinear functional analysis novel concepts formulation. In the seminal paper of L.A. Lusternik [1] the main directions of this field paved the anevue for the branching theory of nolinear equations development. The classic works of A.M. Lyapunov, A.I. Nekrasov, J.H. Poincaré, M.A. Krasnoselsky [2], J. Toland and others contributed to this field. In their studies, reductions of the given nonlinear models to the finite-dimensional systems with parameters were used. The finite-dimensional equivalent system is now known as Lyapunov-Schmidt branching system and the corresponding method is known as Lyapunov-Schmidt (LS) method. In the review [3] and monograph of M.M. Vainberg and V.A. Trenogin [4] the basement of the analytical theory of branching solutions in Banach spaces with applications is given. These works contributed to the modern functional analysis development with applications to concrete classes of equations, see [5–19].

Studies of the integral equations in mechanics played the principal role not only in LS method development, but also initiated the way for functional analysis new chapters construction. Studies of integral equations' bifurcation points are necessary in various mathematical models in the various fields of natural sciences. For some parameter values there the loss of stability may occur and another series of solutions branches off from that bifurcation point.

Such problems include both classical problems of the critical load of the rod and the formation, the emergence of new forms of equilibrium of rotating fluids, a branch at a critical wind speed from

the horizontal surface of the waves, and a number of novel challenging bifurcation problems in biochemistry, plasma physics, electrical engineering and many other applied fields. Over the past decade, the branching theory of solutions of nonlinear equations with parameters and its applications have received enormous development and practical applications.

The monographs [14,20] and papers [21–26] review of the recent results and number of applications in this fields are given. [21] deals with operator equation  $Bx - R(x, \lambda) = 0$ , where  $B : D(B) \subset E_1 \rightarrow E_2$  is closed Fredholm operator,  $R(x, \lambda) = R_{01}\lambda + \sum_{i+k \geq 2} R_{ik}(x)\lambda^k$  is analytic in the neighborhood of origin:  $x = 0, \lambda = 0$ . For solution of desired  $x \rightarrow 0$  as  $\lambda \rightarrow 0$  the iterative scheme is proposed. In [22], the sufficient conditions of bifurcation of solutions of boundary-value problem for Vlasov-Maxwell system are obtained. The analytical method of Lyapunov-Schmidt-Trenogon is employed. In [23] the nonlinear operator equation with parameter  $B(\lambda)x + R(x, \lambda) = 0$  is studied. In [24] the  $N$ -step iterative method in the theory of the branching of solutions of nonlinear equations also numerical method is discussed. The explicit and implicit parametrizations is employed in [25] in the construction of branching solutions by iterative methods. The methods are correspondingly used for solution of the Hammerstein and the Volterra integral equations in the irregular case in [26] and in [27].

The special attention has been paid to the theory development in term of the Sobolev-Schwartz theory of distributions [28,29]. Applications of group methods [30] in bifurcation theory are given in [7,13,31,32].

Despite the abundance of literature in the last 20 to 30 years and interesting results focused on the theory of branching solutions, the formulation and proof of the general existence theorems in nonlinear non-standard models with parameters is still an open problem. The problem of approximate methods development in the neighborhood of critical points is still open. The clarity of the methods and results presentation using the elementary methods is also important. The objective of this article to fill the gap between abstract theory development and concrete problems solution.

It is to be noted that only some part of the total set of results in this field we discuss due to the limited size of the article. Applications and many other outstanding results including cosymmetry by Yudovich, projective-iterative techniques, center manifold reduction, global existence theorems, have remained beyond its scope.

The remainder of this paper is structured as follows. Section 2 demonstrates the construction of the main part of the branching Lyapunov-Schmidt equation and its analysis. The existence theorems of bifurcation points and bifurcation manifolds of real solutions are proved. These theorems generalizes the number of well-known theorems on bifurcation points. Examples of solving integral equations with bifurcation points and points of enhanced bifurcation are given. Methods for parameterizing the branches of solutions of nonlinear equations in a neighborhood of branch points are described in Sections 3 and 4. Iterative methods for constructing branches with the choice of a uniformization parameter are provided that ensure uniform convergence of iterative schemes in the neighborhood of the critical parameter values. Regularization and generalizations for intertwined equations as well as illustrative example are discussed in Section 5. Concluding remarks are included in Section 6.

## 2. Existence Theorem of Bifurcation Points and Manifolds of Nonlinear Equations

Let  $X, Y$  are real Banach spaces,  $\Lambda$  is real normed space. We consider the equation

$$Bx = R(x, \lambda), \quad (1)$$

where  $B : D \subset X \rightarrow Y$  is closed Fredholm operator with dense domain  $D$ ,  $\lambda \in \Lambda$ . Nonlinear operator  $R(x, \lambda)$  with values in  $Y$  is defined, continuous, and continuously differentiable in Fréchet sense wrt  $x$  in the neighborhood

$$\Omega = \{x \in X, \lambda \in \Lambda \mid \|x\| < r, \|\lambda\| < \rho\}.$$

We assume that Equation (1) has trivial solution  $x = 0$  for all  $\lambda$  and

$$R(0, \lambda) = 0, R_x(0, 0) = 0.$$

**Definition 1.** Point  $\lambda = 0$  is called bifurcation point of Equation (1) if in any neighborhood of point  $x = 0, \lambda = 0$  exists pair  $(x, \lambda)$  for  $x \neq 0$  which satisfies Equation (1).

**Corollary 1.** If equation  $Bx = R(x, 0)$  has nonisolated trivial solution  $x = 0$  then  $\lambda = 0$  will be bifurcation point for Equation (1).

**Definition 2.** Point  $\lambda = 0$  is called as the strong bifurcation point of Equation (1) if in the arbitrary neighborhood of pair  $x = 0, \lambda = 0$  exists pair  $(x, \lambda)$  such as  $x \neq 0, \lambda \neq 0$  which satisfies Equation (1).

**Corollary 2.** If  $\lambda = 0$  is strong bifurcation point, then  $\lambda = 0$  is bifurcation point.

**Theorem 1.** If order for the point  $\lambda = 0$  to be a bifurcation point, it is necessary that homogenous linear equation  $Bx = 0$  has nontrivial solution.

**Proof.** If equation  $Bx = 0$  has only trivial solution, then Fredholm operator  $B$  has bounded inverse  $B^{-1}$  and Equation (1) can be reduced to equation

$$x = B^{-1}R(x, \lambda) \quad (2)$$

which meets the condition of the contraction mapping principal in the small neighborhood of pair  $x = 0, \lambda = 0$ . Therefore, equation enjoys unique solution in that neighborhood. Because of imposed conditions  $R(0, \lambda) = 0$  Equation (2) has only trivial solution in the small neighborhood of the point  $x = 0, \lambda = 0$ . Theorem is proved.  $\square$

Let us now focus on the sufficient conditions of bifurcation points existence. We introduce the basis  $\{\phi_i\}_1^n$  in subspace  $N(B)$ , basis  $\{\psi_i\}_1^n$  in  $N(B^*)$  and system  $\{\gamma_i\}_1^n \in X^*, \{z_i\}_1^n \in Y$  which are biorthogonal to these bases, i.e.

$$\langle \phi_i, \gamma_k \rangle = \begin{cases} 1, & i = k \\ 0, & i \neq k, \end{cases}$$

$$\langle z_i, \psi_k \rangle = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

Then from the Schmidt – Trenogin Lemma (see Lemma 2.1.1 in [4]) it follows that operator  $\hat{B} = B + \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i$  is continuously invertible. Let  $\Gamma = \hat{B}^{-1}$ . Then  $\Gamma z_i = \phi_i, i = 1, 2, \dots, n$ . Let us introduce the projectors

$$P = \sum_{i=1}^n \langle \cdot, \gamma_i \rangle \phi_i,$$

$$Q = \sum_{i=1}^n \langle \cdot, \psi_i \rangle z_i$$

and direct decompositions  $X = X_n + X_{\infty-n}, Y = Y_n + Y_{\infty-n}$ , where  $X_n = PX = N(B), X_{\infty-n} = (I - P)X, Y_n = QY = \text{span}\{z_1, \dots, z_n\}, Y_{\infty-n} = (I - Q)Y = R(B)$ . Obviously,  $X_{\infty-n} = \{x \in X : \langle x, \gamma_i \rangle = 0, i = 1, \dots, n\}, Y_{\infty-n} = \{y \in Y : \langle y, \psi_i \rangle = 0, i = 1, \dots, n\}$ . Let us rewrite Equation (1) as following system

$$\hat{B}x = R(x, \lambda) + \sum_{s=1}^n \xi_s z_s, \quad (3)$$

$$\xi_s = \langle x, \gamma_s \rangle, s = 1, \dots, n. \quad (4)$$

Equation (3) by multiplication with operator  $\Gamma$  can be reduced to

$$x = \Gamma R(x, \lambda) + \sum_{s=1}^n \zeta_s \phi_s. \tag{5}$$

Using change

$$x = \sum_{s=1}^n \zeta_s \phi_s + u(\zeta, \lambda) \tag{6}$$

Equation (5) can be reduced to the following equation

$$u(\zeta, \lambda) = \Gamma R(\zeta \phi + u(\zeta, \lambda), \lambda). \tag{7}$$

For sake of clarity let us assume  $\sum_{s=1}^n \zeta_s \phi_s = \zeta \phi$ . For arbitrary  $\zeta, \lambda$  from small neighborhood of origin due to contraction mapping principal the sequence  $u_m = \Gamma R(\zeta \phi + u_{m-1}, \lambda)$ ,  $u_0 = 0$  converges to unique solution  $u(\zeta, \lambda)$  of Equation (7). In that case  $u(0, \lambda) = 0$ . Because of

$$\frac{\partial u}{\partial \zeta_i} = \Gamma R_x(\zeta \phi + u, \lambda) \left( \phi_i + \frac{\partial u}{\partial \zeta_i} \right),$$

then, taking into account function  $u$  continuity wrt  $\zeta, \lambda$  and equality  $R_x(0, 0) = 0$  in the small neighborhood of origin, we get the following equality

$$\frac{\partial u}{\partial \zeta_i} = [I - \Gamma R_x(\zeta \phi + u, \lambda)]^{-1} \Gamma R_x(\zeta \phi + u, \lambda) \phi_i, \quad i = 1, \dots, n.$$

This formula can be presented as follows

$$\frac{\partial u}{\partial \zeta_i} = \sum_{n=1}^{\infty} (\Gamma R_x(\zeta \phi + u, \lambda))^n \phi_i$$

as  $\|R_x(\zeta \phi + u, \lambda)\| \leq q < 1$ . Therefore, using the Taylor formula, the desired function  $u(\zeta, \lambda)$  in the problem of bifurcation point search in the neighborhood of point  $\zeta = 0$  can be represented as following series

$$u(\zeta, \lambda) = \sum_{k=1}^{\infty} (\Gamma R_x(0, \lambda))^k \sum_{s=1}^n \zeta_s \phi_s + r(\zeta, \lambda),$$

where  $\|r(\zeta, \lambda)\| = o(\|\zeta\|)$ . Taking into account (4), (6) and (7), the following finite-dimensional branching system of Lyapunov-Schmidt (LS) can be derived

$$L_k(\zeta, \lambda) := \left\langle \sum_{i=1}^{\infty} (\Gamma R_x(0, \lambda))^i \sum_{s=1}^n \zeta_s \phi_s + r(\zeta, \lambda), \gamma_k \right\rangle = 0, \quad k = 1, \dots, n. \tag{8}$$

Taking into account equality  $\Gamma^* \gamma_k = \psi_k$ , branching system (8) can be presented as follows

$$\sum_{s=1}^n \left\langle R_x(0, \lambda) (I - \Gamma R_x(0, \lambda))^{-1} \phi_s, \psi_k \right\rangle \zeta_k + \rho_k(\zeta, \lambda) = 0, \quad k = 1, 2, \dots, n$$

or, briefly, in the matrix form

$$L(\zeta, \lambda) := M(\lambda) \zeta + b(\zeta, \lambda) = 0. \tag{9}$$

Here  $M(\lambda) = [\langle R_x(0, \lambda) (I - \Gamma R_x(0, \lambda))^{-1} \phi_s, \psi_k \rangle]_{s,k=1}^n$ ,  $b(\zeta, \lambda) = (\rho_1(\zeta, \lambda), \dots, \rho_n(\zeta, \lambda))^T$ ,  $\|b\| = o(\|\zeta\|)$ . Let us employ matrix  $M(\lambda)$  to get the sufficient conditions for point  $\lambda = 0$  to be such a

bifurcation. We introduce the set  $\{\lambda \in \Lambda : \det M(\lambda) = 0\}$  which contain the possible bifurcation point  $\lambda = 0$ .

Let us introduce the condition

**Condition 1.** Let in the neighborhood of point  $\lambda = 0$  there exists set  $S$  which is Jordan continuum, and  $S = S_+ \cup S_-$  and  $0 \in \partial S_+ \cap \partial S_-$ . There let exists continuous mapping  $\lambda(t)$  as  $t \in [-1, 1]$  with values in  $S$  such as  $\lambda : [-1, 0) \rightarrow S_-$ ,  $\lambda : (0, 1] \rightarrow S_+$ ,  $\lambda(0) = 0$ . Moreover, let  $\det M(\lambda) = \alpha(t)$ , where  $\alpha(t) : [-1, 1] \rightarrow \mathbb{R}^1$  is continuous function which is zero only for  $t = 0$ .

**Theorem 2.** (Sufficient bifurcation condition) Let condition 1 be fulfilled, where  $\alpha(t)$  is monotone increasing function. Then  $\lambda = 0$  is bifurcation point of Equation (1). Moreover, if trivial solution of equation  $Bx = R(x, 0)$  is isolated, then  $\lambda = 0$  will be such stronger bifurcation.

**Proof.** Let  $\lambda = \lambda(2(\theta - 1)\delta)$  in branching system (21) for arbitrary small  $\delta > 0$  and  $\theta \in [0, 1]$  and consider the continuous vector field

$$H(\xi, \theta) := \mathcal{L}(\xi, \lambda((2\theta - 1)\delta)) : \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

defined for  $\xi, \theta \in M$ , where  $M = \{\xi, \theta \mid \|\xi\| = r, 0 \leq \theta \leq 1\}$ ,  $r > 0$  is as small as possible.

Case 1. Let us assume  $H(\xi^*, \theta^*) = 0$  where  $(\xi^*, \theta^*) \in M$ . Then by Definition 1  $\lambda = 0$  will be bifurcation point of Equation (1).

Case 2. Let us assume  $H(\xi, \theta) \neq 0$  for  $\forall(\xi, \theta) \in M$  and consequently null of the space  $\Lambda$  is not bifurcation point. Then, due to continuity, fields  $H(\xi, 0)$ ,  $H(\xi, 1)$  are homotopic on the sphere  $\|\xi\| = r$ . Then its rotations are equal:

$$J(H(\xi, 0), \|\xi\| = r) = J(H(\xi, 1), \|\xi\| = r). \tag{10}$$

Due to condition 1 and assumption  $H(\xi, \theta) \neq 0$  for  $\forall(\xi, \theta) \in M$  fields  $H(\xi, 0)$ ,  $H(\xi, 1)$  are homotopic to their linear parts  $M(\lambda(-\delta))\xi$ ,  $M(\lambda(+\delta))\xi$ . Then

$$J(H(\xi, 0), \|\xi\| = r) = J(M(\lambda(-\delta))\xi, \|\xi\| = r), \tag{11}$$

$$J(H(\xi, 1), \|\xi\| = r) = J(M(\lambda(+\delta))\xi, \|\xi\| = r). \tag{12}$$

Because  $\det M(\lambda) = \alpha(t)$  then using index Konecker the following equalities are fulfilled

$$J(M(\lambda(-\delta))\xi, \|\xi\| = r) = \text{sign } \alpha(-\delta),$$

$$J(M(\lambda(+\delta))\xi, \|\xi\| = r) = \text{sign } \alpha(+\delta).$$

Due to Condition 1,  $\alpha(-\delta) < 0$ ,  $\alpha(+\delta) > 0$  then equality (10) is not satisfied and therefore there exists point  $(\xi^*, \theta^*) \in M$  for which  $H(\xi^*, \theta^*) = 0$  and point  $\lambda = 0$  is bifurcation point for Equation (1). Theorem 1 is proved.  $\square$

**Corollary 3.** Let condition 1 is fulfilled for  $\forall \lambda \in \Omega_0$ . Then  $\Omega_0$  will be bifurcation set of Equation (1). If in that case  $\Omega_0$  is connected set and each points belongs to the neighborhood of homeomorphic some set of space  $\mathbb{R}^n$  then  $\Omega_0$  will be  $n$ -dimensional manifolds of bifurcation of Equation (1).

From Theorem 1 it follows the known streithern of the known Theorem of M.A. Krasnoselsky on bifurcation point of odd mutiplicity.

**Definition 3.** System of branching equations of Lyapunov-Schmidt we call potential if  $L(\xi, \lambda) = \text{grad}_{\xi} v(\xi, \lambda)$ .

Obviously, (8) is potential system if matrix  $\left[\frac{\partial L_k(\xi, \lambda)}{\partial \xi_i}\right]_{i,k=1}^n$  is symmetric for  $\forall(\xi, \lambda)$  from neighborhood of null. Let us outline that

$$\left[\frac{\partial L_k}{\partial \xi_i}\right]_{i,k=1}^n = \langle R_x(\xi\phi + u(\xi, \lambda), \lambda)[I - \Gamma R_x(\xi\phi + u(\xi, \lambda), \lambda)]^{-1}\phi_i, \psi_k \rangle_{i,k=1}^n.$$

Then we have the following lemma on potentiality of Equation (9)

**Lemma 1.** *The branching Equation (9) is potential if and only if all the matrices*

$$[\langle R_x(\Gamma R_x)^m \phi_i, \psi_k \rangle]_{i,k=1}^n,$$

$m = 0, 1, \dots$  are symmetric in the neighborhood of  $(0, 0)$ .

**Corollary 4.** *Let  $X = Y = H$ ,  $H$  is Hilbert space. Let operator  $B$  is symmetric in  $D$ , and operator  $R_x(x, \lambda)$  is symmetric for  $\forall(x, \lambda)$  from zero neighborhood. Then LS Equation (9) is potential.*

**Proof.** In conditions of the symmetric operators  $B$  and  $R_x$  the equalities  $\phi_i = \psi_i$ ,  $i = 1, \dots, n$  are valid and

$$\langle R_x \phi_i, \phi_k \rangle = \langle \phi_i, R_x \phi_k \rangle.$$

Since  $\Gamma = \Gamma^*$ , then

$$\langle R_x(\Gamma R_x)^m R_x \phi_i, \phi_k \rangle = \langle \phi_i, R_x(\Gamma R_x)^m \phi_k \rangle$$

for  $m = 1, 2, \dots$ . Therefore, for arbitrary  $\xi, \lambda$  from zero neighborhood there following equalities are valid

$$\frac{\partial L_k}{\partial \xi_i} = \frac{\partial L_i}{\partial \xi_k}, \quad i, k = 1, \dots, n$$

and LS is potential in sense of Definition 3.  $\square$

Let us find the corresponding potential  $U(\xi, \lambda)$  of LS system. We introduce the notation

$$a_{sk}(\lambda) = \langle R_x(0, \lambda)(I - \Gamma R_x(0, \lambda))^{-1}\phi_s, \phi_k \rangle, \quad s, k = 1, \dots, n.$$

Then branching system in the potential case is as follows

$$L_k(\xi, \lambda) := \sum_{s=1}^n a_{sk}(\lambda)\xi_s + b_k(\xi, \lambda) = 0, \quad k = 1, \dots, n$$

where  $a_{sk} = a_{ks}$ ,  $\frac{\partial b_k}{\partial \xi_s} = \frac{\partial b_s}{\partial \xi_k}$ ,  $k, s = 1, \dots, n$ . Therefore the desired potential will be function  $U(\xi, \lambda) = \frac{1}{2} \sum_{s,k=1}^n a_{sk}(\lambda)\xi_s\xi_k + w(\xi, \lambda)$  where  $|w(\xi, \lambda)| = \mathcal{O}(\|\xi\|^2)$ .

Let us introduce

**Condition 2.** Let LS Equation (9) is potential and let in the neighborhood of the  $\|\lambda\| < \varepsilon$  there exists space  $S$  containing the point  $\lambda = 0$  and it is Jordan continuum,  $S = S_+ \cup S_-$ ,  $0 \in \partial S_+ \cap \partial S_-$ . Let  $\det[a_{jk}(\lambda)]_{\lambda \in S_+ \cup S_-} \neq 0$  and matrix  $[a_{jk}(\lambda)]$  for  $\lambda \in S_+$  has exactly  $\nu_1$  negative eigenvalues, and for  $\lambda \in S_-$  has exactly  $\nu_2$  negative eigenvalues.

**Lemma 2.** *Let LS Equation (9) is potential, conditions 2 are fulfilled, let  $\nu_1 \neq \nu_2$ . Then for  $\forall \varepsilon > 0$  there exists  $\lambda^*$  in the sphere  $\|\lambda\| < \varepsilon$  such as the potential  $v(\xi, \lambda)$  in the sphere  $\|\xi\| < \varepsilon$  has stationary point  $\xi \neq 0$ .*

**Proof.** (by contradiction) Let  $\text{grad } v(\xi, \lambda) \neq 0$  as  $0 < \|\xi\| < \varepsilon, \|\lambda\| < \varepsilon$ . Then based on homotopical invariance of the generalised index of Morse-Conley [7,8] it is necessary  $v_1 = v_2$  and we observe the contradiction with conditions of Lemma 2.  $\square$

**Remark 1.** In some special cases for  $\lambda \in \mathbb{R}^1$  we have provided the analytical proofs of this lemma using the Rolle theorem, Morse lemma and local coordinates. Using Lemma 2 there following theorem on bifurcation points existence is valid.

**Theorem 3.** Let LS Equation (9) is potential and let condition 2 be fulfilled for  $v_1 \neq v_2$ . Then  $\lambda = 0$  is bifurcation point of Equation (1). If in such a conditions  $x = 0$  is isolated solution of equation  $Bx = R(x, 0)$ , then  $\lambda = 0$  will be the strong bifurcation point of Equation (1).

Proof follows from Lemma 1, Definitions 1 and 2 and from Corollary 1.

**Corollary 5.** Let is condition 2,  $\Lambda = \mathbb{R}^1$ , matrix  $[\alpha_{ik}(\lambda)]$  as  $\lambda \in (0, \varepsilon)$  is positive defined and symmetric matrix, and for  $\lambda \in (-\varepsilon, 0)$  is negative defined and symmetric matrix. Then  $\lambda = 0$  is bifurcation point of Equation (1).

**Example 1.** Let is consider the equation

$$x(t) = \int_a^b K(t, s, x(s), \lambda) ds,$$

where  $K(t, s, x, \lambda) = a(t)a(s)x(s) + \sum_{i=1}^{\infty} \lambda^i K_i(t, s)x(s) + o(\|x\|)$  as  $|\lambda| < \rho, |x| < r$  and all functions are continuous. Let  $\int_a^b a^2(t) dt = 1$ . Assume  $X = Y = C_{[a,b]}$ ,  $\Lambda = \mathbb{R}^1$  and consider this equation as an abstract Equation (1) such as  $Bx = x(t) - \int_a^b a(t)a(s)x(s) ds$ ,  $R_x(0, \lambda)x = \int_a^b \sum_{i=1}^{\infty} \lambda^i K_i(t, s)x(s) ds$ . Let

$$\int_a^b \int_a^b K_i(t, s)a(t)a(s) ds dt = \begin{cases} 0, & \text{if } i = 1, 2, \dots, 2m \\ c \neq 0, & \text{if } i = 2m + 1. \end{cases}$$

Then corresponding LS is as follows  $(c\lambda^{2m+1} + o(\lambda^{2m+2}))\xi + o(\|\xi\|) = 0$ . Therefore, here branching system (9) contains the single equation where function  $M(\lambda) = c\lambda^{2m+1} + o(\lambda^{2m+1})$  changes sign after zero crossing. Therefore, Theorems 2 and 3 conditions are fulfilled and  $\lambda = 0$  is bifurcation point.

Under additional conditions on nonlinear functions in the integral equation the bifurcation point will be the strong bifurcation point and its nontrivial real solutions can be constructed in its half-neighborhood.

Let us consider the equation

$$a(\lambda)x(t) = 3 \int_0^1 ts(x(s) + x^2(s)) ds.$$

1st case:  $a(\lambda_0) = 1, a'(\lambda_0) \neq 0$ .

Using Theorem 2 we can conclude that  $\lambda_0$  is bifurcation point. Moreover, branching equation here is following

$$-a'(\lambda_0)(\lambda - \lambda_0)\xi + \frac{3}{5}\xi^3 + \dots = 0$$

and exists two small real solutions

$$\xi_{\pm} = \pm \sqrt{\frac{5}{3}a'(\lambda_0)(\lambda - \lambda_0) + o((\lambda - \lambda_0)^{1/2})}$$

as  $a'(\lambda_0)(\lambda - \lambda_0) > 0$ . Hence here  $\lambda_0$  is strong bifurcation point. Moreover, integral equation in half-neighborhood of point  $\lambda$  has two real solutions  $x_{\pm}(t) \sim \pm t \sqrt{\frac{5}{3} a'(\lambda_0)(\lambda - \lambda_0)}$ .  
 2nd case:  $a(\lambda_0) = 1, a'(\lambda_0) = 0, a''(\lambda_0) \neq 0$ .

In this case conditions of Theorem 2 are not fulfilled. Moreover, if  $a(\lambda) \neq 1$  as  $\lambda \neq \lambda_0$  then integral equation apart from trivial solution has no other small real solutions. We demonstrated that trivial solution is isolated in this case.

Under conditions of Theorems 2 and 3 equation  $Bx = R(x, \lambda)$  can have nontrivial solution depending on arbitrary small parameters only for  $\lambda = \lambda^*$ , where  $\lambda^*$  is bifurcation point.

**Example 2.** Let us consider the following integral equation

$$(1 + \lambda)x(t) = 3 \int_0^1 tsx(s) ds + a(t) \int_0^1 x^2(s) ds.$$

Let  $\int_0^t ta(t) dt = 0$ . Using Theorems 2 and 3 we can conclude that  $\lambda = 0$  is bifurcation point. All the solutions of this equation can be presented as follows

$$x(t) = \frac{3t}{1 + \lambda} C_1 + a(t) \frac{C_2}{1 + \lambda_1},$$

where  $C_1 = \frac{c_1}{1 + \lambda}, C_2 = \int_0^1 \left( \frac{3s}{1 + \lambda} C_1 + \frac{a(s)}{1 + \lambda} C_2 \right)^2 ds$ .

Then there are two cases:

1st case. Let  $\lambda \neq 0$ . Then  $c_1 = 0, c_2 = \frac{1}{(1 + \lambda)^2} \int_0^1 a^2(s) ds c_2^2$ . If one select  $c_2 = 0$ , then we get the trivial solution  $x(t) = 0$ . If we assume  $c_2 = \frac{(1 + \lambda)^2}{\int_0^1 a^2(s) ds}$  then we get solutions of equations with no small as  $\lambda \rightarrow 0$ . Then  $\lambda = 0$  is unique bifurcation point.

2nd case. Let  $\lambda = 0$ . Then  $x(t) = 3tc_1 + a(t)c_2$ , and  $c_2 = 3c_1^2 + c_2^2 \int_0^1 a^2(s) ds$ . Hence in the second case equation has two c-parametric solutions

$$x_{1,2}(t) = a(t)c \pm 3t \sqrt{c - c^2 \int_0^1 a^2(s) ds}$$

which are real for  $0 \leq c \leq \frac{1}{\int_0^1 a^2(s) ds}$ . Obviously  $x_{1,2} \rightarrow 0$  as  $c \rightarrow 0$ .

Let us consider one more model from mechanics.

**Example 3.** Let us consider the equation

$$F(x, \lambda) := x(t) - 2 \int_0^\pi \sum_{n=1}^\infty \frac{1 + \lambda^{2n}}{n(1 + \lambda^{2n})} \cos nt \cos ns e^{x(s)} ds \left( \int_0^\pi e^{x(s)} ds \right)^{-1} = 0.$$

Operator  $F(x, \lambda)$  is differentiable wrt  $x$  in sense of Fréchet and Theorems 2 and 3 can be applied. Here

$$B(\lambda)x = x(t) - \frac{2}{\pi} \int_0^\pi \sum_{n=1}^\infty \frac{1 - \lambda^{2n}}{n(1 + \lambda^{2n})} \cos nt \cos ns x(s) ds.$$

Operator  $B(\lambda)$  for  $\lambda \neq 0$  has inverse bounded. Then using Theorem 2 only point  $\lambda = 0$  is the only bifurcation point. Equation  $B(0)x = 0$  has nontrivial solution  $\phi(t) = \cos t$ .  $B(\lambda)$  is self-adjoint operator. Branching equation corresponding bifurcation point  $\lambda = 0$  is following

$$L(\xi, \lambda) := \xi \lambda^2 (-2 + r(\xi, \lambda)) = 0,$$

where  $\xi(0, 0) = 0$ . Then for  $\lambda = 0$  parameter  $\xi$  remains arbitrary, and equation  $F(x, 0) = 0$  has nontrivial parametric solution

$$x(t, \xi) = 2 \sum_{n=1}^{\infty} \frac{\cos nt}{n} (\xi/2)^n$$

for  $|\xi| < 2$ .

For construction of parametric solutions in other simple cases it is useful to use the following result.

**Lemma 3.** Let  $B \in \mathcal{L}(X \rightarrow Y)$  is Fredholm operator,  $\{\phi_i\}_1^n$  is basis in  $N(B)$ ,  $\{\psi_i\}_1^n$  is basis in  $N(B^*)$ ,  $R(x)$  is nonlinear operator,  $R(0) = 0$ ,  $R_x(0) = 0$ . Let  $\langle R(x), \psi_i \rangle = 0, i = 1, \dots, n$  for  $\forall x$ . Then equation  $Bx = R(x)$  has  $c$ -parametric small solution

$$x = \sum_{i=1}^n c_i \phi_i + \Gamma u(c)$$

for  $c \rightarrow 0$ . Here  $\Gamma = (B + \sum_1^n \langle \cdot, \gamma_i \rangle z_i)^{-1}$  is bounded operator. Function  $u(t)$  is constructed for small  $|c|$  using method of successive approximations  $u_n = R(\sum_1^n c_i \phi_i + \Gamma u_{n-1}), u(0) = 0, n = 1, 2, \dots$

**Proof.** First of all let us notice that  $B\Gamma u = u$  if  $\langle u, \phi_i \rangle = 0, i = 1, \dots, n$ . Then, taking into account conditions of the Lemma, we get the following equation to find  $u$

$$u = R \left( \sum_{i=1}^n c_i \phi_i + \Gamma u \right).$$

The latter equation for sufficiently small  $|c_i|, i = 1, \dots, n$  using the implicit operator theorem will enjoy unique continuous solution  $u(c) \rightarrow 0$  and this solution can be found using successive approximations  $u_n = R(\sum_{i=1}^n c_i \phi_i + \Gamma u_{n-1}), u_0 = 0. \square$

**Example 4.** Let us consider the equation

$$x(t) - \int_a^b \sum_{i=1}^n a_i(t) a_i(s) x(s) ds = \int_a^b K(t, s) x^2(s) ds,$$

where all the function are continuous,

$$\int_a^b a_i(t) a_j(t) dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Let  $\int_a^b a_i(t) K(t, s) dt \equiv 0, i = 1, \dots, n$ . Then conditions of Lemma 3 are fulfilled for  $\phi_i = \psi_i = a_i(t)$ . Moreover, operator  $\Gamma$  appears to be an identity operator. Then the desired solution can be constructed as following sum  $x = \sum_{i=1}^n c_i a_i(t) + u(t, c_1, \dots, c_n)$ , where function  $u$  defined from equation

$$u(t, c) = \int_a^b K(t, s) \left( \sum_{i=1}^n c_i a_i(s) + u(s, c) \right)^2 ds,$$

using successive approximations for arbitrary  $c_1, \dots, c_n$  from some neighborhood of origin. In order to estimate this neighborhood of convergence, let us employ the method of convex majorants of L. V. Kantorovich [8,19]. Let  $\max_{a \leq t \leq b} \int_a^b |K(t,s)| ds = m$ . We define the majorant system

$$\begin{cases} r = m(\rho + r)^2 \\ 1 = 2m(\rho + r). \end{cases}$$

Then  $r = \rho = 1/4m$ .

Let vector  $c = (c_1, \dots, c_n)^T$  satisfies the estimate

$$\max_{a \leq t \leq b} \sum_{i=1}^n |c_i| |a_i(t)| < \frac{1}{4m}.$$

Then sequence

$$u_m(t, c) = \int_a^b K(t, s) \left( \sum_{i=1}^n c_i a_i(s) + u_{m-1}(s, c) \right)^2 ds, u_0 = 0$$

will converge. The desired  $c$ -parametric solution  $x(t, c)$  satisfies the estimate  $|x(t, c)| < \frac{1}{2m}$ .

### 3. Solutions Parametrization and Iterations in Branch Points Neighborhood

The objective of this section is to describe the iteration scheme with uniformization parameter selection and initial approximations of branches of solution of Equation (1). It is to be noted that in Section 3 condition  $R(0, \lambda) = 0$  can be unsatisfied.

An important role of power geometry [33] and Newton diagram is well known in asymptotic analysis of finite-dimensional systems when implicate theorem's conditions are not fulfilled. Solution of operator Equation (1) reduces to solution of such type finite-dimensional LS system.

The main stages of this approach we describe below. Similar with Section 1 let us consider the equation

$$Bx = R(x, \lambda). \tag{13}$$

But now  $\Lambda = \mathbb{R}^1$ , operator  $R(x, \lambda) = \sum_{i+k \geq 2} R_{ik}(x)\lambda^k + R_{01}\lambda$  is analytic in the neighborhood of origin,  $R_{ik}(tx) = t^i R_{ik}(x)$ ,  $t \in \mathbb{R}^1$ ,  $B$  is Fredholm operator. We have to construct the solution  $x \rightarrow 0$  as  $\lambda \rightarrow 0$ . For iteration scheme construction one needs coefficients of branching LS system. Let us use the change

$$x = \sum_{j=0}^n \xi_j \phi_j + \Gamma y(\xi, \lambda), \tag{14}$$

where

$$\langle y, \psi_i \rangle = 0, i = 1, \dots, n \tag{15}$$

and Equation (13) will be converted to

$$y = R(\xi \phi + \Gamma y, \lambda). \tag{16}$$

Using the Implicit Operator Theorem for small  $\|\xi\|$  and  $|\lambda|$  we have unique small solution

$$y = \sum_{m \geq 2} y_{m0}(\xi \phi) + \sum_{m \geq 0} \sum_{v \geq 1} y_{mv}(\xi \phi) \lambda^v. \tag{17}$$

Since  $\xi\phi = \sum_{i=1}^n \xi_i\phi$  and  $y_{mv}(\xi\phi)$  are  $m$ -homogenous in  $\xi$ , then for coefficient calculation we construct recurrent formulae

$$\begin{aligned}
 y_{20}(\xi\phi) &= R_{20}(\xi\phi), \\
 y_{m0} &= \frac{1}{m!} \frac{d^m}{d\mu^m} \sum_{i \geq 2}^m R_{i0} \left( \Gamma \sum_{i=1}^{m-1} y_{i0} \mu^i \right)_{\mu=0}, \quad m = 3, 4, \dots, \\
 \Gamma y_{10} &:= \xi\phi, \\
 y_{10} &:= \sum_{i=1}^n \xi_i z_i, \quad y_{01} = R_{01}, \\
 y_{0n} &= \frac{1}{n!} \frac{d^n}{d\lambda^n} \sum_{i+k \geq 2}^n R_{ik} \left( \Gamma \sum_{s=1}^{n-1} y_{0s} \lambda^s \right) \lambda^k \Big|_{\lambda=0}, \quad n = 2, 3, \dots, \\
 y_{n-j,j} &= \frac{1}{(n-j)!j!} \frac{\partial^n}{\partial \lambda^j \partial \mu^{n-j}} \sum_{i+k \geq 2}^n \lambda^k R_{ik} \left( \Gamma \sum_{i+k \geq 1}^{k-1} y_{ik} \mu^i \lambda^k + \sum_{s=0}^{j-1} \Gamma y_{n-s,s} \mu^{n-s} \lambda^s \right) \Big|_{\lambda=\mu=0},
 \end{aligned}$$

$n = 2, 3, \dots, j = 1, \dots, n - 1$ . The sequence  $\{y_n\}$ ,  $y_0 = 0$  converges to solution (17), where

$$y_n = R(\xi\phi + \Gamma y_{n-1}, \lambda), \quad n = 1, 2, \dots$$

Substitution of solution (17) into (15) gives the following branching LS system

$$L^j(\xi, \lambda) := \sum_{m \geq 2} L_{m0}^j(\xi) + \sum_{m \geq 0} \sum_{v \geq 1} L_{mv}^j(\xi) \lambda^v = 0, \quad j = 1, \dots, n, \tag{18}$$

where

$$L_{mv}^j = \langle y_{mv}(\xi\phi), \psi_j \rangle = \sum_{m_1 + \dots + m_n = m} L_{m_1, \dots, m_n, v}^j \xi_1^{m_1}, \dots, \xi_n^{m_n}.$$

For symmetry let us put  $\lambda = \xi_{n+1}$  in (18). Let  $L(\xi_1, \dots, \xi_{n+1})$  be one of the left hand sides of system (18). Eliminate on the corresponding power  $\xi_i$  for sake of clarity we assume

$$L(\xi_1, \dots, o_i, \dots, \xi_{n+1}) \neq 0$$

for  $i = 1, \dots, n + 1$ . Let  $\text{supp } L = \{i | i \in \mathbb{N}_+^{n+1}, L_i \neq 0\}$ ,  $\mathbb{N}_+$  is set of positive integer numbers.

Let us introduce

**Condition 3.** We fix positive  $\alpha_1, \dots, \alpha_{n+1}, \theta_1, \dots, \theta_n$  such as for  $\xi_i = \varepsilon^{\alpha_i} v_i, i = 1, \dots, n + 1$  and  $\varepsilon \rightarrow 0$

$$L^j = \varepsilon^{\theta_j} (l_j(v_1, \dots, v_{n+1}) + r_j(v, \varepsilon)), \tag{19}$$

where  $l_j = \sum_{(i,\alpha)=\theta_j} L_i^j v^i, i = (i_1, \dots, i_{n+1}), \alpha = (\alpha_1, \dots, \alpha_{n+1}), v^i = v_1^{i_1}, \dots, v_{n+1}^{i_{n+1}}, r_i(v, 0) = 0$ .

Let us now introduce the following

**Definition 4.** Hyperplane

$$l : \{\xi \in \mathbb{R}_+^{n+1} | (\xi, \alpha) = \theta\}$$

we call support plane for  $\text{supp } L$ , if

1.  $(\xi, \alpha) \geq \theta$  for  $\xi \in \text{supp } L$ ,
2.  $l \cap \text{supp } L \neq \emptyset$ .

Then algebraic Condition 3 from geometrical point of view means that hyperplanes  $l_j = \{\xi \in \mathbb{R}_+^{n+1} \mid (\xi, \alpha) = \theta_j\}$ ,  $j = 1, \dots, n$  are correspondingly support hyperplanes for  $supp L^j$ ,  $j = 1, \dots, m$ . In case of symmetry when  $\alpha_1 = \dots = \alpha_m$ ,  $\alpha_{m+1} = \dots = \alpha_{n+1}$ , the hyperplane  $l_j$  is symmetric wrt axis  $\xi_1, \dots, \xi_m$  and axis  $\xi_{m+1}, \dots, \xi_{n+1}$ . In such case for verification of Condition 1 the Newton digramm can be employed. The method of numbers  $\alpha_i$ ,  $\theta_i$  selection proposed by Bruno gaurantees the satisfaction of Condition 3 in the general case.

**Condition 4.** Let an algebraic system

$$l_j(v_1, \dots, v_{n+1}) = 0, j = 1, \dots, n \tag{20}$$

has solution  $v^0 = (v_1^0, \dots, v_{n+1}^0)$ , and for  $v = v^0$

$$\det \left[ \frac{\partial l_j}{\partial v_i} \right]_{j=1, \dots, n, i=(1, \dots, n+1) \setminus * } \neq 0,$$

where  $(1, \dots, n + 1) \setminus * := (1, \dots, * - 1, * + 1, \dots, n^*)$ .

Solution  $v^0 \neq 0$  we call as full rank solution for system (20). Here index  $*$  fixes rank minor of the matrix  $\left[ \frac{\partial l_j}{\partial v_i} \right]$  for  $v = v^0$ .

**Lemma 4.** Let Conditions 3 and 4 are fulfilled. Then branching system (18) has small solutions as  $\varepsilon \rightarrow 0$

$$\begin{aligned} \xi_i &= \varepsilon^{\alpha_i} (v_i^0 + o(1)), i = (1, \dots, n + 1) \setminus *, \\ \xi_* &= \varepsilon^{\alpha_*} v_*^0, \end{aligned}$$

where  $\xi_{n+1} := \lambda(\varepsilon)$ .

Proof follows from Implicit Function Theorem due to Conditions 3 and 4. Using substitution of determined  $\xi_i(\varepsilon)$ ,  $i = 1, \dots, n$  and  $\lambda(\varepsilon)$  into (17) and taking into account (2) we get the desired pair  $x(\varepsilon)$ ,  $\lambda(\varepsilon)$  satisfies Equation (13). Then the following Theorem takes place

**Theorem 4.** Let Conditions 3 and 4 be fulfilled. Then Equation (13) enjoys small solution  $x = x(\varepsilon) \rightarrow 0$   $\lambda = \lambda(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

System (20) can contain several solutions and choice of vectors  $\alpha, \theta$  is not unique in general case, the Equation (13) can contain several solutions.

Let us consider the basic case on analytical calculation of the solution for Equation (13) using the method of undetermined coefficients. Let nonlinear system Equation (20) is as follows

$$l_j(v_1, \dots, v_{n+1}) : b_j(v_1, \dots, v_{n+1}) \sum_{k=1}^{n+1} a_{jk} v_k = 0, j = 1, \dots, n.$$

Here  $rank[a_{jk}]_{k=1, \dots, n+1, j=1, \dots, n} = n$ . Then branching Equation (18) call quasilinear. In this case it is easy to construct the asymptotic of solution for Equation (13). Indeed, fixing rank minor  $[a_{jk}]_{k=(1, \dots, n+1), j=1, \dots, n}$  we can construct nontrivial solution  $v^0 = (v_1^0, \dots, v_{n+1}^0)$  of the following system of linear algebraic equations

$$\sum_{k=1}^{n+1} a_{jk} v_k = 0, j = 1, \dots, n$$

for selected  $v_*^0$ . Vector  $v^0$  obviously satisfies system (20). Let us assume  $b_j(v_1^0, \dots, v_{n+1}^0) \neq 0$ ,  $j = 1, \dots, n$ . Then, using Implicit Function Theorem, branching system (18) has small solution with

asymptotics  $v_i \sim \varepsilon^{\alpha_i} v_i^0$ ,  $i = 1, \dots, n+1$ . Using formulae (17) and (2), we can conclude that asymptotics of the corresponding small solution of Equation (13) satisfies the following estimates

$$\lambda(\varepsilon) \sim \varepsilon^{\alpha_{n+1}} v_{n+1}^0,$$

$$Px(\varepsilon) \sim \sum_{i=1}^n \varepsilon^{\alpha_i} v_i^0 \phi_i,$$

where  $P := \sum_{i=1}^n \langle \cdot, \gamma_i \rangle \phi_i$  is the projector on subspace  $N(B)$ . Under the certain conditions, analytical solution of Equation (13) can be effectively constructed using the method of undetermined coefficients as series  $x(\lambda) = \sum_{i=1}^{\infty} x_i \lambda^i$ .

Indeed, let in Equation (13)

$$R(x, \lambda) = \sum_{k=1}^{\infty} R_{1k} \lambda^k x + \sum_{i+k=3}^{\infty} R_{ik}(x) \lambda^k + \sum_{k=1}^{\infty} R_{0k} \lambda^k,$$

$$\det[\langle R_{11} \phi_k, \psi_i \rangle]_{k=1, \dots, n} \neq 0,$$

$$\langle R_{01}, \psi_i \rangle = 0, i = 1, \dots, n.$$

Then branching equation will be quasilinear for  $\alpha_1 = \dots = \alpha_{n+1} = 1$ ,  $p = 2$ . Solution of Equation (13) we construct as series  $x = \sum_{i=1}^{\infty} x_i \lambda^i$ . For calculation of the coefficients  $x_i$  using method of undetermined coefficients we obtain the following recurrent sequence of linear equations

$$Bx_1 = R_{01},$$

$$Bx_2 = R_{11}x_1 + R_{02},$$

$$Bx_m = R_{11}x_{m-1} + f_m(x_1, \dots, x_{m-2}), m = 3, 4, \dots$$

Hence,  $x_1 = \sum_{i=1}^n c_{1i} \phi_i + \Gamma R_{01}$ , where  $\Gamma = (B + \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i)^{-1}$ . Vector  $c_1 = (c_{11}, \dots, c_{1n})'$  can be uniquely defined from the following system of linear algebraic equations

$$\sum_{k=1}^n \langle R_{11} \phi_k, \psi_i \rangle c_{1k} + \langle R_{11} \Gamma R_{01}, \psi_i \rangle + \langle R_{02}, \psi_i \rangle = 0, i = 1, \dots, n.$$

which corresponds to the resolving conditions of the 2nd equation of the sequence. Similarly,  $x_m = \sum_{i=1}^n c_{1m} \phi_i + \hat{x}_m$ , where vector  $c_m = (c_{m+1}, \dots, c_{m+n})'$  is defined from the system of linear algebraic equations, element  $\hat{x}_m$  we uniquely construct in the subspace  $X_{\infty-n}$  using operator  $\Gamma$ .

As result, the following statement can be formulated concerning the existence and construction of the analytical solution of Equation (13).

Let  $R(x, \lambda) = \sum_{k=p}^{\infty} R_{1k} \lambda^k x + \sum_{i+k=p+2}^{\infty} R_{ik}(x) \lambda^k + \sum_{k=1}^{\infty} R_{0k} \lambda^k$ ,  $\det[\langle R_{1p} \phi_i, \psi_k \rangle]_{i,k=1, \dots, n} \neq 0$ ,  $\langle R_{0k}, \psi_i \rangle = 0$ ,  $k = 1, \dots, p$ ,  $i = 1, \dots, n$ .

Then Equation (13) enjoys an analytical solution  $x = \sum_{i=1}^{\infty} x_i \lambda^i$ , where  $x_i = \sum_{k=1}^n c_{ik} \phi_k + \hat{x}_i$ ,  $i = 1, 2, \dots$ . Vectors  $c_i = (c_{i1}, \dots, c_{in})'$  can be uniquely calculated from the system of linear algebraic equations with matrix  $[\langle R_{1p} \phi_i, \psi_k \rangle]_{i,k=1, \dots, n}$ , elements  $\hat{x}_i$  can be determined in the subspace  $X_{\infty-n}$  uniquely.

#### 4. N-Step Iteration Scheme for Construction of The Solution of Equation (13)

Let Conditions 3 and 4 are fulfilled for branching system (18) and  $\zeta^0$  is solution of system (20) is full rank solution. Let  $r = \min(\alpha_1, \dots, \alpha_{n+1})$ . Solution  $x(\varepsilon)$ ,  $\lambda(\varepsilon)$  of Equation (13) we seek in the form

$$x(\varepsilon) = \zeta(\varepsilon) \phi + \Gamma \varepsilon^r y(\varepsilon), \quad \lambda(\varepsilon) = \varepsilon^{\alpha_{n+1}} \zeta_{n+1}(\varepsilon), \quad (21)$$

where  $\zeta(\varepsilon)\phi := \sum_{i=1}^n \varepsilon^{\alpha_i} \zeta_i(\varepsilon)$ ,  $\zeta_i(0) = v_i^0$ ,  $i = 1, \dots, n + 1$ .

$$y(0) = \begin{cases} 0, & r < \alpha_{n+1}, \\ R_{01}v_{n+1}^0, & r = \alpha_{n+1}. \end{cases}$$

Lets put in (21)  $\zeta_* = v_*^0$ . The rest  $\zeta_i(\varepsilon)$ ,  $i \neq *$  and  $y(\varepsilon)$  continuous in zero, we define from system

$$\varepsilon^r y = R(\zeta(\varepsilon)\phi + \Gamma \varepsilon^r y, \varepsilon^{\alpha_{n+1}} \zeta_{n+1}(\varepsilon)) := \Phi(\varepsilon^e y, \zeta, \varepsilon), \tag{22}$$

with condition

$$\langle y, \psi_i \rangle = 0, \quad i = 1, \dots, n. \tag{23}$$

Let us outline that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-r} \Phi(\varepsilon^r y, \zeta, \varepsilon) = y(0)$ . Because  $\zeta_*(\varepsilon) \equiv v_*^0$  then in system (22) and (23) functions  $y(\varepsilon)$ ,  $\zeta_i(\varepsilon)$  are unknown for  $i \neq *$ . System (22) and (23) consists of  $(n + 1)$  equations with  $(n + 1)$  unknowns.

Let us transfer system (22) and (23) in order to meet the conditions of Implicite Operator Theorem in the neighborhood of point  $y(0)$ ,  $v_i^0$ ,  $i \neq *$ . Then by introduction of iterative process in system (22) and (23), we get the following system

$$y = \varepsilon^{-r} \underbrace{\Phi(\Phi \dots (\Phi \dots))}_N := \varepsilon^{-r} \Phi^N(\varepsilon^r \Gamma y, \zeta(\varepsilon)), \tag{24}$$

$$\varepsilon^{\theta_j} \langle R(\zeta(\varepsilon)\phi + \Gamma \Phi^N(\varepsilon^r \Gamma y, \nu, \varepsilon), \varepsilon^{\alpha_{n+1}} \nu_{n+1}), \psi_j \rangle = 0, \quad j = 1, \dots, n. \tag{25}$$

Here  $\nu_*(\varepsilon) = v_*^0$ .

$$\Phi^N := R(\zeta(\varepsilon)\phi + \Gamma R(\zeta\phi + \dots + \Gamma R(\zeta\phi + \Gamma \varepsilon^r y, \zeta_{n+1}), \dots, \zeta_{n+1})).$$

Any solution  $y, v_i, i \neq *$  satisfying system (24) and (25) satisfies also (22) and (23). System (24) and (25) we consider as single operator equation

$$K(u, \varepsilon) = 0, \tag{26}$$

where  $K : E \times \mathbb{R}^1 \rightarrow E$ ,  $E = Y \times \mathbb{R}^n$ ,  $\|u\| := \|y\| + |\zeta|$ ,  $u = (y(\varepsilon), \zeta_1(\varepsilon), \dots, \zeta_{*-1}(\varepsilon), \zeta_{n+1}(\varepsilon), \dots, \zeta_{n+1}(\varepsilon))^T$ ,  $\zeta_*(\varepsilon) \equiv v_*^0$ . Using Conditions 3 and 4 for  $N > \max_{1 \leq j \leq n} (\theta_j - r)$  operator  $K$  will be continuous in the neighborhood of the point  $u_0 = (y(0), \zeta^0)$ ,  $\varepsilon = 0$ ,  $K(u_0, 0) = 0$  and exists Fréchet derivative  $K_u(u, \varepsilon)$  continuous in point  $(u_0, 0)$ ,

$$K_u(u_0, 0) = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}.$$

Here matrix  $D = \left[ \frac{\partial l_j(\zeta^0)}{\partial \zeta_i} \right]$  is rank minor of matrix  $\left[ \frac{\partial l_j}{\partial v_i} \right]$  on solution  $v^0$  of algebraic system (20). Due to Condition 4  $\det D \neq 0$  and there exists bounded inverse operator

$$K_u^{-1}(u_0, 0) = \begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix}.$$

Then Equation (26) satisfies all the conditions of Implicite Operator Theorem, Equation (26) enjoys unique continuous solution  $u(\varepsilon)$  such as

$$u_m = u_{m-1} - K^{-1}(u_0, 0)K(u_{m-1}, \varepsilon) \tag{27}$$

converges to this solution in the neighborhood of the point  $\varepsilon = 0$ . Therefore, if conditions 3 and 4 are fulfilled, then Equation (26) enjoys small solution  $x = x(\varepsilon) \rightarrow 0$ ,  $\lambda = \lambda(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , where

$$Px(\varepsilon) = \sum_{i=1}^n \varepsilon^{\alpha_i} (v_i^0 + o(1)) \phi_i,$$

$\lambda(\varepsilon) \sim \varepsilon^{\alpha_{n+1}} v_{n+1}^0$ . Moreover, sequences

$$x_m = \sum_{i=1}^n \varepsilon^{\alpha_i} \zeta_i^m(\varepsilon) + \Gamma \varepsilon^r y_m,$$

$$\lambda_m = \varepsilon^{\alpha_{n+1}} \zeta_{n+1}^m(\varepsilon), \quad m = 1, 2, \dots,$$

where  $y_m$ ,  $\zeta_i^m(\varepsilon)$ ,  $i = 1, \dots, n+1$  are defined iterations (27) converges as  $m \rightarrow \infty$  to desired small solution  $x(\varepsilon) \rightarrow 0$ ,  $\lambda(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Right hand sides of formulae (21) and (22) used in iterations (27) we build an inverse operator  $\Gamma$ . Therefore for element  $x_m$  calculation it is necessary to solve  $N$ -linear equations

$$\left( B + \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i \right) x = f$$

with continuously invertible operator. Therefore, proposed iteration scheme is  $N$ -step on each step.

## 5. Remarks, Regularization and Generalizations

The right hand side of the iteration scheme (27) contains operator  $\Gamma$  introduced by V.A. Trenogin [34] and negative powers of the small parameter  $\varepsilon$ . But this singularity is resolvable. Indeed, in case of polynomial nonlinearity wrt negative powers of  $\varepsilon$  one can eliminate the corresponding powers of parameter  $\varepsilon$ . For more details readers may refer to [8,24,25]. Then, taking into account boundness of operator  $\Gamma$  and its regularizing properties [20,34] convergence of proposed  $N$ -steps method of successive approximations will be uniform in the branch point's neighborhood. If it is not possible to perform explicit eliminations, then for sake of stable computations in case of negative powers of  $\varepsilon$  one can employ the change of  $\varepsilon$  onto  $\varepsilon + \text{sign} \varepsilon \delta^\nu$ , where  $0 < \nu < 1/2p$ ,  $p = \max_{1 \leq j \leq n} (\theta_j - r)$ , where  $\delta$  is maximal absolute error of computations. Then proposed iteration scheme can be classified as Tikhonov-Lavrentiev regularisation algorithm.

Finally, let us outline that in number of applications Condition 4 for branching system is not satisfied. Analysis of corresponding branching solutions depending on free parameters linked with model's symmetry requires methods from [20,25,31]. Usually, in such cases it is assumed the existence of linear bounded operators  $S \in \mathcal{L}(X \rightarrow X)$  and  $K \in \mathcal{L}(Y \rightarrow Y)$  such as  $BS = KB$ ,  $R(SX, \lambda) = KR(x, \lambda)$  for  $\forall x, \lambda \in \Omega$ .

Operators  $S, K$  can be projectors. If problem  $G$ -invariant then  $S, K$  can be parametric representations of  $G$ -group. In that case we say that Equation (13) is  $(S, K)$ -interwined. In [21,31] the iterative approach is implemented and developed using ideas of analytical method of Lyapunov-Schmidt in case of  $(S, K)$ -interwined equations. In this case it is allowed to change the parameter of uniformization of solutions branches.

Because of symmetry (13) with respect to main representatives of rotation group can be employed to transfer to the spheric coordinates and construct the solution depending on free parameters.

**Example 5.** [24] Let us consider the equation

$$x = \frac{1}{\pi} \int_0^{2\pi} \cos(t-s)x(s) ds + \lambda x(t) + x^2. \quad (28)$$

Here  $\phi_1 = \frac{\cos t}{\sqrt{\pi}}$ ,  $\phi_2 = \frac{\sin t}{\sqrt{\pi}}$  and  $\Gamma$  is identity operator. Let us seek the small solution as  $\lambda \rightarrow 0$  in form  $x = \xi_1\phi_1 + \xi_2\phi_2 + y$ , where  $\int_0^{2\pi} y(t)\phi_i(t)dt = 0$ ,  $i = 1, 2$ . In the polar coordinates  $\xi_1 = \rho \cos \alpha$ ,  $\xi_2 = \rho \sin \alpha$ , Branching LS system can be presented as follows

$$\left(\rho \frac{\lambda}{1-\lambda} + \frac{3}{2\pi} \frac{\rho^3}{(1-\lambda)^5} + r(\rho, \lambda)\right) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = 0.$$

Here  $r(\rho, \lambda)$  is an analytical function in the neighborhood of origin, and  $r(\rho, \lambda) = o(\rho^5)$ . Then the desired implicit parametrization of small solution is following

$$x = \rho \frac{\cos(t-\alpha)}{\sqrt{\pi}} + \mathcal{O}(\rho^2),$$

$$\lambda = -\frac{3}{2\pi} \rho^2 + \mathcal{O}(\rho^3),$$

where  $\alpha \in (-\infty, \infty)$  as  $\rho \rightarrow 0$ . After transition to explicit parametrization we can get two  $\alpha$ -parametric small solutions (which are real for  $\lambda < 0$ )

$$x_{\pm} = \pm \sqrt{-\frac{2\lambda}{3}} \cos(t-\alpha) + o(|\lambda|^{1/2}) \tag{29}$$

which are real valued as  $\lambda \rightarrow -0$ . We can see that equation has two  $\alpha$ -parametric branches of small  $2\pi$ -periodic solutions defined for  $\lambda < 0$ . For  $\alpha = 0$

$$x_{\pm} \approx \pm \sqrt{-\frac{2\lambda}{3}} \cos(t). \tag{30}$$

**Remark 2.** The iteration scheme (27) from Section 4 can be employed for the latter Equation (28).

Let us build the solution corresponding the asymptotic expansion (29) for  $\alpha = 0$ . The sequence  $u_n$  is defined as pair

$$u_n := (\varepsilon\varphi + \varepsilon y_n, \lambda_n) \tag{31}$$

which converges to branch  $u_+ = (x_+, \lambda)$  as  $n \rightarrow +\infty$ . Here

$$\lambda_n = -\lambda_{n-1} \langle y_{n-1}, \varphi \rangle - \varepsilon \langle (y_{n-1} + \varphi)^2, \varphi \rangle, \tag{32}$$

$$y_n = \lambda_{n-1} y_{n-1} + \lambda_n \varphi + (y_{n-1} + \varphi)^2 \varepsilon, \tag{33}$$

$y_0 = 0, \lambda_0 = 0$ . For sake of simplicity only positive branch  $u_+$  is discussed here. Since the latter iteration scheme (31)–(33) has no singularity point for  $\varepsilon = 0$  then sequence  $u_n$  converges  $\varepsilon$ -uniformly. The Simpson’s rule for step  $h = \pi/10$ , is employed for method (31)–(33), number of iterations  $N$  is selected as  $\|u_n - u_{n-1}\| < 10^{-5}$ . Table 1 demonstrates calculations results for single branch  $u_+(t) = (x_+(t), \lambda)$  in points  $t = \{0, \pi/2, \pi\}$ .

**Table 1.** Results for Example 5.

$N$	$\varepsilon$	$\lambda$	$x_+(0)$	$x_+(\pi/2)$	$x_+(\pi)$
6	$10^{-3}$	$-0.2197 \times 10^{-6}$	$0.5652 \times 10^{-3}$	$2 \times 10^{-9}$	$-0.5632 \times 10^{-3}$
5	$10^{-2}$	$-0.2194 \times 10^{-6}$	$0.5737 \times 10^{-3}$	$2 \times 10^{-9}$	$-0.5751 \times 10^{-3}$
5	$10^{-1}$	$-2.1331 \times 10^{-3}$	$56.5654 \times 10^{-3}$	0	$-56.6065 \times 10^{-3}$

Following the main term (30) of asymptotic expansion (29) for small  $\lambda$  we have  $x_+(\pi/2) = 0$ . This matches with the calculated values listed in 5th column of the Table 1. From Table 1 we can obviously conclude that iteration process (31)–(33) allows to define the solution  $(x_+(t), \lambda)$  with sufficient accuracy because it matches with asymptotic expansion (29), moreover, it enjoys the *uniform convergence* with respect to  $\varepsilon$ . As footnote, let us outline that proposed method can be used also to find  $\alpha$ -parametric solutions for fixed  $\alpha \neq 0$ .

## 6. Conclusions

In this article we derived the necessary and sufficient conditions (Theorems 1–3) on the parameters for which a nontrivial solutions to the problem appears. Algorithms for constructing solutions are considered in the remaining theorems. Thus, the article gives algorithms for constructing asymptotic solutions and conditions for the convergence of special authorial methods of successive approximations.

The article also includes an overview of the results of the authors, and some of the results presented were only announced or published without proof in previous articles. In order to make the new methods accessible to a wider audience, all theorems are illustrated by solving substantial concrete examples, and an integral equation is presented that simulates one problem from wave theory.

Results of this paper enable applications of the existence theorems for bifurcation points of nonlinear BVP problems and make it possible to construct an appropriate solutions. Our method has been also applied for solution of degenerate operator-differential and integral equations [14,15,18,20,26,32].

Problem of optimal uniformization parameters selection needs to take into account an insight of the problems and it is not yet solved in algebraic form. The formulation and proof of the nonlocal theorems of existence of branching solutions in nonstandart models remains an important problem. For solution of these problems the Trenogin's nonlocal theorems from [5,6] can be employed.

When developing methods of successive approximations and the corresponding numerical schemes, an important problem is to ensure uniform convergence with respect to the bifurcation parameter of convergence in the maximum possible neighborhood of the branch points of solutions. Particularly difficult is the solution to this problem in cases where the branches of solutions depend on free parameters related to the symmetry of the problem. Here, most of the research focuses on numerical experiments in the vicinity of bifurcation points, the calculation results were unstable, which made their interpretation difficult. In this situation, in the vicinity of the branch point, it is effective to solve many substantial problems by applying regularization methods in the sense of Tikhonov using the efficient analytical methods based on ideas of the Lyapunov-Schmidt method [8,9,20].

In our opinion, the proof of theorems on the existence of bifurcation points in modeling biological and biochemical processes has become especially relevant in connection with the growth of infectious diseases in the global world. In order to attack such global challenges it is necessary to involve both advanced machine learning methods and qualitative theory of nonlinear dynamical systems, asymptotics of solutions of kinetic equations in the vicinity of bifurcation points and to study the stability of new branches of solutions. In solving such complex problems in biology and medicine, it will be useful to efficiently employ the accumulated experience of bifurcation analysis in models of hydrodynamics, elasticity theory, and mathematical physics.

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