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Some Relations of Two Type 2 Polynomials and Discrete Harmonic Numbers and Polynomials

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Received: 25 April 2020; Accepted: 18 May 2020; Published: 1 June 2020

Abstract: Harmonic numbers appear, for example, in many expressions involving Riemann zeta functions. Here, among other things, we introduce and study discrete versions of those numbers, namely the discrete harmonic numbers. The aim of this paper is twofold. The first is to find several relations between the Type 2 higher-order degenerate Euler polynomials and the Type 2 high-order Changhee polynomials in connection with the degenerate Stirling numbers of both kinds and Jindalrae–Stirling numbers of both kinds. The second is to define the discrete harmonic numbers and some related polynomials and numbers, and to derive their explicit expressions and an identity.

Keywords: degenerate Stirling number of the first kind; degenerate Stirling number of the second kind; Jindalrae–Stirling number of the first kind; Jindalrae–Stirling number of the second kind; Type 2 degenerate Euler polunomial; Type 2 Changhee polynomial; degenerate harmonic number; degenerate harmonic polynomial; generalized degenerate harmonic number

MSC: 11B73; 11B83; 05A19

1. Introduction

The degenerate Bernoulli and degenerate Euler polynomials were studied by Carlitz [1], as degenerate versions of the usual Bernoulli and Euler polynomials with their arithmetic and combinatorial interest. The present authors and their colleagues have considered various degenerate versions of certain special numbers and polynomials and have discovered many properties of them. To name a few, these include degenerate Bernoulli numbers of the second kind, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Cauchy numbers, and degenerate Bernstein polynomials (see [2–16]). They have been explored by means of different methods such as generating functions, umbral calculus, combinatorial methods, differential equations, probability theory, p-adic integrals, p-adic q-integrals and special functions.

The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and their various generalizations appear in diverse areas such as various branches of number theory, analysis of algorithms in computer science, elementary particle physics, and theoretical physics. The novelty of the present paper is the introduction of degenerate versions of harmonic numbers, called the degenerate harmonic numbers given by $H_{n,\lambda} = \sum_{k=1}^{n} \frac{(1-\lambda)(2-\lambda)\cdots(k-1-\lambda)}{k!}$, and some related polynomials and numbers. The motivation for this definition is natural and simple. Indeed, it follows when we replace the usual logarithm function by the degenerate logarithm function (see Equation (6)) in the generating function of the harmonic numbers (compare Equations (8) and (29)). Three possible applications of our results in other branches of mathematics are outlined in Section 4 with typical examples. We hope that those new numbers will have applications in other areas besides mathematics.

The aim of this paper is twofold. The first is to find several relations between the Type 2 higher-order degenerate Euler polynomials and the Type 2 high-order Changhee polynomials in connection with the degenerate Stirling numbers of both kinds and Jindalrae–Stirling numbers of both kinds. We note here that we use the orthogonality relations of the degenerate Stirling numbers in order to derive several corollaries from the obtained theorems. The second is to introduce the discrete harmonic numbers and the related polynomials and numbers, namely the higher-order degenerate harmonic polynomials and the generalized degenerate harmonic numbers, and to derive their explicit expressions and an identity. We outline the three possible applications of the degenerate harmonic numbers in Section 4.

The outline of this paper is as follows. In Section 1, we go over the stuffs that are necessary throughout paper. These include the discrete exponential functions, the discrete Stirling numbers of both kinds, the discrete logarithm function, the harmonic numbers and their generating function, the degenerate Bell polynomials, Jindalrae–Stirling numbers of both kinds, the Type 2 higher-order degenerate Euler polynomials, and the Type 2 higher-order Changhee polynomials. In Section 2, we show the orthogonality relations for the degenerate Stirling numbers from which the inversion theorem is derived. Then, we prove relations between the Type 2 degenerate Euler polynomials of order r and the Type 2 Changhee polynomials of order r in connection with the degenerate Stirling numbers of both kinds and the Jindalrae–Stirling numbers of both kinds. In addition, we derive a recurrence relation for the Type 2 higher-order degenerate Euler polynomials. In Section 3, we introduce the discrete harmonic numbers. Then, as a natural generalization of these numbers, we introduce the higher-order degenerate harmonic polynomials and an explicit expression for those polynomials. Finally, we introduce the generalized degenerate harmonic numbers and find an identity relating these numbers, the Type 2 higher-order Changhee polynomials, and the degenerate Stirling numbers of the first kind.

For $0 \neq \lambda \in \mathbb{R}$, the degenerate exponential functions are defined by (see [10,13]),

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e_\lambda(t) = e_\lambda^1(t), \tag{1}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n \geq 1)$.

Note that $\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}$. It is well known that the Stirling numbers of first kind are defined by (see [1–25])

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 0), \tag{2}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

The unsigned Stirling number of the first kind $|S_1(n,l)| = (-1)^{n-l} S_1(n,l)$ counts the number of permutations of n elements with l disjoint cycles. As an inversion formula of Equation (2), the Stirling numbers of the second kind are defined by (see [8,10,13,14,17,19–21])

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0). \tag{3}$$

The Stirling number of the second kind $S_2(n,l)$ counts the number of ways to partition a set of n distinct elements into l nonempty subsets. As degenerate versions of Equations (2) and (3), the degenerate Stirling numbers of the first kind are given by (see [14])

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \; (n \geq 0), \tag{4}$$

and the degenerate Stirling numbers of the second kind are defined by (see [8])

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0),. \tag{5}$$

Let $\log_\lambda(t)$ be the compositional inverse of $e_\lambda^x(t)$, called the discrete logarithm function, such that $\log_\lambda(e_\lambda(t)) = e_\lambda\big(\log_\lambda(t)\big) = t$. Then, we have (see [14])

$$\log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}. \tag{6}$$

The harmonic numbers are defined as (see [17,19,25])

$$H_0 = 1, \quad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad (n \geq 1), \tag{7}$$

We note that the generating function of the harmonic numbers are given by (see [17,19,25])

$$\frac{-\log(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n t^n. \tag{8}$$

In [10], the degenerate Bell polynomials are defined by

$$e_\lambda^x(e_\lambda(t) - 1) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{9}$$

when $x = 1$, $B_{n,\lambda} = B_{n,\lambda}(1)$ are called the degenerate Bell numbers. By taking $\lambda \to 0$, we see that $B_n(x) = \lim_{\lambda\to 0} B_{n,\lambda}(x)$ are the Bell polynomials given by $e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$. The Bell number $B_n = B_n(1)$ counts the number of different ways to partition a set of n distinct elements. From Equations (4) and (5), we can derive the generating functions of the degenerate Stirling numbers of both kinds which are given by (see [8])

$$\frac{1}{k!}\big(e_\lambda(t) - 1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{10}$$

and (see [8])

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!},, \tag{11}$$

where k is a nonnegative integer.

In [21], the Jindalrae–Stirling numbers of the first kind are defined by

$$\frac{1}{k!}\big(\log_\lambda(1+\log_\lambda(1+t))\big)^k = \sum_{n=k}^{\infty} S_{J,\lambda}^{(1)}(n,k)\frac{t^n}{n!}. \tag{12}$$

As an inversion formula of Equation (12), the Jindalrae–Stirling numbers of the second kind are given by (see [21])

$$\frac{1}{k!}\left(e_\lambda(e_\lambda(t)-1)-1\right)^k=\sum_{n=k}^{\infty}S_{J,\lambda}^{(2)}(n,k)\frac{t^n}{n!}. \tag{13}$$

Carlitz considered the degenerate Euler polynomials of order r which are given by (see [1])

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},. \tag{14}$$

In particular, for $r=1$, $E_{n,\lambda}(x)=E_{n,\lambda}^{(1)}(x)$ are the degenerate Euler polynomials. The Type 2 degenerate Euler polynomials of order r are defined by

$$\begin{aligned}\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) &= \underbrace{\operatorname{sech}_\lambda(t)\times\cdots\operatorname{sech}_\lambda(t)}_{r-\text{times}}\,e_\lambda^x(t) \\ &= \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad (r\in\mathbb{N}),\end{aligned} \tag{15}$$

where (see [3])

$$\operatorname{sech}_\lambda(t)=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}=\frac{1}{\cosh_\lambda(t)}.$$

In [20], the Type 2 Changehee polynomials of order r are defined by

$$\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r(1+t)^x=\sum_{n=0}^{\infty}C_n^{(r)}(x)\frac{t^n}{n!}. \tag{16}$$

when $x=0$, $C_n^{(r)}=C_n^{(r)}(0)$ are called the Type 2 Changhee numbers of order r.

2. Type 2 Higher-Order Degenerate Euler and Type 2 High-Order Changhee Polynomials

From Equations (4) and (5), we note that

$$\begin{aligned}(x)_{n,\lambda} &= \sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k \\ &= \sum_{k=0}^{n}S_{2,\lambda}(n,k)\sum_{l=0}^{k}S_{1,\lambda}(k,l)(x)_{l,\lambda} \\ &= \sum_{l=0}^{n}\left(\sum_{k=l}^{n}S_{2,\lambda}(n,k)S_{1,\lambda}(k,l)\right)(x)_{l,\lambda}.\end{aligned} \tag{17}$$

Therefore, by comparing the coefficients on both sides of Equation (17), we obtain the following orthogonality relations where the second one follows analogously to Equation (17).

Lemma 1. *For any integers n,l with $n\geq l$, we have*

$$\sum_{k=l}^{n}S_{2,\lambda}(n,k)S_{1,\lambda}(k,l)=\begin{cases}1, & \text{if } l=n,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\sum_{k=l}^{n}S_{1,\lambda}(n,k)S_{2,\lambda}(k,l)=\begin{cases}1, & \text{if } l=n,\\ 0, & \text{otherwise.}\end{cases}$$

For $n \geq 0$, let $f_{n,\lambda} = \sum_{k=0}^{n} g_{k,\lambda} S_{1,\lambda}(n,k)$. Then, we have

$$\begin{aligned} \sum_{k=0}^{n} f_{k,\lambda} S_{2,\lambda}(n,k) &= \sum_{k=0}^{n} \sum_{l=0}^{k} g_{l,\lambda} S_{1,\lambda}(k,l) S_{2,\lambda}(n,k) \\ &= \sum_{l=0}^{n} g_{l,\lambda} \left(\sum_{k=l}^{n} S_{2,\lambda}(n,k) S_{1,\lambda}(k,l) \right) = g_{n,\lambda}. \end{aligned} \tag{18}$$

Therefore, we obtain the following inversion theorem where the converse follows similarly to Equation (18).

Theorem 1. *Let n be a nonnegative integer. Then, we have*

$$f_{n,\lambda} = \sum_{k=0}^{n} g_{k,\lambda} S_{1,\lambda}(n,k) \iff g_{n,\lambda} = \sum_{k=0}^{n} f_{k,\lambda} S_{2,\lambda}(n,k).$$

In Equation (15), replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned} \left(\frac{2}{(1+t)+(1+t)^{-1}} \right)^r (1+t)^x &= \sum_{k=0}^{\infty} \mathcal{E}_k^{(r)}(x) \frac{1}{k!} \left(\log_\lambda(1+t) \right)^k \\ &= \sum_{k=0}^{\infty} \mathcal{E}_k^{(r)}(x) \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{E}_k^{(r)}(x) S_{1,\lambda}(n,k) \right) \frac{t^n}{n!} \end{aligned} \tag{19}$$

Therefore, by Equations (16) and (19), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$C_n^{(r)}(x) = \sum_{k=0}^{n} \mathcal{E}_{k,\lambda}^{(r)}(x) S_{1,\lambda}(n,k), \quad (r \in \mathbb{N}).$$

In particular,

$$C_n^{(r)} = \sum_{k=0}^{n} \mathcal{E}_{k,\lambda}^{(r)} S_{1,\lambda}(n,k), \quad (r \in \mathbb{N}).$$

By using Theorem 1, we obtain the following corollary from Theorem 2.

Corollary 1. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} C_k^{(r)}(x) S_{2,\lambda}(n,k), \quad (r \in \mathbb{N}).$$

In particular,

$$\mathcal{E}_{n,\lambda}^{(r)} = \sum_{k=0}^{n} C_k^{(r)} S_{2,\lambda}(n,k), \quad (r \in \mathbb{N}).$$

Now, we observe that

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x+2)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t)\big(e_\lambda^2(t)+1\big) \tag{20}$$
$$= \left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^{x+1}(t)\big(e_\lambda(t)+e_\lambda^{-1}(t)\big).$$
$$= 2\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{r-1} e_\lambda^{x+1}(t)=2\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{r-1}(x+1)\frac{t^n}{n!}.$$

Thus, we have

$$\mathcal{E}_{n,\lambda}^{(r)}(x+2)+\mathcal{E}_{n,\lambda}^{(r)}(x)=2\mathcal{E}_{n,\lambda}^{(r-1)}(x+1),\quad (n\geq 0,\, r\geq 2). \tag{21}$$

From Equation (21), we note that

$$\begin{aligned}\mathcal{E}_{n,\lambda}^{(r)}(x+2) &= -\mathcal{E}_{n,\lambda}^{(r)}(x)+2\mathcal{E}_{n,\lambda}^{(r-1)}(x+1)\\ &= -\mathcal{E}_{n,\lambda}^{(r)}(x)+2\big(-\mathcal{E}_{n,\lambda}^{(r-1)}(x-1)+2\mathcal{E}_{n,\lambda}^{(r-2)}(x)\big)\\ &= -\mathcal{E}_{n,\lambda}^{(r)}(x)-2\mathcal{E}_{n,\lambda}^{(r-1)}(x-1)+2^2\mathcal{E}_{n,\lambda}^{(r-2)}(x)\\ &= \cdots\\ &= -\sum_{l=0}^{r-1}2^l\mathcal{E}_{n,\lambda}^{(r-l)}(x-l)+2^r(x-r+2)_{n,\lambda}.\end{aligned} \tag{22}$$

Therefore, by Equation (22), we obtain the following theorem.

Theorem 3. *For $n\geq 0,\ r\in\mathbb{N}$, we have*

$$\mathcal{E}_{n,\lambda}^{(r)}(x+2)+\sum_{l=0}^{r-1}2^l\mathcal{E}_{n,\lambda}^{(r-l)}(x-l) = 2^r(x-r+2)_{n,\lambda}.$$

Replacing t by $\log_\lambda\big(1+\log_\lambda(1+t)\big)$ in Equation (15), we get

$$\begin{aligned}&\left(\frac{2}{\big(1+\log_\lambda(1+t)\big)+\big(1+\log_\lambda(1+t)\big)^{-1}}\right)^r\big(1+\log_\lambda(1+t)\big)^x\\ &=\sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(r)}(x)\frac{1}{k!}\big(\log_\lambda(1+\log_\lambda(1+t))\big)^k\\ &=\sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_{J,\lambda}^{(1)}(n,k)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathcal{E}_{n,\lambda}^{(r)}(x)S_{J,\lambda}^{(1)}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

On the other hand, by Equation (16), we get

$$\left(\frac{2}{(1+\log_\lambda(1+t))+(1+\log_\lambda(1+t))^{-1}}\right)^r(1+\log_\lambda(1+t))^x \tag{24}$$
$$=\sum_{k=0}^\infty C_k^{(r)}(x)\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k$$
$$=\sum_{k=0}^\infty C_k^{(r)}(x)\sum_{n=k}^\infty S_{1,\lambda}(n,k)\frac{t^n}{n!}$$
$$=\sum_{n=0}^\infty\left(\sum_{k=0}^n C_k^{(r)}(x)S_{1,\lambda}(n,k)\right)\frac{t^n}{n!}$$

Therefore, by Equations (23) and (24), we obtain the following theorem.

Theorem 4. *For $n\geq 0$, we have*

$$\sum_{k=0}^n C_k^{(r)}(x)S_{1,\lambda}(n,k) = \sum_{k=0}^n \mathcal{E}_{k,\lambda}^{(r)}(x)S_{J,\lambda}^{(1)}(n,k).$$

In particular,

$$\sum_{k=0}^n C_k^{(r)}S_{1,\lambda}(n,k) = \sum_{k=0}^n \mathcal{E}_{k,\lambda}^{(r)}S_{J,\lambda}^{(1)}(n,k).$$

From Theorem 1, we obtain the following corollary.

Corollary 2. *For $n\geq 0$, we have*

$$C_n^{(r)}(x) = \sum_{k=0}^n\sum_{l=0}^k \mathcal{E}_{l,\lambda}^{(r)}(x)S_{J,\lambda}^{(1)}(k,l)S_{2,\lambda}(n,k).$$

In Equation (16), replacing t by $e_\lambda(e_\lambda(t)-1)-1$, we get

$$\left(\frac{2}{e_\lambda(e_\lambda(t)-1)+e_\lambda^{-1}(e_\lambda(t)-1)}\right)^r e_\lambda^x(e_\lambda(t)-1) \tag{25}$$
$$=\sum_{k=0}^\infty C_k^{(r)}(x)\frac{1}{k!}\left(e_\lambda(e_\lambda(t)-1)-1\right)^k$$
$$=\sum_{k=0}^\infty C_k^{(r)}(x)\sum_{n=k}^\infty S_{J,\lambda}^{(2)}(n,k)\frac{t^n}{n!}$$
$$=\sum_{n=0}^\infty\left(\sum_{k=0}^n C_k^{(r)}(x)S_{J,\lambda}^{(2)}(n,k)\right)\frac{t^n}{n!}.$$

On the other hand,

$$\begin{aligned}&\left(\frac{2}{e_\lambda(e_\lambda(t)-1)+e_\lambda^{-1}(e_\lambda(t)-1)}\right)^r e_\lambda^x(e_\lambda(t)-1) \\ &=\sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(r)}(x)\frac{1}{k!}(e_\lambda(t)-1)^k \\ &=\sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathcal{E}_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{26}$$

Therefore, by Equations (25) and (26), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}S_{J,\lambda}^{(2)}(n,k)C_k^{(r)}(x) = \sum_{k=0}^{n}\mathcal{E}_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k).$$

From Theorem 1, we have the following corollary.

Corollary 3. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}S_{1,\lambda}(n,k)S_{J,\lambda}^{(2)}(k,l)C_l^{(r)}(x).$$

From Equation (9), we note that

$$\begin{aligned}&\left(\frac{2}{e_\lambda(e_\lambda(t)-1)+e_\lambda^{-1}(e_\lambda(t)-1)}\right)^r e_\lambda^x(e_\lambda(t)-1) \\ &=\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{(r)}\frac{1}{l!}(e_\lambda(t)-1)^l\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!} \\ &=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}^{(r)}S_{2,\lambda}(k,l)\frac{t^k}{k!}\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{t^m}{m!} \\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}^{(r)}S_{2,\lambda}(k,l)B_{n-k,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Therefore, by Equations (25) and (27), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}S_{J,\lambda}^{(2)}(n,k)C_k^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\mathcal{E}_{l,\lambda}^{(r)}S_{2,\lambda}(k,l)B_{n-k,\lambda}(x).$$

3. Discrete Harmonic Numbers and Related Polynomials and Numbers

From Equation (6), we note that

$$\lim_{\lambda\to 0}\log_\lambda(1+t) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}t^n = \log(1+t). \tag{28}$$

In view of Equations (8) and (28), we may consider the degenerate harmonic numbers given by

$$-\frac{\log_\lambda(1-t)}{1-t} = \sum_{n=1}^{\infty} H_{n,\lambda} t^n. \tag{29}$$

Note that

$$\lim_{\lambda\to 0} H_{n,\lambda} = H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}, \quad (n\in\mathbb{N}).$$

From Equations (6) and (29), we note that

$$\begin{aligned}\frac{-\log_\lambda(1-t)}{1-t} &= \sum_{m=0}^{\infty} t^m \sum_{k=1}^{\infty} (-\lambda)^{k-1}(1)_{k,1/\lambda}\frac{t^k}{k!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-\lambda)^{k-1}(1)_{k,1/\lambda}}{k!}\right)t^n.\end{aligned} \tag{30}$$

Thus, by Equations (29) and (30), we get

$$H_{n,\lambda} = \sum_{k=1}^{n}\frac{(-\lambda)^{k-1}(1)_{k,1/\lambda}}{k!} = \sum_{k=1}^{n}\frac{(1-\lambda)(2-\lambda)\cdots(k-1-\lambda)}{k!}, \quad (n\in\mathbb{N}).$$

Indeed,

$$\lim_{\lambda\to 0} H_{n,\lambda} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = H_n.$$

For $r\in\mathbb{N}$, the degenerate harmonic polynomials $H_{n,\lambda}^{(r)}(x)$ of order r are defined by

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}(x)t^n = \frac{\big(-\log_\lambda(1-t)\big)^{r+1}}{t(1-t)}(1-t)^x. \tag{31}$$

when $x=0$, $H_{n,\lambda}^{(r)} = H_{n,\lambda}^{(r)}(0)$ are called the degenerate harmonic numbers of order r.

From Equation (31), we note that

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)} t^n = \frac{-\log_\lambda(1-t)}{t(1-t)} = \frac{1}{t}\sum_{n=1}^{\infty} H_{n,\lambda}t^n = \sum_{n=0}^{\infty} H_{n+1,\lambda}t^n. \tag{32}$$

Comparing the coefficients on both sides of Equation (32), we obtain

$$H_{n,\lambda}^{(0)} = H_{n+1,\lambda}, \quad (n\geq 0). \tag{33}$$

From Equations (31) and (33), we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)}(x)t^n &= \frac{-\log_\lambda(1-t)}{t(1-t)}(1-t)^x \\ &= \sum_{m=0}^{\infty} H_{m+1,\lambda}t^m \sum_{l=0}^{\infty}\binom{x}{l}(-1)^l x^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} H_{m+1,\lambda}\binom{x}{n-m}(-1)^{n-m}\right)t^n.\end{aligned} \tag{34}$$

By Equation (34), we get

$$H^{(0)}_{n,\lambda}(x) \;=\; \sum_{m=0}^{n} H_{m+1,\lambda}\binom{x}{n-m}(-1)^{n-m}, \quad (n \geq 0).$$

Now, we observe that

$$\begin{aligned}
\frac{\big(-\log_\lambda(1-t)\big)^{r+1}}{t(1-t)} &= \frac{1}{t(1-t)}\left(\sum_{l=1}^{\infty}\frac{(-\lambda)^{l-1}}{l!}(1)_{l,1/\lambda}t^l\right)^{r+1} \qquad (35)\\
&= \left(\frac{1}{t}+\frac{1}{1-t}\right)\sum_{l=r+1}^{\infty}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}}{l_1!l_2!\cdots l_{r+1}!}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}\,t^l\\
&= \sum_{n=r}^{\infty}\sum_{l_1+\cdots+l_{r+1}=n+1}\frac{(-\lambda)^{n-r}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\,t^n\\
&\quad+\sum_{n=r+1}^{\infty}\sum_{l=r+1}^{n}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\,t^n\\
&= \sum_{l_1+\cdots+l_{r+1}=r+1}\frac{(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}t^r\\
&\quad+\sum_{n=r+1}^{\infty}\sum_{l_1+\cdots+l_{r+1}=n+1}\frac{(-\lambda)^{n-r}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\,t^n\\
&\quad+\sum_{n=r+1}^{\infty}\sum_{l=r+1}^{n}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\,t^n\\
&= \sum_{l_1+\cdots+l_{r+1}=r+1}\frac{(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}t^r\\
&\quad+\sum_{n=r+1}^{\infty}\left(\sum_{l=r+1}^{n+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}}{l_1!l_2!\cdots l_{r+1}!}(1)_{l_1,1/\lambda}(1)_{l_2,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}\right)t^n\\
&= \sum_{n=r}^{\infty}\left(\sum_{l=r+1}^{n+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}}{l_1!l_2!\cdots l_{r+1}!}(1)_{l_1,1/\lambda}(1)_{l_2,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}\right)t^n.
\end{aligned}$$

Therefore, by Equations (31) and (35), we obtain the following equation.

$$H^{(r)}_{n,\lambda} = \begin{cases} \displaystyle\sum_{l=r+1}^{n+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}}{l_1!l_2!\cdots l_{r+1}!}(1)_{l_1,1/\lambda}(1)_{l_2,1/\lambda},\cdots(1)_{l_{r+1},1/\lambda}, & \text{if } n \geq r,\\ 0, & \text{otherwise.}\end{cases} \qquad (36)$$

By Equations (31) and (35), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}H^{(r)}_{n,\lambda}(x)t^n &= \frac{\big(-\log_\lambda(1-t)\big)^{r+1}}{t(1-t)}(1-t)^x \qquad (37)\\
&= \sum_{k=r}^{\infty}\left(\sum_{l=r+1}^{k+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\right)t^k\sum_{m=0}^{\infty}\binom{x}{m}(-1)^m t^m\\
&= \sum_{n=r}^{\infty}\left(\sum_{k=r}^{n}\binom{x}{n-k}(-1)^{n-k}\sum_{l=r+1}^{k+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\right)t^n.
\end{aligned}$$

By comparing the coefficients on both sides of Equation (37), we get

$$H_{n,\lambda}^{(r)}(x)=\begin{cases}\sum_{k=r}^{n}\binom{x}{n-k}(-1)^{n-k}\sum_{l=r+1}^{k+1}\sum_{l_1+\cdots+l_{r+1}=l}\frac{(-\lambda)^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}, & \text{if } n\geq r,\\ 0, & \text{otherwise.}\end{cases} \tag{38}$$

Let us consider the generalized degenerate harmonic numbers which are given by

$$\sum_{n=0}^{\infty}H_{\lambda}(n+r+1,r)t^{n} \;=\; \frac{\big(-\log_{\lambda}(1-t)\big)^{r+1}}{t^{r+1}(1-t)}. \tag{39}$$

Replacing t by $-t$, we get

$$\sum_{n=0}^{\infty}H_{\lambda}(n+r+1,r)(-t)^{n} \;=\; \frac{\big(\log_{\lambda}(1+t)\big)^{r+1}}{t^{r+1}(1+t)}. \tag{40}$$

From Equation (6), we note that

$$\begin{aligned}\big(\log_{\lambda}(1+t)\big)^{r+1} \;&=\; \left(\sum_{l=1}^{\infty}\lambda^{l-1}(1)_{l,1/\lambda}\frac{t^{l}}{l!}\right)^{r+1}\\ &=\sum_{l=r+1}^{\infty}\left(\sum_{l_1+\cdots+l_{r+1}=l}\frac{\lambda^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!l_2!\cdots l_{r+1}!}\right)t^{l}.\end{aligned} \tag{41}$$

By Equation (41), we get

$$\begin{aligned}\frac{\big(\log_{\lambda}(1+t)\big)^{r+1}}{t^{r+1}(1+t)}&=\frac{1}{t^{r+1}}\sum_{l=r+1}^{\infty}\sum_{l_1+\cdots+l_{r+1}=l}\frac{\lambda^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}t^{l}\sum_{m=0}^{\infty}(-1)^{m}t^{m}\\ &=\frac{1}{t^{r+1}}\sum_{n=r+1}^{\infty}\sum_{l=r+1}^{n}(-1)^{n-l}\sum_{l_1+\cdots+l_{r+1}=l}\frac{\lambda^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\,t^{n}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=r+1}^{n+r+1}(-1)^{n+r+1-l}\sum_{l_1+\cdots+l_{r+1}=l}\frac{\lambda^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}\right)t^{n}.\end{aligned} \tag{42}$$

Therefore, by Equations (40) and (42), we get

$$H_{\lambda}(n+r+1,r)=\sum_{l=r+1}^{n+r+1}(-1)^{r+1-l}\sum_{l_1+\cdots+l_{r+1}=l}\frac{\lambda^{l-r-1}(1)_{l_1,1/\lambda}\cdots(1)_{l_{r+1},1/\lambda}}{l_1!\cdots l_{r+1}!}. \tag{43}$$

Let r be a positive integer. We observe that

$$
\begin{aligned}
&\frac{\left(\log_\lambda(1+t)\right)^{r+1}}{t^{r+1}(1+t)}\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r(1+t)^x \\
&=\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r(1+t)^{x-1}\frac{1}{t^{r+1}}\left(\log_\lambda(1+t)\right)^{r+1} \\
&=\sum_{l=0}^{\infty}C_l^{(r)}(x-1)\frac{t^l}{l!}\sum_{k=0}^{\infty}S_{1,\lambda}(k+r+1,r+1)\frac{k!(r+1)!}{(k+r+1)!}\frac{t^k}{k!} \\
&=\sum_{l=0}^{\infty}C_l^{(r)}(x-1)\frac{t^l}{l!}\sum_{k=0}^{\infty}\frac{S_{1,\lambda}(k+r+1,r+1)}{\binom{k+r+1}{k}}\frac{t^k}{k!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\binom{n}{k}S_{1,\lambda}(k+r+1,r+1)}{\binom{k+r+1}{k}}C_{n-k}^{(r)}(x-1)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{44}
$$

On the other hand,

$$
\begin{aligned}
&\frac{\left(\log_\lambda(1+t)\right)^{r+1}}{t^{r+1}(1+t)}\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r(1+t)^x \\
&=\sum_{l=0}^{\infty}H_\lambda(l+r+1)(-t)^l\sum_{m=0}^{\infty}C_m^{(r)}(x)\frac{t^m}{m!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}l!H_\lambda(l+r+1,r)(-1)^lC_{n-l}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{45}
$$

Therefore, by Equations (44) and (45), we obtain the following equation.

$$
\sum_{k=0}^{n}\frac{\binom{n}{k}S_{1,\lambda}(k+r+1,r+1)}{\binom{k+r+1}{k}}C_{n-k}^{(r)}(x-1)=\sum_{l=0}^{n}\binom{n}{l}l!H_\lambda(l+r+1,r)(-1)^lC_{n-l}^{(r)}(x). \tag{46}
$$

Finally, we would like to mention that the generating function method used in harmonic polynomials has also been applied in other fields of discrete applied mathematics such as network theory (see [26,27]).

4. Further Remark

Before we conclude our paper in the following section, we would like to mention three possible applications of our results to other areas.

The first possibility is their applications to differential equations. In [18], certain infinite families of nonlinear differential equations, satisfied by the generating functions of some degenerate polynomials, are derived to find new combinatorial identities for those polynomials.

For example, it is shown that, for any positive integer N, the following nonlinear differential equation with respect to t is satisfied by $F=F(t)=\frac{1}{\log(1+t)}$:

$$
F^{(N)}(t)=\frac{(-1)^N}{(1+t)^N}\sum_{j=2}^{N+1}(j-1)!\,(N-1)!H_{N-1,j-2}F^j, \tag{47}
$$

where

$$
\begin{aligned}
&H_{N,0}=1, \quad \text{for all } N, \\
&H_{N,1}=H_N=1+\frac{1}{2}+\cdots+\frac{1}{N}, \\
&H_{N,j}=\frac{H_{N-1,j-1}}{N}+\frac{H_{N-2,j-1}}{N-1}+\cdots+\frac{H_{0,j-1}}{1}, \quad H_{0,j-1}=0 \quad (2\le j\le N).
\end{aligned}
$$

Then, from Equation (47), it is possible to derive the next combinatorial identity: For $n \geq 0$, we have

$$(-1)^N \sum_{j=0}^{\min\{n,N-1\}} (N-j)!\,(N-1)!H_{N-1,N-1-j}\,(n)_j\, b_{n-j}^{(N+1-j)} \tag{48}$$
$$= \begin{cases} (-1)^N N!\,(N)_n\,, & \text{if } 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l}\,(n)_{l+N+1}\,, & \text{if } n \geq N+1. \end{cases}$$

Here, $b_n^{(k)}$ are the Bernoulli numbers of the second kind of order k defined by

$$\left(\frac{t}{\log(1+t)}\right)^k = \sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!}.$$

In particular, for $k = 1$, $b_n = b_n^{(1)}$ are the Bernoulli numbers of the second kind.

The second possibility is their applications to probability theory. For instance, in [9] (see also, [11]), by using the generating functions of the moments of certain random variables, new identities connecting some special numbers and moments of random variables are deduced.

To state this, we first recall the following definitions. A random variable X is said to be the Poisson random variable with parameter $\alpha > 0$, if the probability mass function of X is given by

$$p(i) = P\{X = i\} = e^{-\alpha}\frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \cdots.$$

In addition, for $\lambda \in \mathbb{R}$, X_λ is the degenerate Poisson random variable with parameter $\alpha > 0$, if the probability mass function of X_λ is given by

$$P_\lambda(i) = P\{X_\lambda = i\} = e_\lambda^{-1}(\alpha)\frac{\alpha^i (1)_{i,\lambda}}{i!}, \quad i = 0, 1, 2, \cdots.$$

Furthermore, the degenerate Bell polynomials and the new type degenerate Bell polynomials are, respectively, defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad e_\lambda^{-1}(x)e_\lambda(xe^t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}.$$

Then, it was shown that

$$E[X_\lambda^n] = \beta_{n,\lambda}(\alpha), \quad E[(X)_{n,\lambda}] = Bel_{n,\lambda}(\alpha), \quad (n \geq 1), \tag{49}$$

where X_λ and X are, respectively, the degenerate Poisson random variable with parameter $\alpha > 0$, and the Poisson random variable with parameter $\alpha > 0$.

The third possibility is their applications to identities of symmetry. For example, in [7] (see also [28]), abundant identities of symmetry are obtained for various degenerate versions of many special polynomials by using p-adic fermionic integrals.

Let $\tau_k(\lambda, n)$ be the alternating generalized falling factorial sum given by

$$\tau_k(\lambda, n) = \sum_{i=0}^{n} (-1)^i\,(i)_{k,\lambda}, \quad (n \geq 0).$$

Then, for any odd positive integers w_1, w_2, we have:

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}E_{k,\frac{\lambda}{w_2}}(w_1y_1)\tau_{n-k}(\frac{\lambda}{w_1},w_2-1)w_1^{n-k}w_2^{k}\\
=&\sum_{k=0}^{n}\binom{n}{k}E_{k,\frac{\lambda}{w_1}}(w_2y_1)\tau_{n-k}(\frac{\lambda}{w_2},w_1-1)w_2^{n-k}w_1^{k}\\
=&w_1^n\sum_{i=0}^{w_1-1}(-1)^iE_{n,\frac{\lambda}{w_1}}(w_2y_1+\frac{w_2}{w_1}i)\\
=&w_2^n\sum_{i=0}^{w_2-1}(-1)^iE_{n,\frac{\lambda}{w_2}}(w_1y_1+\frac{w_1}{w_2}i)\\
=&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\frac{\lambda}{w_1w_2}}(y_1)\tau_l(\frac{\lambda}{w_2},w_1-1)\tau_m(\frac{\lambda}{w_1},w_2-1)w_1^{k+m}w_2^{k+l}\\
=&w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}(-1)^iE_{k,\frac{\lambda}{w_1w_2}}(y_1+\frac{i}{w_1})\tau_{n-k}(\frac{\lambda}{w_1},w_2-1)w_2^{k}\\
=&w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}(-1)^iE_{k,\frac{\lambda}{w_1w_2}}(y_1+\frac{i}{w_2})\tau_{n-k}(\frac{\lambda}{w_2},w_1-1)w_1^{k}\\
=&(w_1w_2)^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}(-1)^{i+j}E_{n,\frac{\lambda}{w_1w_2}}(y_1+\frac{i}{w_1}+\frac{j}{w_2}).
\end{aligned}
\tag{50}
$$

5. Conclusions

In Section 2, the orthogonality relations are shown for the degenerate Stirling numbers from which the inversion theorem was derived. Then, in connection with the degenerate Stirling numbers of both kinds and the Jindalrae–Stirling numbers of both kinds, several relations are proved between the Type 2 degenerate Euler polynomials of order r and the Type 2 Changhee polynomials of order r. In addition, a recurrence relation is deduced for the Type 2 higher-order degenerate Euler polynomials. In Section 3, the discrete harmonic numbers are introduced as a degenerate version of the usual harmonic numbers. Then, as a natural generalization of these numbers, the higher-order degenerate harmonic polynomials are considered and an explicit expression for them is obtained. Finally, the generalized degenerate harmonic numbers are constructed so that an identity involving these numbers, the Type 2 higher-order Changhee polynomials, and the degenerate Stirling numbers of the first kind is derived.

In Section 4, we mentioned three possibilities for applications of our results to other areas. The first one is their applications to differential equations, the second one is their applications to probability theory, and the third one is their applications to identities of symmetry. For each possibility, we provide a typical example (see Equations (47)–(50)).

It is one of our future projects to continue to study various degenerate versions of some special polynomials and numbers, and to find some of their possible applications to mathematics, science, and engineering.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to thank the referees for their detailed and valuable suggestions that helped improve the original manuscript in its present form.

Conflicts of Interest: The authors declare that they have no competing interests.

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