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# Analysis of Homotopy Decomposition Varieties in Quotient Topological Spaces

**Susmit Bagchi**

Department of Aerospace and Software Engineering (Informatics), Gyeongsang National University, Jinju 660701, Korea; profsbagchi@gmail.com

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**Abstract:** The fundamental groups and homotopy decompositions of algebraic topology have applications in systems involving symmetry breaking with topological excitations. The main aim of this paper is to analyze the properties of homotopy decompositions in quotient topological spaces depending on the connectedness of the space and the fundamental groups. This paper presents constructions and analysis of two varieties of homotopy decompositions depending on the variations in topological connectedness of decomposed subspaces. The proposed homotopy decomposition considers connected fundamental groups, where the homotopy equivalences are relaxed and the homeomorphisms between the fundamental groups are maintained. It is considered that one fundamental group is strictly homotopy equivalent to a set of 1-spheres on a plane and as a result it is homotopy rigid. The other fundamental group is topologically homeomorphic to the first one within the connected space and it is not homotopy rigid. The homotopy decompositions are analyzed in quotient topological spaces, where the base space and the quotient space are separable topological spaces. In specific cases, the decomposed quotient space symmetrically extends Sierpinski space with respect to origin. The connectedness of fundamental groups in the topological space is maintained by open curve embeddings without enforcing the conditions of homotopy classes on it. The extended decomposed quotient topological space preserves the trivial group structure of Sierpinski space.

**Keywords:** topological spaces; quotient topology; fundamental groups; homotopy; embeddings

**MSC:** 54E15; 55Q05; 55Q52; 55MXX

## 1. Introduction

The symmetry breaking in any system involves a wide variety of topological excitations and it contains the associated topological constraints. The preparations of homotopy groups and related decompositions provide meaningful insights into the phases of a system after the symmetry is broken [1]. The homotopy decomposition in the classifying space is constructed considering that the space contains torsion-free groups [2]. The special homotopy classes in a category of based spaces are proposed allowing the decomposition of stable homotopy. The formulation is based on positive filtration of the space and the Toda bracket [3]. The decomposition and related decomposed homotopy types for metrizable $LC^n$ spaces are formulated in [4]. Interestingly, often the homotopy type of original space and the decomposed space are very similar in $LC^n$ spaces. Moreover, the cellular decomposition $G$ of $S^n$ ($n$-sphere) in higher dimension ($n \geq 5$) results in the formation of $n$-manifold denoted by $S^n/G$, which is also homeomorphic to $S^n$. If $X, Y$ are metric continua with the absolute neighbourhood retract then there is a isomorphic functional continua $f_n$ between the fundamental groups $\pi_n(X)$ and $\pi_n(Y)$. It indicates that if $X$ is compact as well as connected space and $G$ is an upper semicontinuous decomposition of $X$ into the corresponding compact quotient space $X/G$, then $X$ and $X/G$ have same homotopy type. However, this observation holds if and only if the quotient space $X/G$ is finite dimensional in nature.

There are varieties of fundamental groups based on the nature of topological spaces. It is known that in a connected and compact metric space, the fundamental groups can be finitely generated [5]. In the loop based topological space $X$, the quasitopological fundamental group $\pi_1^{qtop}(X, x_0)$ is a fundamental group variety with inherited quotient topology [6]. Interestingly, if a topological space $X$ does not have any universal cover then the fundamental groups are non-discrete. Moreover, the quasitopological fundamental group is homotopy invariant in nature. It is illustrated that a locally path connected metric space $X$ is $\pi_1$-shape injective if the fundamental groups given by $\pi_1^{qtop}(X, x_0)$ are separated. Hence, a locally path connected space is homotopically path-Hausdorff if the fundamental group $\pi_1^{qtop}(X, x_0)$ follows separation axioms in $T_1$ topological space. The Peano continuity is defined in the compact metric space, which is a connected space (including the locally connectedness property). There exists homeomorphism between a fundamental group in one-dimensional Peano continuum and another fundamental group in the Peano continuum on a plane if the map is continuous [7]. Moreover, a set of homotopy fixed points derived from a planar Peano continuum coincides with a point set representing a space, which is not a (locally) simply connected space. In one-dimensional wild space with Peano continuum, the fundamental group determines the respective homeomorphism type. Note that the loops of fundamental groups in a planar space (set) are homotopy rigid [7].

In this paper, the two different varieties of homotopy decompositions are proposed in a connected topological space. The aim is to analyze the variations in algebraic and topological properties of homotopy decompositions in quotient topological spaces depending on the connectedness of decomposed subspaces as well as fundamental groups. The main difference between the proposed decomposition varieties is the variations of connectedness of the decomposed subspaces. However, every variety of homotopy decomposition considers connected fundamental groups having different base points. The analysis of decomposed homotopy within quotient topological space is presented maintaining Hausdorff property. The decomposed quotient topological space extends Sierpinski space symmetrically with respect to origin in specific case. In the following subsections (Sections 1.1 and 1.2), brief descriptions about the concepts of homotopy decomposition and motivation for this work are presented.

### *1.1. Homotopy and Decomposition*

The fundamental groups in a space can be formulated in various different ways. In general, the fundamental group $F$ of a group manifold denoted by $G$ is Abelian. It is shown that if the fundamental group of homogeneous space is solvable then it can be finitely generated [8]. If $E$ is the exterior of a knot based at point $x$, then the fundamental group of knot exterior can be defined as $\gamma_K : [0,1]^2 \to (E, x)$ and in such case the homotopies fix the boundary points [9]. The Kampen fundamental group is variety of fundamental groups in a topological space, where such groups are dependent on the homeomorphism in subspaces [10]. In other words, the Kampen fundamental group needs identification of homeomorphic subsets of underlying topological space. It is important to note that, Kampen fundamental group considers separable and regular topological spaces. As the simplexes are constructed with the possibility of inclusion of deformation, hence the underlying topological space is considered to be an arc-wise connected space. Interestingly, the Kampen fundamental groups can be generated as arc-wise connected components by using countable generators. If $X, Y$ are two one-connected spaces, then the Postnikov homotopy decomposition of $f : X \to Y$ reduces space $Y$ into a point [11]. A topological space is not separable if it is path connected and most possibly a convex space. However, this concept is further refined in hyperspace topological structures. If $X$ is a space then the topology in hyperspace is in $2^X$ containing every closed subsets equipped with Vietoris topology of exponential type [12]. The compactness of Vietoris topological hyperspace and the compactness of original space $X$ are equivalent in nature. The ordered arc maintains connectedness in a hyperspace by following the ordered set inclusion principle. As a result, the continuum in a topological hyperspace is decomposable if and only if it is a union two proper subcontinua.

The compact Lie groups form classifying spaces and the corresponding homotopy theory is developed for those classifying spaces. The homotopy decomposition of such classifying spaces are constructed considering that maximal torus $T$ exists in such spaces [13]. The similar homotopy decomposition results were proposed by A. Borel considering the existence of normalizer $NT$ of torus forming the Weyl group given by $NT/T$. It is important to note that the Lie group is a connected group supporting isomorphism and cohomology of primes. The corresponding homotopy decomposition applies the algebraic functors of category theory [13]. Specifically, the functors maintain left-adjoint property. From the geometric point of view, the groups of symplectomorphisms on symplectic manifolds are considered for generating homotopy on the topological groups [14]. This means that symplectomorphic groups of manifolds are considered as a class of topological groups. The initial results are formulated based on symplectomorphic group actions on contractible spaces. In such case, a specific condition is maintained that spaces should be compatible to almost complex symplectic manifolds. The constructions employ various aspects of homotopy pushout decomposition, canonical projections and amalgamation of topological groups [14]. Interestingly, the concept of tubular neighbourhood in a contractible space is introduced to analyze homotopy. Note that, in this case the homology commutes with sequential colimits of $T_1$ topological spaces with closed inclusion [14]. Moreover, the homotopy decomposition follows that the homotopy colimits maintain weak contraction and a weak equivalence relation in contractible spaces, generating a quotient space.

### *1.2. Motivation*

The theories of homotopy and fundamental groups of algebraic topology have several applications. The properties of fundamental groups and homotopy differ depending on planarity, connectedness and retraction within a space. In general, the homotopy decomposition considers different forms of symmetries and homeomorphisms. However, an interesting question is: if such symmetry and homeomorphism is relaxed in a connected topological space, then what would be the properties of decompositions? What would be the nature of homotopy decompositions within the quotient topological spaces if the fundamental groups are connected? Moreover, if the connected fundamental groups differ in homotopic rigidity, then what would be the structural properties of connected quotient topological spaces? These questions are addressed in this paper in relative details. It is illustrated that, various homotopy decompositions with different forms of connectedness as well as homotopic rigidity give rise to different sets of properties in the decomposed quotient topological spaces. Interestingly, in specific cases, the decomposed quotient space generated from homotopy decomposition extends Sierpinski space symmetrically with respect to origin. However, the trivial group structure in Sierpinski space is preserved within the extended decomposed quotient topological space.

Rest of the paper is organized as follows. The preliminary concepts are presented in Section 2. The proposed definitions and main results are presented in Sections 3 and 4, respectively. The discussion about interrelation between the decomposed quotient space and Sierpinski space is illustrated in Section 5. Finally, Section 6 concludes the paper.

## 2. Preliminary Concepts

In this section a set of basic definitions and preliminary concepts are presented to establish the notions about topological spaces and homotopy. Note that the symbol $S^1$ denotes 1-sphere in a complex plane by following the standard representation. Let $X$ be a point set and $\tau_X \subseteq P(X)$ be a subset of power set of $X$. The structure $(X, \tau_X)$ is called a topological space if the following axioms are satisfied by it: (I) $\{\phi, X\} \subset \tau_X$, (II) $[\{A_i : i \in Z^+\} \subset \tau_X] \Rightarrow \left[\underset{i \in Z^+}{\cup} A_i \in \tau_X\right]$ and, (III) $[\{A_i, A_k\} \subset \tau_X] \Rightarrow [(A_i \cap A_k) \in \tau_X]$. It indicates that the topological space includes indiscrete topology, countably infinite union of subspaces and finite intersection between subspaces. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ between two topological spaces is continuous if $\forall x \in X, \exists y \in Y$ such that $f(U_x) \subset U_y$, where $U_x, U_y$ are open neighbourhoods of $x, y$ respectively. Let two continuous functions be defined between two spaces $X, Y$ as, $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Y$. The functions $f_1, f_2$ are called homotopic if there exists continuous $F : X \times [0,1] \rightarrow Y$ such that, $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$. If we consider that $\{p_i \big| p_i : [0,1] \rightarrow X\}$ is a set of continuous functions then it prepares two base points $p_i(0) \in X, p_i(1) \in X$ in the space for some $i \in Z^+$. If we consider two such continuous functions, $p_1 : [0,1] \rightarrow X$ and $p_2 : [0,1] \rightarrow X$ then $F_p : [0,1]^2 \rightarrow X$ is called a path homotopy if following properties are satisfied: (I) $F_p(s \in [0,1], 0) = p_1(s)$, (II) $F_p(s \in [0,1], 1) = p_2(s)$, (III) $F_p(0, t \in [0,1]) = p_1(0) = p_2(0)$ and (IV) $F_p(1, t \in [0,1]) = p_1(1) = p_2(1)$.

The fundamental group in $(X, \tau_X)$ at a base point $x_b \in X$ is given by $\pi_1(X, x_b \in X)$ containing the homotopy class $[p]_{xb}$ and $*_H$ is the homotopy product such that $\forall p_i, p_k \in [p]_{xb}, p_i *_H p_k$ is a loop at base point $x_b$. If $\{x_0, x_1, x_2\} \subset X$ are distinct base points in path homotopies (not necessarily fundamental groups) then the homotopy product in topological space is given as: $p_i *_H p_k$, where $p_i(0) = x_0, p_i(1) = x_1$ and $p_k(0) = x_1, p_k(1) = x_2$. Note that, the homotopy functions $p_i, p_k$ may belong to different path homotopy classes, which are equivalent classes each. Moreover, the homotopy product maintains the algebraic condition given by, $[p] *_H [q] = [p *_H q]$, where $[p], [q]$ are two homotopy classes in $(X, \tau_X)$.

Two circle functions $f_{c1} : [0, 2\pi] \rightarrow S^1, f_{c2} : [0, 2\pi] \rightarrow S^1$ are called circle homotopic if there exits a homotopy given by $F_c : [0, 2\pi] \times [0,1] \rightarrow S^1$ such that the following condition is maintained: $\forall t \in [0,1], F_c(0,t) = F_c(2\pi, t)$. In the topological space $(X, \tau_X)$, if $A \subset X$ is a subspace then $v : X \rightarrow A$ is a retract if and only if $v(.)$ is continuous and the identity $v(a \in A) = a$ is maintained.

## 3. Connected Fundamental Groups and Decomposed Homotopy

In this section, a set of definitions are formulated for topological path embeddings, homotopy decompositions, and the generation of decomposed quotient topological spaces. In this paper, sets $\Re$ and $Z$ represent sets of reals and integers, respectively. Note that, we consider two separable Hausdorff topological spaces $(X, \tau_X)$ and $(Y, \tau_Y)$ for the constructions. The open set $U_x \subset X$ denotes neighbourhood of $x \in X$ and the similar concept also applies to $(Y, \tau_Y)$ for some point in the corresponding topological space. If $A$ is topologically homeomorphic to $B$ then it is denoted by $\hom(A, B)$. It is important to note that, the proposed definitions and results consider multiple connected as well as homeomorphic fundamental groups, where at least one group is circle homotopy rigid. This indicates that, if we consider $\pi_1(X, x_\alpha)$ as a fundamental group in $(X, \tau_X)$, then it is homotopy rigid with respect to a set of circle groups $G_S = \{S^1\big|_i : i \in Z^+\}$ such that $\forall h \in [f]_{x\alpha}$ is closed and there exists a homotopy equivalence $\chi : S^1\big|_i \rightarrow (A \subset X)$ with homeomorphism $\hom(S^1, A)$, where the equivalence relation is preserved as $A \cong h$.

### 3.1. Topological Path Embedding

The topological path embedding closely follows the concept of generalized curve embeddings within topological spaces [15]. Let $A$ be an arbitrary set and $(X, \tau_X)$ be a Hausdorff topological space. The continuous injective function $l : A \rightarrow X$ is a topological path embedding such that $l(a \in A) \neq l(b \in A)$ if $a \neq b$. The functional composition involving such topological path embedding is employed in later sections to construct homeomorphic curve embedding and associated structures for homotopy decompositions. In particular, we are interested in arbitrary open paths (i.e., not loops) generated from the corresponding arbitrary set. This means that, the topological open path embedding does not involve any homotopy class. However, in order to generate such path embedding with homeomorphism to a given arbitrary curve represented by a function, the open paths are composed from the function with homeomorphism as defined below.

### 3.2. Homeomorphic Embedding of Curve

Let $f : [0,1] \to A$ be a continuous open curve. The topological embedding $(l \circ f) : [0,1] \to (X, \tau_X)$ is a homeomorphic open path embedding if $\forall x \in (l \circ f), \exists y \in A$ such that $U_y \subseteq l^{-1}(U_x)$, where $(l \circ f)(.)$ is injective and continuous in $(X, \tau_X)$.

Specifically, we consider $\exists \{x_0\}, \{x_1\} \in \tau_X$ such that $x_0 = (l \circ f)(0)$ and $x_1 = (l \circ f)(1)$ are distinct points in the embedded open curve. This paper considers multiple fundamental groups in $(X, \tau_X)$, which are connected through such points.

### 3.3. Connected Fundamental Groups

Let $\pi_1(X, x_\alpha)$ and $\pi_1(X, x_\beta)$ be two fundamental groups in $(X, \tau_X)$ such that $\{A_\alpha, A_\beta\} \subset \tau_X$ with $\{x_\alpha\} \subset A_\alpha$ and $\{x_\beta\} \subset A_\beta$. The fundamental groups $\pi_1(X, x_\alpha)$ and $\pi_1(X, x_\beta)$ are called *connected* in $(X, \tau_X)$ if $x_0 = x_\alpha$ and $x_1 = x_\beta$.

It is considered that $\exists W \subset X, V \subset X$ such that $W \cap V \neq \phi$ and $A_\alpha \subset W, A_\beta \subset V$, where $A_\alpha \cap A_\beta = \phi$. Thus, the fundamental groups are placed within the separation of subspaces in a path-connected topological space.

**Remark 1:** Evidently, there exist two path-homotopies $F_0 : [0,1]^2 \to (A_0 \subset X, \tau_X)$ and $F_1 : [0,1]^2 \to (A_1 \subset X, \tau_X)$ generating the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ in $(X, \tau_X)$. The corresponding homotopy classes associated to $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are denoted by $[h]_{x0}$ and $[h]_{x1}$ respectively. For the simplicity of notation, the fundamental groups in $(X, \tau_X)$ are distinguished by indicated base points within the topological space.

### 3.4. Decomposed Homotopy Loop

If the homotopy loop $h_{i \in Z_0^+} \in [h]_{x0}$ then the decomposition of $h_i$ is given by $D_{hi} = \{E_{i1}, E_{i2}, \{x_0\}\}$ such that:

$$\begin{aligned} &\bigcup_B (B \in D_{hi}) = h_i, \\ &E_{i1} \cap E_{i2} \neq \phi, \\ &x_0 \notin E_{i1} \cup E_{i2}. \end{aligned} \tag{1}$$

**Remark 2:** In this paper we consider the equivalence relation $h_i \cong S^1$ allowing homotopy rigid. However, it is maintained that $\hom(h_k, S^1)$ for $h_{k \in Z_0^+} \in [h]_{x1}$ if not specified otherwise in some cases. Moreover, the connected variety of decomposition is prepared in a way so that $|E_{i1} \cap E_{i2}| = 1$, where $E_{i1}, E_{i2}$ are half-open. Note that, if the decomposition is fully disconnected then $E_{i1} \cap E_{i2} = \phi$ and the decomposed components are open. In any case, the decomposition maintains the condition given as $E_{i1} \subset S^1, E_{i2} \subset S^1$ by following homotopy rigid.

Once the decomposition of a homotopy loop is completed, the quotient topological maps can be performed to generate quotient topology under decomposition. In this case, two separable (i.e., not path connected) topological spaces are considered which are denoted by $(X, \tau_X)$ and $(Y, \tau_Y)$, where $X \cap Y = \phi$.

### 3.5. Decomposed Quotient Topology

Let the two separable Hausdorff topological spaces be given by $(X, \tau_X)$ and $(Y, \tau_Y)$. The surjective quotient map $\gamma : D_{hi} \to (Y, \tau_Y)$ is called a decomposed quotient map if the following conditions are satisfied:

$$\begin{aligned} &\{A_0, A_{in}\} \subset \tau_Y, n \in \{1, 2\}, \\ &\gamma(E_{in}) = a_{in} \in A_{in}, \\ &\gamma(\{x_0\}) = a_0 \in A_0. \end{aligned} \tag{2}$$

The concept of decomposed quotient map enforces surjective property of the map while generating quotient topology from the partitioned set-valued domain. It is illustrated in later sections of this paper that the generated quotient topology under decomposition can retain discrete group algebraic structures. However, first, we present a definition of the related cyclic generator and the corresponding concept of cycle number.

### *3.6. Cyclic Generator and Cycle Number*

Let $h_i \in [h]_{x0}$ be a homotopy loop in $\pi_1(X, x_0)$ in $(X, \tau_X)$ and $\{g\}$ be an isolated point set. The function $\mu : h_i \to \{g\}$ is defined as:

$$\begin{aligned} &m, n, p \in Z^+, \mu(x \in h_i) = g, \\ &g^m \equiv a_{i1}, m = 1, 4, \dots, \\ &g^n \equiv a_{i2}, n = 2, 5, \dots, \\ &g^p \equiv a_0, p = 3, 6, \dots \end{aligned} \tag{3}$$

Hence, the element $g$ is the cyclic generator of $\gamma(D_{hi})$ within $(Y, \tau_Y)$ preparing a cyclic group itself represented as $G_g = (\mu(h_i), \cdot)$. The cycle number $r \in Z^+$ denotes the repetition of cycle $g^{4r}$ in $\gamma(D_{hi})$.

**Remark 3:** It is easy to verify that if $r = 1$ then $g^4$ completes a full closed cyclic sequence in $\gamma(D_{hi})$. Note that $g^3$ is identified with the identity element.

## 4. Main Results

A set of main results are presented in this section. It is important to note that the decomposed quotient map between two separable topological spaces does not inherently generate group structure. If the homotopy decomposition is a connected variety then the quotient map cannot be a uniform surjection and the co-domain cannot induce a group structure. This observation is presented in next theorem.

**Theorem 1:** *The decomposed quotient topology $\gamma(D_{hi})$ cannot induce group structure from $h_i \cong S^1$ in $(X, \tau_X)$ to $(Y, \tau_Y)$ maintaining uniform surjection.*

**Proof:** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two Hausdorff topological spaces such that $X \cap Y = \phi$. Let $\pi_1(X, x_0)$ be a fundamental group in subspace $(A_0 \subset X, \tau_X)$ associated to homotopy class $[h]_{x0}$ with $h_i \cong S^1$, where $h_i \in [h]_{x0}$. If $\gamma : D_{hi} \to (Y, \tau_Y)$ is a decomposed quotient map, then $\exists B_1 \subset E_{i1}$ and $\exists B_2 \subset E_{i2}$ such that $\gamma(B_1) = a_{i1} \in A_{i1}$ and $\gamma(B_2) = a_{i2} \in A_{i2}$, where $\{A_{i1}, A_{i2}\} \subset \tau_Y$. Suppose, $\exists a_0 \in A_0 \in \tau_Y$ such that it maintains $\gamma(\{x_0\}) = a_0$ in $(Y, \tau_Y)$. Let us assume that, $G = (\gamma(D_{hi}), *_Y)$ be a group of order 3, where $*_Y : \gamma^2 \to \gamma$ is closed in $(Y, \tau_Y)$. Furthermore, let us consider that in the decomposed quotient topological subspace the following group algebraic properties hold as, $(a_{i1} \in \gamma) = (a_{i2}^{-1} \in \gamma)$ and $a_0 \in \gamma, (a_{in} *_Y a_0) = (a_0 *_Y a_{in}) = a_{in}$ with $n \in \{1, 2\}$. However, the definition of $D_{hi}$ indicates that $\exists x \in h_i$ such that $\gamma(\{x\}) = A_x \in \tau_Y$, where $A_x = \{a_{i1}, a_{i2}\} \subseteq A_{i1} \cup A_{i2}$ and $x \in E_{i1} \cap E_{i2}$. If $h_i \cong S^1$ condition is maintained in $\pi_1(X, x_0)$, then $\gamma(.)$ is not invertible and it is a multi-valued surjection. This leads to the contradiction about formation of induced $G = (\gamma(D_{hi}), *_Y)$ due to decomposed quotient map, because it is not a uniform surjection. Hence, as a consequence $G = (\gamma(D_{hi}), *_Y)$ is not an induced group structure in the decomposed quotient topology in $(Y, \tau_Y)$ under uniform surjection. □

**Example 1:** The following example illustrates the concept. Let $S^1$ be a unit circle group in the complex z-plane and $h_i \cong S^1$. In $G = (S^1, \cdot)$ multiplicative group structure if $x_\theta = e^{i\theta}$ in $h_i \cong S^1$ then $\forall x_\theta \in h_i, \exists x_\theta^{-1} \in h_i$ such that $x_\theta . x_\theta^{-1} = x_\theta^{-1} . x_\theta = e^{2i\pi} \in S^1$. Moreover, it is true that $\forall e^{\pm i\theta} \in S^1, e^{\pm i\theta} . e^{2i\pi} = e^{2i\pi} . e^{\pm i\theta} = e^{\pm i\theta}$. Suppose, $E_{i1}, E_{i2}$ are half-open in $S^1$ and $\{x_0\} = \{e^{2i\pi}\}$ for $\pi_1(X, x_0)$. If $E_{i1} \cap E_{i2} = \{e^{i\pi}\}$ condition is maintained in $D_{hi}$ by following concept of connected decomposition then it leads to $e^{i\pi} . e^{i\pi} = e^{2i\pi} \in S^1$. Moreover, in $D_{hi}$, one can select $x_\theta \in E_{i1} \backslash \{e^{i\pi}\}$ and $x_\theta^{-1} \in E_{i2} \backslash \{e^{i\pi}\}$.

As a result, if we consider $\gamma(D_{hi}) = \{y_\theta, y_\theta^{-1}, y_0\} \subset Y$ such that $y_\theta = \gamma(\{x_\theta\})$ and $y_\theta^{-1} = \gamma(\{x_\theta^{-1}\})$, then assuming $G = (h_i \cong S^1, \cdot)$ exists one can conclude that $y_0 = \gamma(\{x_0\}) = \gamma(\{x_\theta.x_\theta^{-1}\})$, where $y_0 = \gamma(\{x_\theta\}).\gamma(\{x_\theta^{-1}\}) = y_\theta.y_\theta^{-1}$. However, this leads to contradiction because $\gamma(\{e^{i\pi}\}) = \{y_\theta, y_\theta^{-1}\}$ is a multi-valued surjection maintaining $E_{i1} \cap E_{i2} = \{e^{i\pi}\}$ and $y_\theta.y_\theta^{-1} = y_0$. Hence $\gamma(.)$ is not a uniform surjection inducing a group structure in $(Y, \tau_Y)$.

However, if the homotopy decomposition is a disconnected variety then the surjection is uniform, and the decomposed quotient map induces a group structure. This property is presented as a corollary below.

**Corollary 1:** If the decomposition of $h_i \in [h]_{x0}$ is fully disconnected such that $E_{i1} \cap E_{i2} = \phi$ then there is uniform quotient surjection $\gamma(.)$ inducing a group $G = (\gamma(D_{hi}^1), *)$ of order 3 in $(Y, \tau_Y)$, where $D_{hi}^1$ is the disconnected decomposition.

**Proof:** Let $D_{hi}^1$ be a disconnected decomposition of $h_i \in [h]_{x0}$ considering fundamental group $\pi_1(X, x_0)$ in the topological space $(X, \tau_X)$. Thus, in $D_{hi}^1$ the decomposition maintains $E_{i1} \cap E_{i2} = \phi$, where both $E_{i1}, E_{i2}$ are open sets. If the disconnected decomposition is formulated as, $D_{hi}^1 = \{E_{i1}, E_{i2}, \{x_0, x_1\}\}$ such that $\bigcup_B (B \in D_{hi}^1) = h_i$ and the corresponding uniform surjection is given by $\gamma : D_{hi}^1 \to (Y, \tau_Y)$ generating quotient topology such that $\gamma(E_{i1}) = a_{i1}, \gamma(E_{i2}) = a_{i2}$ and $\gamma(\{x_0, x_1\}) = a_0$ then $G = (\gamma(D_{hi}^1), *)$ is a group of order 3 if and only if $a_{i1} = a_{i2}^{-1}$ and $a_{in} * a_0 = a_0 * a_{in} = a_{in}$, where $n \in \{1, 2\}$. □

**Remark 4:** It is interesting to note that, the multi-valued surjection (i.e., non-uniform surjection) can be transformed under function composition to induce a group structure in quotient topological space. Let $P(A_i)$ be a power set of $A_i = \{a_{i1}, a_{i2}, a_0\}$ and $\delta : P(A_i) \to A_i$ be a single valued function following the concept of axiom of choice [16,17]. If we restrict $\delta : P(A_i) \to A_i$ such that $\delta(B \in P(A_i)) = b \in B$ if $|B| = 1$ and apply axiom of choice if $|B| > 1$, then $(\delta \circ \gamma) : D_{hi} \to (Y, \tau_Y)$ can induce a group $G = ((\delta \circ \gamma)(D_{hi}), *)$ in quotient subspace in $(Y, \tau_Y)$.

In general, the orientations of various sets of path homotopies influence the behaviour of homotopic product $(*_H)$ of functions in homotopy classes in the fundamental groups. For example, $h_i *_H \overline{h}_i = x_0$ in $\pi_1(X, x_0)$, where $\overline{h}_i$ is the orientation reversing. However, the orientations in decomposed homotopy paths also influence the formation of induced groups in quotient topological space. The appropriately oriented homotopy paths in decomposed homotopy loops can directly induce additive group of order 3 in the decomposed quotient topological spaces. This observation is presented in next theorem.

**Theorem 2:** *If $s : [0,1] \to h_i$ and $q : [0,1] \to h_i$ are two orientation reversing paths in decomposed homotopy loop in $[h]_{x0}$ then there exists decomposed quotient map $\gamma : D_{hi} \to (Y, \tau_Y)$ generating $G = (\gamma(D_{hi}), +)$ in quotient topological space.*

**Proof:** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two Hausdorff topological spaces, which are not path connected. Let $\pi_1(X, x_0)$ be a fundamental group in $(X, \tau_X)$ and $h_i \in [h]_{x0}$ in the homotopy class such that $h_i \cong S^1$. If $s : [0,1] \to h_i$ and $q : [0,1] \to h_i$ are two orientation reversing paths in decomposed homotopy loop $D_{hi}$ of $h_i \in [h]_{x0}$, then one can define them as:

$$
\begin{aligned}
s(t) &= e^{i\pi t}, t \in [0,1], \\
q(t) &= e^{-i\pi t}, t \in [0,1].
\end{aligned}
\tag{4}
$$

Let us prepare the corresponding decomposition $D_{hi}$ as given below:

$$\begin{aligned} &E_{i1} = \{s(t) : t \in (0,1]\}, \\ &E_{i2} = \{q(t) : t \in (0,1]\}, \\ &x_0 = s(0) = q(0). \end{aligned} \tag{5}$$

This preserves the connected decomposition condition that, $E_{i1} \cap E_{i2} = \{s(1)\} = \{q(1)\}$. Suppose $\gamma : D_{hi} \to (Y \subset Z, \tau_Y)$ is a decomposed quotient map in integer space (Z) such that $\gamma(E_{i1}) = (i\pi t)^{-1} \ln s(t)$ and $\gamma(E_{i2}) = (i\pi t)^{-1} \ln q(t)$. Thus, it leads to $\gamma(E_{i1} \cup E_{i2}) = \{-1, 1\} \subset Y$ and $\gamma(E_{i1} \cap E_{i2}) \subset \{-1, 1\}$. Moreover, suppose the decomposed quotient map maintains $\gamma(\{x_0\}) = 0 \in Y$. Hence, the generated quotient topological subspace from orientation reversing paths in the decomposed homotopy loop induces a group $G = (\gamma(D_{hi}), +)$, where $\gamma(D_{hi}) = \{-1, 0, 1\}$ in $(Y \subset Z, \tau_Y)$. □

The induced group structure formation in quotient topological spaces from the decomposed homotopy loop can be further extended involving multiple fundamental groups in a connected topological space. Suppose $(X, \tau_X)$ is a path connected topological space and two disjoint fundamental groups are presented by $\pi_1(X, x_0)$, $\pi_1(X, x_1)$ within two separable subspaces (because $(X, \tau_X)$ topological space is Hausdorff). It can be illustrated that the second quotient topological subspace can acquire a group structure from the first one if a bijective composition can be established between the two subspaces. This is explained in the next theorem.

**Theorem 3:** *If $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are two fundamental groups in a path connected topological space $(X, \tau_X)$ with two decomposed quotient maps $\gamma : D_{hi} \to (Y, \tau_Y)$, $\gamma : D_{hk} \to (Y, \tau_Y)$ for the two corresponding homotopy loops $h_i \in [h]_{x0}$, $h_k \in [h]_{x1}$ then there exists a bijection $\beta : \gamma(D_{hk}) \to \gamma(D_{hi})$ such that $G^1 = ((\beta \circ \gamma)(D_{hk}), +)$ is a group of order 3.*

**Proof:** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two Hausdorff separated topological spaces ($X \cap Y = \phi$). Let $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ be two fundamental groups in $(X, \tau_X)$, where $\gamma : D_{hi} \to (Y, \tau_Y)$ and $\gamma : D_{hk} \to (Y, \tau_Y)$ are two decomposed quotient maps from $(X, \tau_X)$ to $(Y, \tau_Y)$. Suppose, the two corresponding disjoint homotopy loops are $h_i \in [h]_{x0}$ and $h_k \in [h]_{x1}$. If the decompositions are denoted by $D_{hi} = \{E_{i1}, E_{i2}, \{x_0\}\}$ and $D_{hk} = \{E_{k1}, E_{k2}, \{x_1\}\}$. Let there be a bijection $\beta : \gamma(D_{hk}) \to \gamma(D_{hi})$ between the quotient topological subspaces. One can formulate the bijection as:

$$\begin{aligned} &a_{kn}, a_1 \in \gamma(D_{hk}), n \in \{1, 2\}, \\ &\beta(a_{kn}) = \beta(a_{in}), \\ &\beta(a_1) = a_0. \end{aligned} \tag{6}$$

Recall that $G = (\gamma(D_{hi}), +)$ is a group of order 3. This leads to the conclusion that:

$$\begin{aligned} &n, n+1 \in \{1, 2\}, \\ &\beta(a_{kn}) + \beta(a_{k(n+1)}) = \beta(a_1), \\ &\beta(a_1) + \beta(a_{kn}) = \beta(a_{kn}), \\ &\beta(a_1) + \beta(a_{k(n+1)}) = \beta(a_{k(n+1)}), \\ &\beta(a_1) + \beta(a_1) = \beta(a_1). \end{aligned} \tag{7}$$

Hence, the bijective composition structure $G^1 = ((\beta \circ \gamma)(D_{hk}), +)$ is also a group in $(Y \subset Z, \tau_Y)$. □

**Remark 5:** It is important to note that, in the above theorem we have considered that $h_i \cong S^1$; however we have not put condition that $h_k \cong S^1$ in $[h]_{x1}$. The above theorem is valid as long as $\hom(h_k, S^1)$ condition is maintained.

Evidently, multiple fundamental groups in a topological space can induce distributed disjoint groups within decomposed quotient spaces. The following corollary illustrates that there exists an isomorphism between groups in decomposed quotient spaces if they have equal order.

**Corollary 2:** If $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are topologically homeomorphic in $(X, \tau_X)$ then $G = (\gamma(D_{hi}), +)$ and $G^1 = ((\beta \circ \gamma)(D_{hk}), +)$ are isomorphic in $(Y, \tau_Y)$.

**Proof:** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ are two Hausdorff topological spaces, where $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are disjoint topologically homeomorphic fundamental groups in $(X, \tau_X)$. Suppose homotopy loop $h_i \in [h]_{x0}$ exists such that $h_i \cong S^1$. If $f_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a homeomorphism then $f_*[h]_{x0} = [h]_{x1}$. Note that $G = (\gamma(D_{hi}), +)$ with order 3 is generated in decomposed quotient topological space of $h_i \in [h]_{x0}$. However, due to homeomorphism it is true that $f_*(h_i) = h_k \in [h]_{x1}$ and the structure $G^1 = ((\beta \circ \gamma)(D_{hk}), +)$ also has order 3 in $(Y, \tau_Y)$. This leads to conclusion that, $G^1 \cong G$ (isomorphic) in topological space $(Y, \tau_Y)$. □

We have so far dealt with the properties of multiple fundamental groups and associated decomposed quotient mappings within the topological spaces. However, earlier it is mentioned that we are considering connected fundamental groups and such groups are placed in a connected topological space. Moreover, the decomposed quotient spaces are generated in a separable topological space. Hence, it is interesting to analyze the inherent locality of homeomorphism of the topological spaces in this structural setting. The following theorem shows that an equivalence relation between the two path embeddings exists.

**Theorem 4:** *If* $\eta_X : (A \subset X, \tau_X) \to (B \subset Y, \tau_Y)$ *is a local homeomorphism in topological subspaces such that* $(l \circ f)([0, 1]) \subset A$ *then there exists* $\eta_Y : [0, 1] \to B$ *such that* $\eta_X(A)\big|_{(l \circ f)} \cong \eta_Y([0, 1])$.

**Proof:** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two separated Hausdorff topological spaces, where $\eta_X : A \to B$ is a local homeomorphism in topological subspaces $A \subset X, B \subset Y$. Suppose $(l \circ f) : [0, 1] \to A$ is a continuous embedding such that $(l \circ f)(0) = x_0$ and $(l \circ f)(1) = x_1$, where $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are two fundamental groups. As $\eta_X : A \to B$ is a local homeomorphism in topological subspaces, so $\forall b \in \eta_X(.), \exists a \in A$ such that $U_a \subseteq \eta_X^{-1}(U_b)$ and $\eta_X(.)$ is a continuous bijection. Let $\eta_Y : [0, 1] \to B$ be a continuous function such that $\eta_Y(0) = y_0 \in Y$ and $\eta_Y(1) = y_1 \in Y$, where $\eta_X(x_0)\big|_{(l \circ f)} = \eta_Y(0)$ and $\eta_X(x_1)\big|_{(l \circ f)} = \eta_Y(1)$. This leads to the conclusion that $\forall b \in (l \circ f)(.), \exists a \in \eta_Y(.)$ such that $U_b \subseteq \eta_X^{-1}(U_a)$. As a result, the equivalence relation $\eta_X(A)\big|_{(l \circ f)} \cong \eta_Y([0, 1])$ is maintained in respective topological subspaces. □

**Lemma 1:** The following observations can be made further from the above theorem: if $\gamma(D_{hi}) \subset Y_A \subset Y, \gamma(D_{hk}) \subset Y_B \subset Y$ such that $Y_A \cap Y_B = \phi$ then $G = (\gamma(D_{hi}), +)$ and $G^1 = ((\beta \circ \gamma)(D_{hk}), +)$ are not path connected, however if $U_{a0} \subset Y_A, a_0 \in U_{a0}$ and $U_{a1} \subset Y_B, a_1 \in U_{a1}$ are each locally path connected open neighbourhoods, then $U_{a0} \cup U_{a1}$ is a connected subspace in $(Y, \tau_Y)$. As a result, $U_{a0}, U_{a1}$ are locally dense sets.

**Proof:** The topological space $(Y, \tau_Y)$ is Hausdorff and $\exists U_{a0}, U_{i1}, U_{i2} \subset Y_A$ such that $\overline{U_{a0}} \cap \overline{U_{i1}} = \phi, \overline{U_{a0}} \cap \overline{U_{i2}} = \phi$ and $\overline{U_{i1}} \cap \overline{U_{i2}} = \phi$. The similar property holds for $Y_B$. Thus the subspaces $Y_A$ and $Y_B$ are consisting of countable dense sets if $\forall x \in \gamma(D_{hi}) \cup \gamma(D_{hk})$ the open neighbourhoods maintain $U_x \cup \partial U_x = \overline{U_x}$. Hence, the subspaces $U_{a0} \subset Y_A, a_0 \in U_{a0}$ and $U_{a1} \subset Y_B, a_1 \in U_{a1}$ are connected by $\eta_Y([0, 1])$. Moreover, $U_{a0}, U_{a1}$ are locally dense and are also locally path connected. □

There exists a set of homotopically equivalent paths between topological spaces containing connected fundamental groups and decomposed quotient subspaces. The following theorem presents this property considering $*_H$ as homotopic function product. Note that $\cong_H$ denotes the equivalence of path homotopy between two homotopic paths in the topological space.

**Theorem 5:** *If $f_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a homeomorphism in $(X, \tau_X)$ with $\alpha \cong (l \circ f)([0,1])$ and $\beta_G : G \to G^1$ is a bijection in $(Y, \tau_Y)$ then there are oriented homotopy paths in $p_0 : \pi_1(X, x_1) \to \pi_1(X, x_0)$ and $p_1 : \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that $(\beta_G \circ \gamma)(w) = \gamma(v)$, where $w \equiv \alpha *_H f_*[h]_{x0} *_H \overline{\alpha}$ and $v \equiv \overline{\alpha} *_H [h]_{x0} *_H \alpha$.*

**Proof:** Let $f_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ be a homeomorphism in topological space $(X, \tau_X)$ such that $\alpha \cong (l \circ f)([0,1])$ is an isomorphic path in $(X, \tau_X)$. Let $\beta_G : G \to G^1$ be a bijection in $(Y, \tau_Y)$ and $\gamma(.)$ is a decomposed quotient map between $(X, \tau_X)$ and $(Y, \tau_Y)$. Suppose one defines the oriented homotopy paths in $p_0 : \pi_1(X, x_1) \to \pi_1(X, x_0)$ and $p_1 : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as $p_0[h]_{x1} = ([\alpha] *_H [\overline{\alpha}]) *_H [h]_{x0}$ and $p_1[h]_{x0} = ([\overline{\alpha}] *_H [\alpha]) *_H [h]_{x1}$ respectively. However, as $f_*[h]_{x0} = [h]_{x1}$ thus $p_1[h]_{x0} = ([\overline{\alpha}] *_H [\alpha]) *_H f_*[h]_{x0}$ is in $(X, \tau_X)$. Moreover, if the constant path is defined as $e_{xn} : [0,1] \to (x_n \in X), n \in \{0,1\}$ such that $e_{x0}(t \in [0,1]) = x_0 = ([\alpha] *_H [\overline{\alpha}])$ and $e_{x1}(t \in [0,1]) = x_1 = ([\overline{\alpha}] *_H [\alpha])$ then there are equivalences of path homotopies denoted by $e_{x1}([0,1]) \cong_H p_0[h]_{x1} *_H [\alpha]$ and $e_{x0}([0,1]) \cong_H p_1[h]_{x0} *_H [\overline{\alpha}]$. Thus, if one selects two homotopic paths $w \equiv \alpha *_H f_*[h]_{x0} *_H \overline{\alpha}$ and $v \equiv \overline{\alpha} *_H [h]_{x0} *_H \alpha$, then $w = x_0$ and $v = x_1$. This leads to the conclusion that $(\beta_G \circ \gamma)(w) = \gamma(v)$ in $(Y, \tau_Y)$. □

**Remark 6:** Note that in the homotopy classes the following condition is maintained: $f_*([\overline{\alpha}] *_H [h]_{x0}) = f_*[\overline{\alpha}] *_H f_*[h]_{x0}$. However, it is easy to verify that $f_*[h]_{x0} = p_1[h]_{x0}$ and $p_1[h]_{x0} = ([\overline{\alpha}] *_H [\alpha]) *_H [h]_{x1}$. This leads to the condition that $f_*[\overline{\alpha}] = f_*[\overline{\alpha}] *_H ([\overline{\alpha}] *_H [\alpha])$. Hence, one can conclude that $f_*[\overline{\alpha}] = f_*[\overline{\alpha}] *_H e_{x1}([0,1])$ and $[\overline{\alpha}] *_H [\alpha] \neq [\alpha] *_H [\overline{\alpha}]$. Moreover, the following commutativity is maintained: $f_*[\overline{\alpha}] *_H e_{x1}([0,1]) = e_{x1}([0,1]) *_H f_*[\overline{\alpha}]$.

The connectedness of fundamental groups gives rise to an interesting property. It indicates that it is possible to formulate a homotopically equivalent path involving multiple homotopy classes. This property is illustrated in next theorem.

**Theorem 6:** *If $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are two connected fundamental groups in topological space $(X, \tau_X)$ with $h_{il} \in [h]_{x0}, h_{km} \in [h]_{x1}$ and $\alpha \cong (l \circ f)([0,1])$ then $\exists -a, b \in \mathfrak{R}$ such that $\lambda : [-a, b] \to (X, \tau_X)$ exists, where $\lambda([-a, b]) \cong_H h_{il} *_H \alpha *_H h_{km}$ for some $l, m \in Z^+$.*

**Proof:** Let $(X, \tau_X)$ be a topological space and $\pi_1(X, x_0), \pi_1(X, x_1)$ are two connected fundamental groups. Consider $\alpha \cong (l \circ f)([0,1])$ as a path for the corresponding embedding in $(X, \tau_X)$. Suppose $\exists -a, b \in \mathfrak{R}, b > 1$ such that there exist two disjoint path homotopies given by $F_i : [-a, 0]^2 \to (X, \tau_X)$ and $F_k : [1, b]^2 \to (X, \tau_X)$. As result, one can consider $l, m \in Z^+$, $\forall h_{il} \in [h]_{x0}, \forall h_{km} \in [h]_{x1}$ to formulate the sets of homotopic functions, which is given as:

$$\begin{aligned}
&l \leq L, m \leq M, \\
&h_{il} : [-a, 0] \to (X, \tau_X), \\
&h_{km} : [1, b] \to (X, \tau_X), \\
&h_{il}(-la) = h_{il}(0) = x_0, \\
&h_{km}(mb) = h_{km}(1) = x_1.
\end{aligned} \tag{8}$$

Suppose a continuous function $\lambda : [-a, b] \to (X, \tau_X)$ exists such that $\lambda(-a) = x_0, \lambda(b) = x_1$. Moreover, the continuous function maintains the condition, $\lambda([0,1]) = (l \circ f)([0,1])$ in the topological space $(X, \tau_X)$. Furthermore, the continuous function $\lambda : [-a, b] \to (X, \tau_X)$ maintains the following properties:

$$\begin{aligned}
&\lambda(t \in [-a, 0]) = h_{il}(lt), l \in [1, L], \\
&\lambda(t \in [1, b]) = h_{km}(mt), m \in [1, M].
\end{aligned} \tag{9}$$

Let $A \subset X$ be a subspace and there are subspaces given by $B_i = \{x \in X : x \in h_{il} \in [h]_{x0}\}, B_k = \{x \in X : x \in h_{km} \in [h]_{x1}\}$ such that the combined subspace is given by, $A = (l \circ f)([0,1]) \cup B_i \cup B_k$. Thus $\lambda : [-a, b] \to A$ is continuous in the corresponding topological subspace. As a result, if we consider a homotopy path $h_{il} *_H \alpha *_H h_{km}$ in $A \subset X$ then it will maintain equivalence of path homotopy as $\lambda([-a, b]) \cong_H h_{il} *_H \alpha *_H h_{km}$.

There exists a relation between isolated point set $\{g\}$ and the decomposed quotient topological space if the element of the point set is a generator of the induced group in the decomposed quotient space. This relation has an effect on the finiteness of cycle number. This observation is presented in the following theorem. □

**Theorem 7:** *If* $h_i : [0,1] \to (X, \tau_X)$ *generates a path homotopy in fundamental group* $\pi_1(X, x_0)$ *then* $\prod_{t\in[0,1]} \mu(h_i(t))$ *induces infinite cycle number, i.e.,* $r \to +\infty$ *in* $\gamma(D_{hi})$.

**Proof:** Let $h_i : [0,1] \to (X, \tau_X)$ be a path homotopy in the fundamental group $\pi_1(X, x_0)$ in the topological space $(X, \tau_X)$. Note that, the interval $[0,1] \subset \Re^+$ is uncountable and compact real subspace in nature. As $h_i(0) = h_i(1)$ in $\pi_1(X, x_0)$, hence $\mu(h_i(0)).\mu(h_i(1)) = g^2$. Moreover, $\forall t \in (0,1)$ it is true that $\mu(h_i(t)) = g$ by the definition. If one chose $\varepsilon > 0, \varepsilon \in \Re^+$ and the corresponding interval $[t-\varepsilon, t+\varepsilon]$, then the interval is also uncountable. This leads to conclusion that, $\exists v \in Z^+$ and $\prod_{u\in[t-\varepsilon,t+\varepsilon]} \mu(h_i(u)) = \underset{v\to+\infty}{lt} g^v \in G_g$, where $G_g = (\mu(h_i), \cdot)$ in $(Y, \tau_Y)$. However, this further leads to the equality involving the cycle number of the cyclic generator that $r = \underset{v\to+\infty}{lt}(v/4)$, where $g^m = a_{i1}, g^n = a_{i2}, g^p = a_0$ and $m > 1, n > 1, p > 1$. Hence, the product $\prod_{t\in[0,1]} \mu(h_i(t))$ induces cycle number $r \to +\infty$ in $\gamma(D_{hi})$. □

This indicates that a continuous homotopic path in the fundamental group generates a countable infinite cycle number of cyclic generators of the group in the decomposed quotient topological subspace.

## 5. Discussion: Relation to Sierpinski Space

The Krull–Schmidt property of modular decomposition states that an object decomposes in a unique way such that a set of further indecomposable objects is generated [18]. This concept is maintained in the homotopy decomposition varieties presented in this paper in the quotient topological spaces. There is a relationship between the Sierpinski space and the proposed decomposed homotopy in quotient topological space (referred to as DHQ space). The proposed decomposed quotient space generated from the decomposed homotopy loop extends the Sierpinski space [19]. If the Sierpinski set is given by $S_P$ then the one-point extension of Sierpinski set is the set $S_{P\gamma} = S_P \cup \{-1\}$ supporting the additive group $G = (\gamma(D_{hi}), +)$ within the decomposed quotient topological subspace. Note that, the group in Sierpinski space is trivial, in general. However, the induced additive group in decomposed quotient topological is not necessarily trivial in nature. The one-point extension of Sierpinski space by the proposed decomposed quotient topological space is symmetric with respect to origin. As a result, the Sierpinski topology is also extended in decomposed quotient space due to symmetric one-point extension of the basis set. The extension of topological space is given by $E = \{\{-1\}, S_{P\gamma}\}$, which can be included in the original Sierpinski topological space. The summary of interrelationship between Sierpinski space and its extension in the decomposed quotient space is given in Table 1.

The extension is null if the space is the original Sierpinski space. Otherwise, the basis set is symmetrically extended with respect to origin and the topological space is extended by inclusion of resulting topological subspaces. Note that both the spaces are connected in nature in $\Re$. The interesting distinction between the two spaces is based on the uniqueness of non-zero sequence. In case of Sierpinski space, the non-zero sequence generates a unique cluster point and it is a constant sequence.

However, in the case of extended Sierpinski space, the non-zero sequences are not unique and as a result the cluster points are also not necessarily unique.

**Table 1.** Summary of interrelationship between Sierpinski space and DHQ space.

| Space/Set | Extension | Topology | Group | Connectedness | Compact | Sequentially Complete (Non-Zero Sequence) | Non-Zero Sequence |
|---|---|---|---|---|---|---|---|
| Sierpinski | $\{\phi\}$ | Sierpinski topology, $\tau_{SP}$ | Trivial | Yes | Compactible | Always | Unique cluster-point |
| Decomposed homotopy quotient | $E$ | $\tau_{SP\gamma} = \tau_{SP} \cup E$ | Non-trivial | Yes | Compactible | Not always | Not necessarily unique cluster-point |

Note that, the non-zero convergent sequences in extended Sierpinski space are constants alike original space. Interestingly, the Sierpinski space is sequentially complete in terms of non-zero sequence. However, it may not be always true in extended Sierpinski space.

## 6. Conclusions

The algebraic as well as topological properties of homotopy decomposition vary depending on the connectedness of the topological space. This paper proposes the homotopy decompositions of path connected fundamental groups into quotient topological spaces. The proposed decompositions have two varieties with respect to the connectedness of decomposed subspaces. The resulting decomposed quotient spaces in a topological space with separation from the base space containing fundamental groups give rise to the variations in algebraic and topological properties depending on the connectedness of homotopy decompositions. It is shown that, if the decomposed subspaces maintain path connectedness within the components, then the decomposed quotient topological space cannot induce an additive group structure maintaining uniform surjection. However, if the decomposition of homotopy is disconnected variety then it maintains uniform surjection and successfully induces an additive group structure within the decomposed quotient topological spaces. The proposed formulation is supported by fundamental group which is homotopy rigid with respect to a set of circle groups in a complex plane. It is illustrated that there exists bijection between multiple decomposed quotient topological spaces allowing formation of group structures under function composition. Interestingly, the proposed disconnected homotopy decomposition under quotient map extends the Sierpinski space within the decomposed quotient topological space while preserving the trivial group of Sierpinski space. The one-point extension of the Sierpinski space is symmetric with respect to origin. Furthermore, the cyclic generator of the group in the decomposed quotient topological space attains infinite cycle number when the decomposed homotopy generates locally connected open path components in homotopy class.

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