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# Trapezium-Type Inequalities for Raina's Fractional Integrals Operator Using Generalized Convex Functions

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**Abstract:** The authors have reviewed a wide production of scientific articles dealing with the evolution of the concept of convexity and its various applications, and based on this they have detected the relationship that can be established between trapezoidal inequalities, generalized convex functions, and special functions, in particular with the so-called Raina function, which generalizes other better known ones such as the hypergeometric function and the Mittag–Leffler function. The authors approach this situation by studying the Hermite–Hadamard inequality, establishing a useful identity using Raina's fractional integral operator in the setting of $\phi$-convex functions, obtaining some integral inequalities connected with the right-hand side of Hermite–Hadamard-type inequalities for Raina's fractional integrals. Various special cases have been identified.

**Keywords:** Hermite–Hadamard inequality; Raina's fractional integral operator; Hölder inequality; power mean inequality; generalized convexity

## 1. Introduction

In recent decades, the concept of convexity has had a more general evolution due to its wide application in various fields of science, as demonstrated in the following works [1–4]. Among the types of generalized convexity and its applications are some such as $h$-convexity, $MT$-convexity, $\eta-$convexity, $(s,m)$-convexity and others, which can be found in the following references [5–9]. Also, the use of special functions, in addition to those involving fractional integral operators, have been related to this topic [6,10–12]. The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.** *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $p, q \in I$ with $p < q$. Then the following inequality holds:*

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)dx \leq \frac{f(p)+f(q)}{2}. \tag{1}$$

This inequality has remained an area of great interest due to its wide applications, by example in the field of statistics and probability theory [3] and other fields of mathematical analysis [2]. In recent decades, some researchers have studied (1) under the premises of new definitions that have emerged in the development of the concept of convex function. Interested readers see the references [7,13–18].

Fractional calculus has also had a great interest in the scientific community and some applications has been studied by J. Hristov [19], D. Kumar et al. in [20] and K.M. Owolabi in [21]. In conjunction with the study of fractional integral inequalities, a line of research has developed, so that the literature about it has grown as seen in the following papers [5,6,10–12,22–26].

Also, the growing development of the concept of convex function has been related in the field of integral inequalities as shown in the following research papers [8,9,27–30]. From the perspective developed by M. A. Noor, in [7], in relation to $\phi$-convex functions, this work establishes some inequalities of the Hermite–Hadamard type.

Motivated by the excellent works mentioned above, we developed the following work establishing Hermite–Hadamard inequality and an identity for Raina's fractional integrals operator via generalized $\phi$-convex functions and, from these, some integral inequalities connected with the right-hand side of Hermite–Hadamard type inequalities for Raina's fractional integrals are established.

## 2. Preliminaries

In the work of Noor, M.A. [7] we find the following definitions related with the notion of $\phi$-convexity.

Let $K$ be a non-empty closed set in $\mathbb{R}^n$, and $K^\circ$ the interior of $K$. We denote by $\langle .,. \rangle$ and $\|.\|$ the inner product and norm on $\mathbb{R}$ respectively. Let $\phi : K \to \mathbb{R}$ be continuous mappings.

**Definition 1.** *Let $u \in K$. The set $K$ is said to be $\phi$-convex, if there exists a function $\phi$ such that*

$$u + te^{i\phi}(u - v) \in K$$

*for all $u, v \in K$ and $t \in [0,1]$.*

**Definition 2.** *A function $f$ defined on the $\phi$-convex set $K$ is said to be $\phi$-convex, if*

$$f(p + te^{i\phi}(q - p)) \leq (1 - t)f(p) + tf(q), \quad \forall\, p, q \in K, \quad t \in [0,1].$$

*The function $f$ is said to be $\phi$-concave iff $(-f)$ is $\phi$-convex. Please note that every convex function is $\phi$-convex but the converse does not hold in general.*

In [31], Raina R.K. introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),..}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \tag{2}$$

where $\rho, \lambda > 0, x \in \mathbb{C}$ with $|x| < R$, where is $R$ is a real positive constant, and $\sigma = (\sigma(1), .., \sigma(k), ..)$ is a bounded sequence of positive real numbers. Please note that if we take in (2) $\rho = 1, \lambda = 0$ and $\sigma(k) = ((\alpha)_k(\beta)_k)/(\gamma)_k)$ for $k = 0, 1, 2, \ldots$ , where $\alpha, \beta$ and $\gamma$ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol $(a)_k$ denote the quantity

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1) \ldots (a + k - 1), \quad k = 0, 1, \ldots,$$

and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we have the classical Hypergeometric Function (see [32,33]), that is

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k$$

also, if $\sigma(k) = (1, 1, 1, \ldots)$ with $\rho = \alpha, (Re(\alpha) > 0)$ , $\lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in (2) then we have the classical Mittag–Leffler function (see [32,33])

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k.$$

When it is provided that the series converges uniformly then we can differentiate term wise, also integrate, to obtain

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(wx^\rho) = x^{\lambda-n-1} \mathcal{F}^{\sigma}_{\rho,\lambda-n}(wx^\rho)$$

and

$$\int_0^x \cdots \int_0^x t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(wt^\rho)(dt)^n = x^{\lambda+n-1} \mathcal{F}^{\sigma}_{\rho,\lambda+n}(wx^\rho).$$

Also, the same author defined the following left-sided fractional integral operators as follows

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-t)^\rho\right] \varphi(t)dt, \quad (x > a) \tag{3}$$

where $a \in \mathbb{R}^+$; and in [34], Agarwal R.P., Luo M-J, and Raina, R.K. defined the right sided fractional integral operator as

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(t-x)^\rho\right] \varphi(t)dt, \quad (x < b)\,, \tag{4}$$

where $\lambda, \rho, b > 0$, $w \in \mathbb{R}$ and $\varphi$ is such that the integral on the right side exits.

Both $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi$ are bounded integral operators on $L_p(a,b)$,$(1 \leq p \leq \infty)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^\rho\right] < \infty;$$

indeed, for $\varphi \in L_p\left((a,b)\right)$ we have

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p$$

and

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p$$

where

$$\left\|\varphi\right\|_p = \left(\int_a^b |\varphi(x)|^p \, dx\right)^{1/p}.$$

Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann–Liouville fractional integrals $I^\alpha_{a+}$ and $I^\alpha_{b-}$ of order $\alpha$.

$$\left(I^\alpha_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t)dt, \qquad (x > a, \alpha > 0)$$

and

$$\left(I^\alpha_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \varphi(t)dt, \qquad (x < b, \alpha > 0)$$

follow from (3) and (4) setting $\lambda = \alpha, \sigma = (1, 0, 0, \ldots)$ and $w = 1$.

The main objective of this paper is to establish Hermite–Hadamard's inequalities for Raina's fractional integral operator using a similar method in [35] via generalized $\phi$-convex functions and we will investigate some integral inequalities connected with the right-hand side of the Hermite–Hadamard type inequalities for Raina's fractional integrals. At the end, a briefly conclusion is given.

## 3. Results

We define the following generalized $\phi$-convex set and function, using the class of functions defined by (2).

**Definition 3.** *A non-empty set $K$ is said to be a generalized $\phi$-convex set, if*

$$p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p) \in K, \qquad \forall\, p,q \in K, \quad t \in [0,1], \tag{5}$$

*where $\nu,\mu > 0$ and $\eta = (\eta(1),..,\eta(k),..)$ is a bounded sequence of positive real numbers.*

Please note that the expression $\mathcal{F}^{\eta}_{\nu,\mu}(q-p)$ indicates the evaluation of $q-p$ under the action of Raina's function. If $\mu = \nu = 1$, $\eta(1) = 1$ and $\eta(k) = 0$ for all $k \neq 1$, then the above $\phi$-convex set coincides with the classical convex set.

**Definition 4.** *A real-valued function $f$ defined on a $\phi$-convex set $K$ is said to be generalized $\phi$-convex, if*

$$f(p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \leq (1-t)f(p) + tf(q), \qquad \forall\, p,q \in K, \quad t \in [0,1]. \tag{6}$$

Again, if $\mu = \nu = 1$, $\eta(1) = 1$ and $\eta(k) = 0$ for all $k \neq 1$, then the generalized $\phi$-convex functions coincides with the classical convex function.

Hermite–Hadamard's inequalities using generalized $\phi$-convex function can be represented in Raina's fractional integral form as follows:

**Theorem 2.** *Let $\nu,\mu > 0$, $\eta = \{\eta(k)\}_{k=0}^{\infty}$ a bounded sequence of real numbers and $f : \left[p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right] \to \mathbb{R}$ be a non-negative, with $p,q \in \mathbb{R}$ and $p < q$ and $\mathcal{F}^{\eta}_{\nu,\mu}(q-p) > 0$.*

*If $f$ is a generalized $\phi$-convex function on $\left[p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right]$, then the following inequalities for Raina's fractional integral operator hold*

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\frac{1}{2}\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] f\left(\frac{2p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right) \\
&\leq \frac{1}{2}\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+,w}f\right)(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-,w}f\right)(p)\right] \\
&\leq \left[\frac{f(p)+f(q)}{2}\right] \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],
\end{aligned} \tag{7}$$

*for $\rho,\lambda > 0$, $w \in \mathbb{R}$ and a bounded sequence of positive real numbers $\sigma$ such that the function $\mathcal{F}^{\sigma}_{\rho,\lambda}$ be a non-negative function.*

**Proof.** Taking into account that the integral of a non-negative function is greater than any value of the function at any point of the integration interval, and since $f$ is a non-negative generalized $\phi$-convex function, then, using the definition of Raina's fractional integral operator and the change of variable $u = p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)$ we have

$$\begin{aligned}
&\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+,w}f\right)\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\\
&=\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\int_{p}^{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-u\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-u\right)^{\rho}\right]f(u)du\\
&=\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]f\left(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt\\
&\geq\left(\frac{1}{2}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\frac{1}{2}\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]f\left(\frac{2p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right)
\end{aligned}\tag{8}$$

Similarly, using the change of variable $u=p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)$ we have

$$\begin{aligned}
&\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-,w}f\right)(p)\\
&=\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\int_{p}^{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}(u-p)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(u-p)^{\rho}\right]f(u)du\\
&=\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]f\left(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt\\
&\geq\left(\frac{1}{2}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\frac{1}{2}\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]f\left(\frac{2p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right).
\end{aligned}\tag{9}$$

Adding inequalities (8) and (9) we obtain the left side of the inequality (7):

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\frac{1}{2}\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]f\left(\frac{2p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right)\\
&\leq\frac{1}{2}\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+,w}f\right)\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+\mathcal{F}^{\eta}_{\nu,\nu}(q-p)-,w}f\right)(p)\right].
\end{aligned}$$

For the proof of the right-side inequality in (7) we first note that if $f$ is generalized $\phi$-convex function, then

$$f\left(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\leq tf(p)+(1-t)f(q)$$

and

$$f\left(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\leq(1-t)f(p)+tf(q).$$

By adding these inequalities, we have

$$f\left(p+(1-t)\mathcal{F}^{\sigma}_{\rho,\lambda}(q-p)\right)+f\left(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\leq f(p)+f(q).\tag{10}$$

Then multiplying both sides of (10) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]f\left(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt\\
&\quad+\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\nu}(q-p)\right)^{\rho}t^{\rho}\right]f\left(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt\\
&\qquad\leq[f(p)+f(q)]\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]dt,
\end{aligned}$$

i.e.,

$$\frac{1}{2}\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\lambda}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+,w}f\right)\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda;p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-,w}f\right)(p)\right]$$
$$\leq \frac{[f(p)+f(q)]}{2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right].$$

The proof of this theorem is complete. □

**Remark 1.** *Letting $\lambda=\alpha$, $w=1$ and $\sigma(0)=1$ and $\sigma(k)=0, k>0$, then we have the Hermite–Hadamard inequality for generalized $\phi$-convex functions using the Riemann–Liouville fractional integral*

$$\frac{1}{2^{\alpha-1}\Gamma(\alpha)}f\left(\frac{2p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right)$$
$$\leq \frac{1}{2}\left(\frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\right)^{\alpha}\left[\left(\mathcal{I}^{\alpha}_{p+}f\right)\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+\left(\mathcal{I}^{\alpha}_{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-}f\right)(p)\right]$$
$$\leq \frac{[f(p)+f(q)]}{2\Gamma(\alpha+1)},$$

*and if $\alpha=1$ then we have the inequality for the classical Riemann integral*

$$f\left(\frac{2p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}{2}\right)\leq \frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\int_{p}^{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f(u)du \leq \frac{1}{2}\left[f(p)+f(q)\right].$$

*In addition, if $\nu=\mu=1$, $\eta(1)=1$ and $\eta(k)=0$ for $k\neq 0$ we have the classical Hermite–Hadamard inequality* (1)

## 4. Raina's Fractional Integral Inequalities for Generalized $\phi$-Convex Functions

For establishing some new results regarding the right side of Hermite–Hadamard type inequalities for Raina's fractional integrals via generalized $\phi$-convex functions we need to prove the following lemma.

**Lemma 1.** *Let $\nu,\mu>0$, $\eta=\{\eta(k)\}_{k=0}^{\infty}$ a bounded sequence of real numbers, and $p,q\in\mathbb{R}$ with $p<q$, and $f:\left[p,p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right]\to\mathbb{R}$ be a differentiable on $(p,p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))$, where $\mathcal{F}^{\eta}_{\nu,\mu}(q-p))>0$. Then the following equality holds*

$$\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right)$$
$$-\frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\right)$$
$$=\int_0^1(1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt$$
$$-\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt,$$

*for $\rho,\lambda>0$, $w\in\mathbb{R}$ and a bounded sequence of positive real numbers $\sigma$.*

**Proof.** Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\lambda}\, \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right] f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) \\ &- \frac{1}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\int_0^1 (1-t)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right] f\left(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt \end{aligned}$$

and with the change of variable $x = p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)$ it is obtained

$$\begin{aligned} I_1 &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) \\ &\qquad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\int_p^{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} (x-p)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\,(x-p)^{\rho}\right] f\,(x)\,dx \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) \\ &\qquad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)-;w} f\right)(p). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho} t^{\rho}\right] f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)dt \\ &= -\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\,(p) \\ &\qquad + \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w} f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)). \end{aligned}$$

Therefore,

$$\begin{aligned} & I_1 - I_2 \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) \\ &\qquad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)-;w} f\right)(p) \\ &+ \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f\,(p) \\ &\qquad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w} f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) \\ &- \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w} f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w} f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right). \end{aligned}$$

The proof is complete. □

Using Lemma 1, it can be obtained some interesting inequalities.

**Theorem 3.** *Let $\nu, \mu > 0$, $\eta = \{\eta(k)\}_{k=0}^{\infty}$ a bounded sequence of real numbers, and $p, q \in \mathbb{R}$ with $p < q$, and $f : \left[p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right] \to \mathbb{R}$ be a differentiable on $(p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p))$, where $\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) > 0$. If $|f'|^r$ is generalized $\phi$-convex function on $[p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)]$ for $r \geq 1$, then the following inequality for Raina's fractional integral operator holds*

$$\Bigg| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p))} \left(f\left(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) + f(p)\right)$$
$$- \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}} \left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\Bigg|$$

$$\leq \begin{cases} \left(|f'(p)| + |f'(q)|\right)\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3} + \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\right)\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right], & r = 1 \\ 2^{1/s}\left(|f'(q)|^r + |f'(p)|^r\right)^{1/r} \mathcal{F}^{\sigma_2^s}_{\rho,\lambda}[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}], & r > 1,\ \frac{1}{r} + \frac{1}{s} = 1, \end{cases}$$

*where*

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho + \lambda + 2} \text{ and } \sigma_2^s(k) = \frac{\sigma(k)}{(s(k\rho+\lambda)+1)^{1/s}} \text{ for } k = 1, 2, \cdots,$$

*for $\rho, \lambda > 0$, $w \in \mathbb{R}$ and a bounded sequence of positive real numbers $\sigma$.*

**Proof.** Let $r = 1$. Using Lemma 1, the triangular inequality and the generalized $\phi$-convexity of $|f'|$, we have

$$\Bigg| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} \left(f\left(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right) + f(p)\right)$$
$$- \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}} \left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\Bigg|$$

$$\begin{aligned}
&\leq \int_0^1 (1-t)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right] |f'\left(p + (1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)| dt \\
&\quad + \int_0^1 t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho} t^{\rho}\right] |f'\left(p + (1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)| dt \\
&\leq \int_0^1 (1-t)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right] \left(t|f'(p)| + (1-t)|f'(q)|\right) dt \\
&\quad + \int_0^1 t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho} t^{\rho}\right] \left(t|f'(p)| + (1-t)|f'(q)|\right) dt. \qquad (11)
\end{aligned}$$

Now, it can be seen that

$$\begin{aligned}\int_0^1 (1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}&\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]dt\\ &=\int_0^1 t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]dt\\ &=\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],\end{aligned} \tag{12}$$

where

$$\sigma_1(k)=\frac{\sigma(k)}{k\rho+\lambda+2}\text{ for }k=0,1,2,\cdots,$$

and

$$\begin{aligned}\int_0^1 (1-t)^{\lambda}t\mathcal{F}^{\sigma}_{\rho,\lambda+1}&\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]dt\\ &=\int_0^1 t^{\lambda}(1-t)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]dt\\ &=\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right].\end{aligned} \tag{13}$$

By replacement of (12) and (13) in (11) we have

$$\begin{aligned}&\left|\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right)\right.\\ &\left.-\frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\right|\\ &\le|f'(p)|\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)\\ &\quad+|f'(q)|\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)\\ &=\left(|f'(p)|+|f'(q)|\right)\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}+\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\right)\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right].\end{aligned}$$

For the case of $r>1$ it will be used Lemma 1, the triangular inequality , Hölder's inequality and the generalized $\phi$-convexity of $|f'|^r$. Then, we can observe that

$$\begin{aligned}&\left|\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right)\right.\\ &\left.-\frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\right|\\ &\le\int_0^1(1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|\,dt\\ &\qquad+\int_0^1 t^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|\,dt\\ &=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho}\int_0^1(1-t)^{k\rho+\lambda}|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|\,dt\\ &+\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho}\int_0^1 t^{k\rho+\lambda}|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|dt\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k \left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho} \times$$
$$\left(\int_0^1 (1-t)^{k\rho+\lambda} |f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)| dt + \int_0^1 t^{k\rho+\lambda} |f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)| dt\right)$$
$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k \left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho} \times$$
$$\left[\left(\int_0^1 (1-t)^{s(k\rho+\lambda)} dt\right)^{1/s} \left(\int_0^1 |f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|^r dt\right)^{1/r} \right.$$
$$\left. + \left(\int_0^1 t^{s(k\rho+\lambda)} dt\right)^{1/s} \left(\int_0^1 |f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|^r dt\right)^{1/r}\right]$$
$$= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k \left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho} \left(\int_0^1 |f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|^r dt\right)^{1/r} \times$$
$$\left(\left(\int_0^1 (1-t)^{s(k\rho+\lambda)} dt\right)^{1/s} + \left(\int_0^1 t^{s(k\rho+\lambda)} dt\right)^{1/s}\right)$$
$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k \left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{k\rho} \left[\left(|f'(p)|^r + |f'(q)|^r\right)^{1/r} \left(\frac{2}{s(k\rho+\lambda)+1}\right)^{1/s}\right]$$
$$= 2^{1/s} \left(|f'(p)|^r + |f'(q)|^r\right)^{1/r} \mathcal{F}^{\sigma_2^s}_{\rho,\lambda}[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}],$$

where

$$\frac{1}{r}+\frac{1}{s}=1 \text{ and } \sigma_2^s(k) = \frac{\sigma(k)}{(s(k\rho+\lambda)+1)^{1/s}} \text{ for } k=1,2,3,\cdots.$$

In the above expression it was used

$$\left(\int_0^1 (1-t)^{s(k\rho+\lambda)} dt\right)^{1/s} = \left(\int_0^1 t^{s(k\rho+\lambda)} dt\right)^{1/s} = \frac{1}{(s(k\rho+\lambda)+1)^{1/s}}.$$

The proof is complete. □

**Corollary 1.** *With the notations in Theorem 3, if $|f'| \leq K$, we have*

$$\left| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} \left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) \right.$$
$$\left. - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}} \left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w} f\right)(p) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w} f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right) \right|$$
$$\leq \begin{cases} 2K\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}+\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\right)\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right], & r=1 \\ 2K\mathcal{F}^{\sigma_2^s}_{\rho,\lambda}[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}], & r>1,\ \frac{1}{r}+\frac{1}{s}=1, \end{cases}$$

*where*

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho+\lambda+2} \text{ and } \sigma_2^s(k) = \frac{\sigma(k)}{(s(k\rho+\lambda)+1)^{1/s}} \text{ for } k=1,2,3,\ldots.$$

**Remark 2.** *If in Theorem 3 we choose* $\lambda = \alpha$, $\sigma = \{1, 0, 0, \cdots\}$ *and* $w = 1$ *it is obtained the following inequality for the Riemann–Liouville fractional integral*

$$\left| \frac{1}{\alpha}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) - \left(\left(\mathcal{I}^{\alpha}_{(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)-}f\right)(p) + \left(\mathcal{I}^{\alpha}_{p+}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\right|$$
$$\leq \begin{cases} (|f'(p)|+|f'(q)|)\left(\frac{\alpha+3}{\Gamma(\alpha+3)}\right), & r=1 \\ \\ (|f'(q)|^r+|f'(p)|^r)^{1/r}\,\frac{1}{(s\alpha+1)\Gamma(\alpha)}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1, \end{cases}$$

*If* $\alpha = 1$ *then it is attained the following inequality for the classical Riemann integral*

$$\left|(f(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))+f(p)) - \int_p^{p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)} f(x)dx\right|$$
$$\leq \begin{cases} \frac{2}{3}\left(|f'(p)|+|f'(q)|\right), & r=1 \\ \\ (|f'(q)|^r+|f'(p)|^r)^{1/r}\,\frac{1}{(s+1)}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1. \end{cases}$$

**Theorem 4.** *Let* $\nu, \mu > 0$, $\eta = \{\eta(k)\}_{k=0}^{\infty}$ *a bounded sequence of real numbers, and* $p, q \in \mathbb{R}$ *with* $p < q$, *and* $f : \left[p, p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right] \to \mathbb{R}$ *be a differentiable on* $(p, p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))$, *where* $\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) > 0$. *If* $|f'|^r$ *is generalized* $\phi$*-convex function on* $[p, p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)]$ *for* $r \geq 1$. *If* $|f'|^r$ *is generalized* $\phi$*-convex function on* $\left[p, p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right]$ *for* $r \geq 1$, *then the following inequality for Raina's fractional integral operator holds*

$$\left| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right)\right.$$
$$\left. - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\right|$$

$$\leq \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\left[\left(A(\rho,\lambda,\sigma_1)|f'(p)|^r + B(\rho,\lambda,\sigma)|f'(q)|^r\right)^{1/r}\right.$$
$$\left. + \left(B(\rho,\lambda,\sigma)|f'(p)|^r + A(\rho,\lambda,\sigma_1)|f'(q)|^r\right)^{1/r}\right],$$

*where*

$$A(\rho,\lambda,\sigma_1) = \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],\ B(\rho,\lambda,\sigma) = \mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],$$

*and*

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho+\lambda+2}\ \text{for}\ k = 1, 2, \cdots.$$

*for* $\rho, \lambda > 0$, $w \in \mathbb{R}$ *and a bounded sequence of positive real numbers* $\sigma$.

**Proof.** Let $r \geq 1$. Using Lemma 1, the triangular inequality, the power mean inequality and the generalized $\phi$-convexity of $|f'|$, we have

$$\begin{aligned}
&\Bigg| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) \\
&\quad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\Bigg| \\
&\leq \left(\int_0^1 (1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]dt\right)^{1-1/r}\times \\
&\quad \left(\int_0^1 (1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}(1-t)^{\rho}\right]|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|^{r}dt\right)^{1/r} \\
&+ \left(\int_0^1 t^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]dt\right)^{1-1/r}\times \\
&\quad \left(\int_0^1 t^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]|f'\left(p+(1-t)\,\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)|^{r}dt\right)^{1/r}.
\end{aligned}$$

Using (12) and (13) (in Theorem 3) it can be written

$$\begin{aligned}
&\Bigg| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) \\
&\quad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)\Bigg| \\
&\leq \left(\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)^{1-1/r}\times \\
&\quad \Bigg[\left(|f'(p)|^{r}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+|f'(q)|^{r}\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)^{1/r} \\
&\quad + \left(|f'(p)|^{r}\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+|f'(q)|^{r}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)^{1/r}\Bigg]
\end{aligned}$$

and setting

$$A(\rho,\lambda,\sigma_1)=\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] \text{ and } B(\rho,\lambda,\sigma)=\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]$$

the desired result is attained. □

**Corollary 2.** *With the notations in Theorem 4, if $|f'| \leq K$, we have*

$$\begin{aligned}
&\Bigg| \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]}{\mathcal{F}^{\eta}_{\nu,\mu}(q-p)}\left(f\left(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)+f(p)\right) \\
&\quad - \frac{1}{\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\lambda+1}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))-;w}f\right)(p)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,p+;w}f\right)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))\right)\Bigg| \\
&\qquad \leq 2K\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\left(A(\rho,\lambda,\sigma_1)+B(\rho,\lambda,\sigma)\right)^{1/r},
\end{aligned}$$

*where*

$$A(\rho,\lambda,\sigma_1)=\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],\ B(\rho,\lambda,\sigma)=\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right],$$

*and*

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho + \lambda + 2} \text{ for } k = 1, 2, \cdots.$$

**Theorem 5.** *Let $\nu, \mu > 0$, $\eta = \{\eta(k)\}_{k=0}^{\infty}$ a bounded sequence of real numbers, and $p, q \in \mathbb{R}$ with $p < q$, and $f, g$ be real-valued, non-negative and generalized $\phi$-convex functions on $(p, p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p))$, where $\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) > 0$. Then*

$$\begin{aligned}(\mathcal{J}^{\sigma}_{\rho,\lambda,p+,w} fg)(p + \mathcal{F}^{\eta}_{\nu,\mu}(q-p)) &+ (\mathcal{J}^{\sigma}_{\rho,\lambda,p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p),w} fg)(p) \\ &\leq M(p,q)\left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] + \mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right) \\ &\quad + N(p,q)\mathcal{F}^{\sigma_2}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] \end{aligned} \tag{14}$$

*where*

$$\sigma_1(k) = \sigma(k)\left(\frac{2}{k\rho + \lambda + 2} + \frac{2}{k\rho + \lambda + 1}\right),$$
$$\sigma_2(k) = \sigma(k)(\lambda + 4).$$

*and*

$$M(p,q) = f(p)g(p) + f(q)g(q), \quad N(p,q) = f(p)g(q) + f(q)g(p),$$

*for $\rho, \lambda > 0$, $w \in \mathbb{R}$ and a bounded sequence of positive real numbers $\sigma$.*

**Proof.** Since $f$ and $g$ are generalized $\phi$-convex functions on $[p, p + \mathcal{F}^{\sigma}_{\rho,\lambda}(q-p)]$, we have

$$f(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \leq t f(p) + (1-t) f(q) \tag{15}$$

and

$$g(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \leq t g(p) + (1-t) g(q). \tag{16}$$

From (15) and (16), we get

$$\begin{aligned} f(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p))&g(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \\ &\leq t^2 f(p)g(p) + (1-t)^2 f(q)g(q) + t(1-t)[f(p)g(q) + f(q)g(p)]. \end{aligned}$$

Similarly,

$$\begin{aligned} f(p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p))&g(p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \\ &\leq (1-t)^2 f(p)g(p) + t^2 f(q)g(q) + t(1-t)[f(p)g(q) + f(q)g(p)]. \end{aligned}$$

By adding the above two inequalities, it follows that

$$\begin{aligned} f(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p))&g(p + (1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \\ &\quad + f(p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p))g(p + t\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) \\ &\leq (2t^2 - 2t + 1)[f(p)g(p) + f(q)g(q)] + 2t(1-t)[f(p)g(q) + f(q)g(p)]. \end{aligned}$$

Multiplying both sides of above inequality by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]\times \\
&\qquad\qquad f(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p))g(p+(1-t)\mathcal{F}^{\eta}_{\nu,\mu}(q-p))dt \\
&+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]f(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p))g(p+t\mathcal{F}^{\eta}_{\nu,\mu}(q-p))dt \\
&\leq [f(p)g(p)+f(q)g(q)]\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right](2t^2-2t+1)]dt \\
&\quad+[f(p)g(q)+f(q)g(p)]\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]2t(1-t)dt.
\end{aligned}$$

Now with a simple calculation we obtain that

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right](2t^2-2t+1)]dt \\
&\quad=\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]
\end{aligned}$$

where

$$\sigma_1(k)=\sigma(k)\left(\frac{2}{k\rho+\lambda+2}-\frac{2}{k\rho+\lambda+1}\right),$$

and

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}t^{\rho}\right]2t(1-t)dt \\
&\quad=\mathcal{F}^{\sigma_2}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]
\end{aligned}$$

where

$$\sigma_2(k)=\sigma(k)(2\lambda+4).$$

Setting

$$M(p,q)=f(p)g(p)+f(q)g(q),\quad N(p,q)=f(p)g(q)+f(q)g(p)$$

and a suitable change of variable the desired result is obtained. □

**Corollary 3.** *With the notations in Theorem 5, if $f=g$, we have*

$$\begin{aligned}
&(\mathcal{J}^{\sigma}_{\rho,\lambda,p+,w}f^2)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p))+(\mathcal{J}^{\sigma}_{\rho,\lambda,p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p),w}f^2)(p) \\
&\quad\leq M(p,q)\left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]+\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right) \\
&\qquad+N(p,q)\mathcal{F}^{\sigma_2}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]
\end{aligned}$$

*where*

$$\sigma_1(k)=\sigma(k)\left(\frac{2}{k\rho+\lambda+2}-\frac{2}{k\rho+\lambda+1}\right),$$
$$\sigma_2(k)=\sigma(k)(2\lambda+4).$$

*and*

$$M(p,q)=f^2(p)+f^2(q),\quad N(p,q)=2f(p)f(q).$$

**Corollary 4.** *With the notations in Theorem 5, if we choose $g \equiv 1$, we have*

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,p+,w}f)(p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p)) + (\mathcal{J}^{\sigma}_{\rho,\lambda,p+\mathcal{F}^{\eta}_{\nu,\mu}(q-p),w}f)(p)$$
$$\leq M(p,q)\left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] + \mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right] + \mathcal{F}^{\sigma_2}_{\rho,\lambda+2}\left[w\left(\mathcal{F}^{\eta}_{\nu,\mu}(q-p)\right)^{\rho}\right]\right)$$

*where*

$$\sigma_1(k) = \sigma(k)\left(\frac{2}{k\rho+\lambda+2} - \frac{2}{k\rho+\lambda+1}\right),$$
$$\sigma_2(k) = \sigma(k)(2\lambda+4),$$

*and*

$$M(p,q) = f(p) + f(q).$$

**Remark 3.** *Using the same considerations in the Remark 1 and 2 it is possible directly to find the inequalities established in Theorems 4 and 5 (and corresponding corollaries) for the Riemann–Liouville fractional integral and the classical Riemann integral.*

## 5. Conclusions

In the development of this work, a special definition of generalized $\phi$-convex functions has been introduced when considering the class of functions defined by R.K. Raina in [31], from which the hypergeometric function and the Mittag–Leffler function and some others are included. Using this definition and Raina's fractional integral operator [34], inequalities of the Hermite–Hadamard type and some special cases have been established. With reference to the definition of generalized $\phi$-convex sets and the corresponding generalized $\phi$-convex functions there is much to investigate in the field of mathematical analysis, in particular with the different possibilities of the parametric values of Raina's function.

The authors hope that these results will serve as a motivation for future work in this area.

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