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Type 2 Degenerate Poly-Euler Polynomials

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Abstract: In recent years, many mathematicians have studied the degenerate versions of many special polynomials and numbers. The polyexponential functions were introduced by Hardy and rediscovered by Kim, as inverses to the polylogarithms functions. The paper is divided two parts. First, we introduce a new type of the type 2 poly-Euler polynomials and numbers constructed from the modified polyexponential function, the so-called type 2 poly-Euler polynomials and numbers. We show various expressions and identities for these polynomials and numbers. Some of them involving the (poly) Euler polynomials and another special numbers and polynomials such as (poly) Bernoulli polynomials, the Stirling numbers of the first kind, the Stirling numbers of the second kind, etc. In final section, we introduce a new type of the type 2 degenerate poly-Euler polynomials and the numbers defined in the previous section. We give explicit expressions and identities involving those polynomials in a similar direction to the previous section.

Keywords: poly-Euler polynomials and numbers; degenerate poly-Euler polynomials and numbers; modified degenerate polyexponential functions; poly-Bernoulli polynomials; the Stirling numbers

1. Introduction

Carlitz [1,2] initiated a study of degenerate versions of the usual Bernoulli and Euler polynomials and numbers with their arithmetic and combinatorial interest. These numbers have been actively investigated and many interesting properties and formulas for them have been discovered (see, e.g., in [3–15]). Recently, many mathematicians studied some arithmetic and combinatorial aspects of various degenerate special polynomials and numbers (see, e.g., in [4–10,16–29]).

As is well known, the Bernoulli polynomials of order α are defined by their generating function as follows (see [1,2,19]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1}$$

We note that for $\alpha = 1$, $B_n(x) = B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials. When $x = 0$, $B_n^{\alpha} = B_n^{\alpha}(0)$ are called the Bernoulli numbers of order α.

The Euler polynomials are defined by (see [1,2,21])

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

The first few values are $E_0(x) = 1, E_1(x) = x - \frac{1}{2}, E_2(x) = x^2 - x, E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$.

For an integer k, the polylogaritm is defined by (see [13,15,16])

$$Li_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}. \tag{3}$$

In 1997, Kaneko [16] introduced poly-Bernoulli numbers, which are defined by the polylogaritm function.

Recently, Kim-Kim [7] introduced the modified polyexponential function as

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \ (k \in \mathbb{Z}). \tag{4}$$

From Equation (4), we note that $\mathrm{Ei}_1(x) = e^x - 1$.

By using these functions,

Kim-Kim-Kim-Jang [9] introduced the degenerate poly-Bernoulli polynomials and numbers as follows,

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{5}$$

When $x = 0$, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers. For $n \geq 0$, the Stirling numbers of the first kind are defined by (see [6–11])

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l. \tag{6}$$

where $(x)_0 = 1, \ (x)_n = x(x-1)\dots(x-n+1)$, for $n \geq 1$.

From (6), we see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}. \tag{7}$$

In the inverse expression to (7), for $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{8}$$

From (8), we note that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}. \tag{9}$$

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as (see [5–11])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{10}$$

In [1,30], Carlitz considered the degenerate Bernoulli polynomials, which are given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}. \tag{11}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Moreover, the degenerate Euler polynomials are defined by

$$\frac{2}{e_{\lambda}(t)+1}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{12}$$

When $x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the degenerate Euler numbers.

In [6], the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_{l},\quad (n\geq 0). \tag{13}$$

As an inversion formula of (13), the degenerate Stirling numbers of the first kind are defined by (see [6])

$$(x)_{n}=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda},\quad (n\geq 0). \tag{14}$$

From (13) and (14), we note that (see [6,28,29])

$$\frac{1}{k!}\big(e_{\lambda}(t)-1\big)^{k}=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^{n}}{n!}, \tag{15}$$

and (see [6,28,29])

$$\frac{1}{k!}\big(\log_{\lambda}(1+t)\big)^{k}=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^{n}}{n!}, \tag{16}$$

where $\log_{\lambda}(t)=\frac{1}{\lambda}(t^{\lambda}-1)$ is the compositional inverse of $e_{\lambda}(t)$ satisfying $\log_{\lambda}\big(e_{\lambda}(t)\big)=e_{\lambda}\big(\log_{\lambda}(t)\big)=t$ (see [22]).

Yoshinori [14] introduced poly-Euler polynomials and numbers as follows,

$$\frac{2Li_{k}(1-e^{-t})}{t(e^{t}+1)}e^{xt}=\sum_{n=0}^{\infty}E_{n}^{(k)}(x)\frac{t^{n}}{n!}. \tag{17}$$

The numbers $E_{n}^{(k)}=E_{n}^{(k)}(0)$ are called poly-Euler numbers. In particular, when $k=1$, $E_{n}^{(1)}(x)=E_{n}(x)$, for any $n\geq 0$.

Studying degenerate version of various special polynomials became an active area of research and produced many interesting arithmetic and combinatorial results. Indeed, many symmetric identities have been studied for degenerate versions of many special polynomials. The poly-exponential functions were first studied by Hardy [31] and reconsidered by Kim [8] in the view of an inverse to the polylogarithm functions which were studied by Jaonquière [16], Lewin [14], and Zagier [32–34]. The paper is divided two parts. In Section 2, we introduce a new type of the type 2 poly-Euler polynomials and numbers constructed from the modified polyexponential function, so called the type 2 poly-Euler polynomials and numbers. We show several identities for these polynomials and numbers. Some of them involving the ordinary (poly) Euler polynomials and another special numbers and polynomials such as (poly) Bernoulli polynomials, and the Stirling numbers of the first kind, the type 2 poly-Euler polynomials, the Stirling numbers of the second kind, etc. We also deduced an expression of the type 2 poly-Euler numbers of the Euler number and values of higher order Bernoulli polynomials at zero. In Section 3, we introduce the type 2 degenerate poly-Euler polynomials and numbers constructed from the modified polyexponential function. We give explicit expressions and identities involving those polynomials in a similar direction in Section 2.

2. Type 2 Poly-Euler Polynomials and Numbers

In this section, we define the type 2 poly-Euler numbers by means of the polylogarithm functions and represent the usual Euler numbers (more precisely, the values of Euler polynomials at 1) when $k=1$. At the same time, we give explicit expressions and identities involving those polynomials.

Now, the type 2 poly-Euler polynomials is defined by

$$\frac{Ei_k(log(1+2t))}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x)\frac{t^n}{n!}. \tag{18}$$

When $x = 0, \mathcal{E}_n^{(k)} = \mathcal{E}_n^{(k)}(0)$ are called type 2 poly-Euler numbers.

As $\mathrm{Ei}_1(\log(1+2t)) = 2t$, we see that $\mathcal{E}_n^{(1)}(x) = E_n(x)$ $(n \geq 0)$ are the Euler polynomials.

The next lemma is intended to be used conveniently to prove some of the theorems below.

Lemma 1. *For $n \geq 0, k \in \mathbb{Z}$, we have*

$$\mathrm{Ei}_k(\log(1+2t)) = \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{1}{l^{k-1}} S_1(n,l)\frac{2^n t^n}{n!}. \tag{19}$$

Proof. From (4), we see that

$$\begin{aligned} \mathrm{Ei}_k(\log(1+2t)) &= \sum_{l=1}^{\infty} \frac{1}{l^{k-1}} \frac{(\log(1+2t))^l}{l!} \\ &= \sum_{l=1}^{\infty} \frac{1}{l^{k-1}} \sum_{n=l}^{\infty} S_1(n,l)\frac{2^n t^n}{n!} = \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{1}{l^{k-1}} S_1(n,l)\frac{2^n t^n}{n!}. \end{aligned} \tag{20}$$

□

Theorem 1. *For $n \geq 0, k \in \mathbb{Z}$, we have*

$$\mathcal{E}_n^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=1}^{m+1} \binom{n}{l}\binom{l}{m} \frac{2^{n-l+m} S_1(m+1,j)}{(l-m+1)(m+1)j^{k-1}} B_{n-l}(\frac{x}{2}). \tag{21}$$

Proof. From (1) and Lemma 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t(e^t+1)}e^{xt} \\ &= \left(\frac{e^{xt}}{t(e^{2t}-1)}\right)\left(e^t-1\right)\left(\sum_{m=1}^{\infty}\sum_{j=1}^{m}\frac{1}{j^{k-1}}S_1(m,j)\frac{2^m t^m}{m!}\right) \\ &= \left(\frac{e^{xt}}{t(e^{2t}-1)}\right)\left(\sum_{i=1}^{\infty}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\sum_{j=1}^{m+1}\frac{1}{j^{k-1}}S_1(m+1,j)\frac{2^{m+1}t^{m+1}}{(m+1)!}\right) \\ &= \left(\frac{2te^{xt}}{e^{2t}-1}\right)\left(\sum_{i=0}^{\infty}\frac{1}{i+1}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\sum_{j=1}^{m+1}\frac{1}{j^{k-1}}S_1(m+1,j)\frac{2^m}{m+1}\frac{t^m}{m!}\right) \\ &= \left(\sum_{s=0}^{\infty}B_s(\frac{x}{2})\right)\frac{2^s t^s}{s!}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{j=1}^{m+1}\binom{l}{m}\frac{2^m S_1(m+1,j)}{(l-m+1)(m+1)j^{k-1}}\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=1}^{m+1}\binom{n}{l}\binom{l}{m}\frac{2^{n-l+m}S_1(m+1,j)}{(l-m+1)(m+1)j^{k-1}}B_{n-l}(\frac{x}{2})\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

Therefore, by comparing the coefficients on both side of (22), we get the desired result. □

Corollary 1. *For $n \in \mathbb{N} \cup \{0\}$ and $x = 0$, we have*

$$\mathcal{E}_n^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \sum_{j=1}^{m+1} \binom{n}{l} \binom{l}{m} \frac{2^{n-l+m+1} S_1(m+1,j)}{(l-m+1)(m+1)j^{k-1}} B_{n-l}. \tag{23}$$

Theorem 2. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_n^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{2^m}{(l+1)^{k-1}} \frac{S_1(m+1,l)}{m+1} E_{n-m}(x). \tag{24}$$

Proof. By using (2) and Lemma 1, we have

$$\begin{aligned}
\frac{Ei_k(log(1+2t))}{t(e^t+1)} e^{xt} &= \frac{1}{t}\left(\frac{1}{e^t+1}e^{xt}\right)\left(\sum_{m=1}^{\infty}\sum_{l=1}^{m} \frac{1}{l^{k-1}} S_1(m,l) \frac{2^m t^m}{m!}\right) \\
&= \left(\frac{2}{e^t+1}e^{xt}\right)\left(\sum_{m=0}^{\infty}\sum_{l=1}^{m+1} \frac{1}{l^{k-1}} S_1(m+1,l) \frac{2^m t^m}{(m+1)!}\right) \\
&= \left(\sum_{j=0}^{\infty} E_j(x)\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=1}^{m+1} \frac{2^m}{l^{k-1}} \frac{S_1(m+1,l)}{m+1} \frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1} \binom{n}{m} \frac{2^m}{(l+1)^{k-1}} \frac{S_1(m+1,l)}{m+1} E_{n-m}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{25}$$

Therefore, by comparing the coefficients on both side of (25), we get the desired result. □

Corollary 2. *For $n \in \mathbb{N} \cup \{0\}$ and $x = 0$, we have*

$$\mathcal{E}_n^{(k)} = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{2^m}{(l+1)^{k-1}} \frac{S_1(m+1,l)}{m+1} E_{n-m}. \tag{26}$$

Theorem 3. *For $n \geq 0$, we have*

$$\mathcal{E}_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{n-m}^{(k)} x^l. \tag{27}$$

Proof. We observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x)\frac{t^n}{n!} = \frac{Ei_k(log(1+2t))}{t(e^t+1)}e^{xt} &= \left(\sum_{m=0}^{\infty}\mathcal{E}_m^{(k)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} x^l \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_m^{(k)}x^{n-l}\right)\frac{t^n}{n!}\left(or = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m}^{(k)}x^l\right)\frac{t^n}{n!}\right).
\end{aligned} \tag{28}$$

Therefore, by comparing the coefficients on both side of (28), we get the desired result. □

Theorem 4. *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$\frac{d}{dx}\mathcal{E}_n^{(k)}(x) = n\mathcal{E}_{n-1}^{(k)}(x). \tag{29}$$

Proof. By using Theorem 6, we observe that

$$\begin{aligned}\frac{d}{dx}\mathcal{E}_n^{(k)}(x) &= \frac{d}{dx}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{n-l}^{(k)}x^l\right)\\ &= \sum_{l=1}^{n}\binom{n}{l}\mathcal{E}_{n-l}^{(k)}lx^{l-1} = \sum_{l=0}^{n-1}\binom{n}{l+1}\mathcal{E}_{n-l}^{(k)}(l+1)x^l \\ &= n\sum_{l=0}^{n-1}\frac{(n-1)!}{l!(n-l-1)!}\mathcal{E}_{n-l-1}^{(k)}x^l = n\mathcal{E}_{n-1}^{(k)}(x).\end{aligned} \tag{30}$$

Thus, we obtain the desired result. □

For the next theorem, we need the following well-known identity,

$$\left(\frac{t}{log(1+t)}\right)^r(1+t)^{x-1} = \sum_{n=0}^{\infty}B_n^{(n-r+1)}(x)\frac{x^n}{n!}, \tag{31}$$

where $B_n^{(n)}$ is the Bernoulli numbers of order n at $x=0$.

Theorem 5. *For $n \geq 0$, $k \in \mathbb{Z}$, we get*

$$\begin{aligned}\mathcal{E}_n^{(k)} &= \sum_{m=0}^{n}\binom{n}{m}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\\ &\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}E_{n-l},\end{aligned} \tag{32}$$

where, $B_m^{(m)}$ is the Bernoulli numbers of order m.

Proof. First, we note that

$$\begin{aligned}\frac{d}{dx}Ei_k(log(1+2x)) &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(log(1+2x))^n}{(n-1)!n^k}\\ &= \frac{2}{1+2x}\sum_{n=1}^{\infty}\frac{n(log(1+2x))^{n-1}}{(n-1)!n^k}\\ &= \frac{2}{(1+2x)log(1+2x)}\sum_{n=1}^{\infty}\frac{(log(1+2x))^n}{(n-1)!n^{k-1}}\\ &= \frac{2}{(1+2x)log(1+2x)}Ei_{k-1}(log(1+2x))dt.\end{aligned} \tag{33}$$

As $Ei_1(log(1+2t)) = 2t$, we obtain the following equation by using (31) and (33).

$$\begin{aligned}
&Ei_k(log(1+2x))\\
&=\int_0^x \frac{2}{(1+2t)log(1+2t)}\\
&\quad\times\underbrace{\int_0^t \frac{2}{(1+2t)log(1+2t)}\cdots\int_0^t \frac{2}{(1+2t)log(1+2t)}}_{(k-2)-\text{times}}\int_0^t \frac{4t}{(1+2t)log(1+2t)}dt\cdots dt\\
&=2x\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\\
&\quad\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}.
\end{aligned}\tag{34}$$

From above Equation (34), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}\frac{t^n}{n!}&=\frac{1}{t(e^t+1)}Ei_k(log(1+2t))\\
&=\frac{2t}{t(e^t+1)}\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\\
&\quad\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}\\
&=\left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\sum_{m=0}^{\infty}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\\
&\quad\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}\frac{x^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\\
&\quad\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}E_{n-l}\frac{x^n}{n!}.
\end{aligned}\tag{35}$$

Therefore, by comparing the coefficients on both side of (35), we get the desired result. □

Corollary 3. *For $k=2$, we have*

$$\mathcal{E}_n^{(2)}=\sum_{l=0}^{n}\binom{n}{l}2^l\frac{B_{l+1}^{(l)}}{l+1}E_{n-l}.\tag{36}$$

Theorem 6. *For $n\geq 1, k\in\mathbb{Z}$, we have*

$$\mathcal{E}_{n-1}^{(k)}(1)+\mathcal{E}_{n-1}^{(k)}=\frac{2^n}{n}\sum_{l=1}^{n}\frac{1}{l^{k-1}}S_1(n,l).\tag{37}$$

Proof. By using (27),

$$\begin{aligned}
Ei_k(log(1+2t)) &= t(e^t+1)\left(\sum_{l=0}^{\infty}\mathcal{E}_l^{(k)}\frac{t^l}{l!}\right) \\
&= t\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}+1\right)\left(\sum_{l=0}^{\infty}\mathcal{E}_l^{(k)}\frac{t^l}{l!}\right) = t\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m}^{(k)}+\mathcal{E}_n^{(k)}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\mathcal{E}_n^{(k)}(1)+\mathcal{E}_n^{(k)}\right)\frac{t^{n+1}}{n!} = n\sum_{n=1}^{\infty}\left(\mathcal{E}_{n-1}^{(k)}(1)+\mathcal{E}_{n-1}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{38}$$

Now, by comparing the coefficients of (20) and (38), we get what we wanted. □

Theorem 7. *For $n \geq 1, k \in \mathbb{Z}$, we have*

$$\sum_{m=1}^{n} 2^n \frac{1}{m^{k-1}} S_1(n,m) = 2\delta_{n,1}, \tag{39}$$

where $\delta_{n,k}$ is the Keroneker delta.

Proof. From (37), we obtain

$$Ei_1(log(1+2t)) = 2t = \sum_{n=1}^{\infty}\sum_{m=1}^{n} 2^n \frac{1}{m^{k-1}} S_1(n,m)\frac{t^n}{n!}. \tag{40}$$

Therefore, by comparing the coefficients of (40) and, we get the desired result. □

Theorem 8. *For $n \geq 0, k \in \mathbb{Z}$, we get*

$$\mathcal{E}_n^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_2(m,l)\mathcal{E}_{n-l}^{(k)}. \tag{41}$$

Proof. By using (18), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x)\frac{t^n}{n!} &= \frac{Ei_k(log(1+2t))}{e^t+1}e^{xt} = \frac{Ei_k(log(1+2t))}{e^t+1}(e^t-1+1)^x \\
&= \frac{Ei_k(log(1+2t))}{e^t+1}\left(\sum_{m=0}^{\infty}\binom{x}{m}(e^t-1)^m\right) \\
&= \frac{Ei_k(log(1+2t))}{e^t+1}\left(\sum_{m=0}^{\infty}(x)_m\frac{(e^t-1)^m}{m!}\right) \\
&= \left(\sum_{i=0}^{\infty}\mathcal{E}_i^{(k)}\frac{x^i}{i!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)_m S_2(m,l)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_2(m,l)\mathcal{E}_{n-l}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{42}$$

Now, by comparing the coefficients of (42), we get what we wanted. □

The following theorem is related the poly-Bernoulli polynomials (5) in [11].

Theorem 9. *For $n \geq 1, k \in \mathbb{Z}$, we get*

$$n(\mathcal{E}_{n-1}^{(k)}(x+1) + n\mathcal{E}_{n-1}^{(k)}(x)) = 2^n(\beta_n^{(k)}(x+2) - \beta_n^{(k)}(x))). \tag{43}$$

Proof. From (18), we get

$$\begin{aligned}\frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t(e^t+1)}t(e^t+1)e^{xt} &= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t(e^t+1)}e^{(x+1)t} + \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t(e^t+1)}e^{xt}t \\ &= \sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x+1)\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x)\frac{t^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty}\left(n\mathcal{E}_{n-1}^{(k)}(x+1) + n\mathcal{E}_{n-1}^{(k)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{44}$$

On the other hand, from (5), we observe that

$$\begin{aligned}\frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{(e^{2t}-1)}(e^{2t}-1)e^{xt} &= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{(e^{2t}-1)}e^{(x+2)t} - \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{e^t-1}e^{xt} \\ &= \sum_{n=0}^{\infty}\left(\beta_n^{(k)}(x+2) - \beta_n^{(k)}(x)\right)\frac{2^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}2^n\big(\beta_n^{(k)}(x+2) - \beta_n^{(k)}(x)\big)\frac{t^n}{n!}.\end{aligned} \tag{45}$$

Now, by comparing the coefficients of (44) and (45), we get what we wanted. □

3. The Type 2 Degenerate Poly-Euler Polynomials and Numbers

In this section, we introduce the type 2 degenerate poly-Euler numbers, which is the degenerate Euler numbers when $k = 1$ (12). At the same time, we give explicit expressions and identities involving those polynomials.

Now, the type 2 degenerate poly-Euler polynomials is defined by

$$\frac{Ei_k(log(1+2t))}{t(e_\lambda(t)+1)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{46}$$

When $x = 0, \mathcal{E}_{n,\lambda}^{(k)} = \mathcal{E}_{n,\lambda}^{(k)}(0)$ are called type 2 degenerate poly-Euler numbers.

As $\mathrm{Ei}_1(\log(1+2t)) = 2t$, we see that $\mathcal{E}_{n,\lambda}^{(1)}(x) = E_{n,\lambda}(x)$ $(n \geq 0)$ are the degenerate Euler polynomials.

Theorem 10. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\mathcal{E}_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{2}{n-l+1}(1)_{n-l+1,\lambda}\beta_{l,\frac{\lambda}{2}}(\frac{x}{2}) \tag{47}$$

Proof. From (5) and (10), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{Ei_k\big(log(1+2t)\big)}{t(e_\lambda(t)+1)}e_\lambda^x(t)\\ &= \frac{Ei_k\big(log(1+2t)\big)}{t(e_\lambda(t)+1)(e_\lambda(t)-1)}e_\lambda^x(t)(e_\lambda(t)-1)\\ &= \frac{1}{t}\left(\frac{Ei_k\big(log(1+2t)\big)}{e_{\frac{\lambda}{2}}(2t)-1}e_\lambda^x(t)\right)(e_\lambda(t)-1)\\ &= \frac{1}{t}\left(\sum_{l=0}^{\infty}\beta_{l,\frac{\lambda}{2}}(\frac{x}{2})\frac{2^l t^l}{l!}\right)\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)\\ &= \frac{1}{t}\left(\sum_{l=0}^{\infty}\beta_{l,\frac{\lambda}{2}}(\frac{x}{2})\frac{2^l t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m+1,\lambda}\frac{t^{m+1}}{(m+1)!}\right)\\ &= \left(\sum_{l=0}^{\infty}2^l\beta_{l,\frac{\lambda}{2}}(\frac{x}{2})\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{m+1}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{2^l}{n-l+1}(1)_{n-l+1,\lambda}\beta_{l,\frac{\lambda}{2}}(\frac{x}{2})\right)\frac{t^n}{n!}.\end{aligned} \tag{48}$$

By comparing the coefficients of (48), we get what we wanted. □

Theorem 11. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\mathcal{E}_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=1}^{n+1}\binom{n}{l}\frac{2^m S_1(m+1,l)}{(m+1)l^{k-1}}E_{n-l,\lambda}. \tag{49}$$

Proof. From (12) and (19), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t(e_\lambda(t)+1)}\\ &= \frac{1}{t(e_\lambda(t)+1)}\left(\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{2^{m+1}}{l^{k-1}}S_1(m+1,l)\frac{t^{m+1}}{(m+1)!}\right)\\ &= \left(\frac{2}{e_\lambda(t)+1}\right)\left(\sum_{l=0}^{\infty}\sum_{m=1}^{l+1}\frac{2^m S_1(m+1,l)}{(m+1)l^{k-1}}\frac{t^m}{m!}\right)\\ &= \left(\sum_{i=0}^{\infty}E_{i,\lambda}\frac{t^i}{i!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=1}^{l+1}\frac{2^m S_1(m+1,l)}{(m+1)l^{k-1}}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{2^m S_1(m+1,l)}{(m+1)l^{k-1}}E_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{50}$$

Now, by comparing the coefficients of (50), we get what we wanted. □

Theorem 12. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\mathcal{E}_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_{2,\lambda}\mathcal{E}_{n-l,\lambda}^{(k)}. \tag{51}$$

Proof. From (18) and (46), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t\big(e_\lambda(t)+1\big)}e_\lambda^x(t) = \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t\big(e_\lambda(t)+1\big)}\big(e_\lambda(t)-1+1\big)^x \\
&= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t\big(e_\lambda(t)+1\big)}\left(\sum_{m=0}^{\infty}\binom{x}{m}\big(e_\lambda(t)-1\big)^m\right) \\
&= \frac{\mathrm{Ei}_k\big(\log(1+2t)\big)}{t\big(e_\lambda(t)+1\big)}\left(\sum_{m=0}^{\infty}(x)_m\sum_{m=l}^{\infty}S_{2,\lambda}(m,l)\frac{t^m}{m!}\right) \\
&= \left(\sum_{i=0}^{\infty}\mathcal{E}_{i,\lambda}^{(k)}\frac{t^i}{i!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)_m S_{2,\lambda}(m,l)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_{2,\lambda}\mathcal{E}_{n-l,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{52}$$

Therefore, by comparing the coefficients of (52), we get what we wanted. □

We can get the following theorem in the same way as Theorem 8.

Theorem 13. *For $n \in \mathbb{N}\cup\{0\}, k \in \mathbb{Z}$, we get*

$$\begin{aligned}
\mathcal{E}_{n,\lambda}^{(k)} &= \sum_{m=0}^{n}\binom{n}{m}2^m\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}} \\
&\times\frac{B_{m_1}^{(m_1)}}{m_1+1}\frac{B_{m_2}^{(m_2)}}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}}{m_1+\cdots+m_{k-1}+1}E_{n-l,\lambda},
\end{aligned} \tag{53}$$

where $B_m^{(m)}$ is the Bernoulli numbers of order m.

Theorem 14. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\mathcal{E}_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{\infty}\binom{n}{l}(x)_{l,\lambda}\mathcal{E}_{n-l,\lambda}^{(k)}. \tag{54}$$

Proof. From (10) and (46), we observe

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{Ei_k\big(log(1+2t)\big)}{t\big(e_\lambda(t)+1\big)}e_\lambda^x(t) \\
&= \Big(\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(k)}\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\Big) = \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}\mathcal{E}_{n-l,\lambda}^{(k)}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{55}$$

by comparing the coefficients of (55), we get the desired result. □

Theorem 15. *For $n \geq 1$ and $k \in \mathbb{Z}$, we get*

$$\mathcal{E}_{n-1,\lambda}^{(k)}(1) + \mathcal{E}_{n-1,\lambda}^{(k)} = \frac{2^n}{n}\left(\sum_{m=1}^{n}\frac{S_1(n,m)}{m^{k-1}}\right). \tag{56}$$

Proof. First, we observe by using (10) and (54) the following,

$$\begin{aligned}
&Ei_k(log(1+2t)) = t(e_\lambda(t)+1)\left(\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)\\
&= t\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}+1\right)\left(\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)\\
&= t\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-m,\lambda}\mathcal{E}_{m,\lambda}^{(k)}+\mathcal{E}_{n,\lambda}^{(k)}\right)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}^{(k)}+\mathcal{E}_{n,\lambda}^{(k)}\right)(n+1)\frac{t^{n+1}}{(n+1)!}\\
&= \sum_{n=1}^{\infty}n\left(\sum_{l=0}^{n-1}\binom{n-1}{l}(1)_{n-1-l,\lambda}\mathcal{E}_{l,\lambda}^{(k)}+\mathcal{E}_{n-1,\lambda}^{(k)}\right)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}n\left(\mathcal{E}_{n-1,\lambda}^{(k)}(1)+\mathcal{E}_{n-1,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{57}$$

Now, by comparing the coefficients of (19) and (57), we get what we wanted. □

Theorem 16. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\begin{aligned}
&\sum_{l=1}^{n}\sum_{m=0}^{n}\binom{n}{l}(-1)^{n-l+1}2^n(n-l-1)!\mathcal{E}_{m,\lambda}^{(k)}S_1(n,m)\\
&\qquad= \sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}\binom{n-m}{l}2^{n-m-1}E_{l,\lambda}S_1(n-m,l).
\end{aligned} \tag{58}$$

Proof. Replace t by $log(1+2t)$. From (7) and (12), we obtain

$$\begin{aligned}
&\frac{Ei_k(t)}{(e_\lambda(log(1+2t))+1)} = \frac{1}{2}\left(\frac{2}{e_\lambda(log(1+2t))+1}\right)\left(\sum_{m=1}^{\infty}\frac{t^m}{(m-1)!m^k}\right)\\
&= \frac{1}{2}\left(\sum_{l=0}^{\infty}E_{l,\lambda}\frac{(log(1+2t))^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{t^m}{(m-1)!m^k}\right)\\
&= \frac{1}{2}\left(\sum_{l=0}^{\infty}E_{l,\lambda}\sum_{j=l}^{\infty}S_1(j,l)\frac{2^jt^j}{j!}\right)\left(\sum_{m=1}^{\infty}\frac{t^m}{(m-1)!m^k}\right)\\
&= \frac{1}{2}\left(\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}E_{l,\lambda}S_1(j,l)2^j\frac{t^j}{j!}\right)\left(\sum_{m=1}^{\infty}\frac{t^m}{(m-1)!m^k}\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{l=0}^{n-m}\binom{n}{m}\binom{n-m}{l}2^{n-m-1}E_{l,\lambda}S_1(n-m,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{59}$$

On the other hand, from (7)

$$\begin{aligned}
&log(1+2t)\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(k)}\frac{(log(1+2t))^m}{m!}\\
&=\left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}2^i}{i}t^i\right)\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(k)}\sum_{l=m}^{\infty}S_1(l,m)\frac{2^l t^l}{l!}\\
&=\left(\sum_{i=1}^{\infty}(-1)^{i+1}2^i(i-1)!\frac{t^i}{i!}\right)\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}2^l\mathcal{E}_{m,\lambda}^{(k)}S_1(l,m)\right)\frac{t^l}{l!}\\
&=\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\sum_{m=0}^{l}\binom{n}{l}(-1)^{n-l+1}2^n(n-l-1)!\mathcal{E}_{m,\lambda}^{(k)}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{60}$$

Therefore, by comparing the coefficients of (59) and (60), we get what we wanted. □

Theorem 17. *For $n \geq 0$ and $k \in \mathbb{Z}$, we get*

$$\sum_{m=0}^{n}\frac{1}{2^m}\mathcal{E}_{m,\lambda}^{(k)}S_2(n,m)=\sum_{\alpha=0}^{n}\sum_{j=0}^{\beta}\sum_{l=0}^{j}\binom{n}{\alpha}\binom{\alpha}{j}\frac{1}{2^l(n-\alpha+1)^k}S_2(j,l)E_{l,\lambda}B_{\alpha-j}. \tag{61}$$

Proof. Replace t by $\frac{e^t-1}{2}$. From Equations (1), (12) and (9), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(k)}\frac{(e^t-1)^n}{2^n n!}&=\frac{2Ei_k(t)}{(e^t-1)(e_\lambda(\frac{e^t-1}{2})+1)}\\
&=\left(\frac{t}{e^t-1}\right)\left(\frac{2}{e_\lambda(\frac{e^t-1}{2})+1}\right)\frac{1}{t}\left(\sum_{m=1}^{\infty}\frac{t^m}{(m-1)!m^k}\right)\\
&=\left(\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\right)\left(\sum_{l=0}^{\infty}E_{l,\lambda}\frac{(e^t-1)^l}{2^l l!}\right)\left(\sum_{m=1}^{\infty}\frac{1}{m^k}\frac{t^{m-1}}{(m-1)!}\right)\\
&=\left(\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\right)\left(\sum_{l=0}^{\infty}E_{l,\lambda}\frac{1}{2^l}\sum_{j=l}^{\infty}S_2(j,l)\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{t^m}{m!}\right)\\
&=\left(\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\right)\left(\sum_{j=0}^{\infty}\sum_{l=0}^{j}\frac{1}{2^l}E_{l,\lambda}S_2(j,l)\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{t^m}{m!}\right)\\
&=\left(\sum_{\alpha=0}^{\infty}\left(\sum_{j=0}^{\beta}\sum_{l=0}^{j}\binom{\alpha}{j}\frac{1}{2^l}E_{l,\lambda}S_2(j,l)B_{\alpha-j}\right)\frac{t^\alpha}{\alpha!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{t^m}{m!}\right)\\
&=\left(\sum_{n=0}^{\infty}\left(\sum_{\alpha=0}^{n}\sum_{j=0}^{\beta}\sum_{l=0}^{j}\binom{n}{\alpha}\binom{\alpha}{j}\frac{1}{2^l(n-\alpha+1)^k}S_2(j,l)E_{l,\lambda}B_{\alpha-j}\right)\frac{t^n}{n!}.\right)
\end{aligned} \tag{62}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(k)}\frac{(e^t-1)^m}{2^m m!}&=\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}^{(k)}\frac{1}{2^m}\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{2^m}\mathcal{E}_{m,\lambda}^{(k)}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{63}$$

Therefore, by comparing the coefficients of (62) and (63), we obtain the desired result. □

4. Conclusions

In this paper, first, we introduced a new type of the type 2 poly-Euler polynomials and numbers constructed from the modified polyexponential function. We studied several identities for these polynomials and number. Specifically, we obtained various expressions of the type 2 poly-Euler polynomials in terms of Bernoulli polynomials and the Stirling numbers of the first kind; Euler polynomials and the Stirling numbers of the first kind; the type 2 poly-Euler polynomials and the Stirling numbers of the second kind; the poly Bernoulli polynomials of [11] in Theorem 2, 3, 12, and 13; etc. We also deduced an expression of the type 2 poly-Euler numbers of the Euler number and values of higher order Bernoulli polynomials at zero in Theorem 8. In Section 3, we derived some identities the type 2 degenerate poly-Euler polynomials in terms of the degenerate Bernoulli polynomials; the degenerate Euler polynomials and the Stirling numbers of the first kind; the degenerate the Stirling numbers of the second kind; the degenerate Euler numbers and the Stirling numbers of the first kind; the degenerate Euler numbers; and the Bernoulli numbers in Theorems 14, 15, 16, 20, and 21.
The field of degenerate versions is widely applied not only to number theory and combinatorics but also to symmetric identities, differential equations and probability theory. In recent years, many symmetric identities have been studied for degenerate versions of many special polynomials [6–10,19–26]. In the future, we hope to continue to study the degenerate versions of various polynomials and help them in other fields such as physics, engineering, etc.

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