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$SU(2) \times SU(2)$ Algebras and the Lorentz Group $O(3,3)$

Martin Walker

Independent Researcher, 3958 Grandis Place, Victoria, BC V8N 4H6, Canada; martinpwalker4691@gmail.com

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Abstract: The Lie algebra of the Lorentz group $O(3,3)$ admits two types of $SU(2) \times SU(2)$ subalgebras: a standard form based on spatial rotation generators and a second form based on temporal rotation generators. The units of measurement for the conserved quantity due to invariance under temporal rotations are investigated and found to be the same units of measure as the Planck constant. The breaking of time reversal symmetry is considered and found to affect the chiral properties of a temporal $SU(2) \times SU(2)$ algebra. Finally, the symmetry between algebras is explored and pairs of algebras are found to be related by $SU(2) \times U(1)$ symmetry, while a group of three algebras are related by $SO(4)$ symmetry.

Keywords: Lie algebra; $O(3,3)$; time rotation; Dirac; Noether

1. Introduction

Spinors were first introduced by Elli Cartan in 1913. The ideas were later adopted into quantum mechanics to describe the intrinsic spin of a fermion and play a fundamental role in Dirac's equation [1]. In group theory, spinors transform under the spin $\frac{1}{2}$ representation of an $SU(2) \times SU(2)$ Lie algebra, which is also the Lie algebra of the proper Lorentz group $O(3,1)$ [2].

This article investigates some aspects of symmetry in the Lorentz group $O(3,3)$. This Lie group can be associated with a six-dimensional mathematical space containing three space dimensions and three time dimensions [3]. The corresponding Lie algebra is $SO(3,3)$ in which the symmetry of time and the symmetry of space are isomorphic. As a result, there are two types of $SU(2) \times SU(2)$ subalgebras: one containing spatial rotation generators and one containing temporal rotation generators.

To better understand the temporal $SU(2) \times SU(2)$ algebras, we investigate the units of measure for the conserved quantity due to invariance under temporal rotations, for a restricted definition of action, in an $O(3,3)$ space. Using Noether's theorem, it is found that the conserved quantity has the same units of measure as the Planck constant.

We also consider the effects of breaking time reversal symmetry. For a temporal $SU(2) \times SU(2)$ algebra, the two chiralities are related by a time reversal transformation. This suggests that breaking time reversal symmetry affects the chiral properties of a temporal $SU(2) \times SU(2)$ algebra.

Finally, we explore symmetries between different algebras in $SO(3,3)$. We find pairs of algebras related by $SU(2) \times U(1)$ symmetry, as well as a group of three algebras related by $SO(4)$ symmetry.

In Section 2, two types of $SU(2) \times SU(2)$ algebras are described. In Section 3, we investigate the units of measure for the conserved quantity due to invariance under temporal rotations. In Section 4, we consider the implications of breaking time reversal symmetry. In Section 5, the symmetry between algebras is explored.

2. $SU(2) \times SU(2)$ Subalgebras

One form of $SU(2) \times SU(2)$ Lie algebra is related to the proper Lorentz group $O(3,1)$. This Lie group can be associated with transformations in a four-dimensional space containing three space dimensions and one time dimension [4]. It has six generators [2],

$$\mathrm{J_1, J_2, J_3, K_1, K_2, K_3} \tag{1}$$

where the J's are spatial rotation generators and the K's are boosts. The commutation relations for this algebra are,

$$[\mathrm{J_j, J_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} \qquad [\mathrm{K_j, K_k}] = -\mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} \qquad [\mathrm{J_j, K_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{K_m} \tag{2}$$

where ϵ is the Levi-Civita symbol, i is the imaginary unit and the indexes j, k, m can assume any value from 1 to 3. Using a complexification and a change of basis the Lie algebra becomes a direct product of two $SU(2)$ algebras [5],

$$\frac{1}{2}(\mathrm{J_1 + iK_1}), \frac{1}{2}(\mathrm{J_2 + iK_2}), \frac{1}{2}(\mathrm{J_3 + iK_3}), \frac{1}{2}(\mathrm{J_1 - iK_1}), \frac{1}{2}(\mathrm{J_2 - iK_2}), \frac{1}{2}(\mathrm{J_3 - iK_3}) \tag{3}$$

with commutation relations

$$\begin{aligned}
&[\tfrac{1}{2}(\mathrm{J_j + iK_j}), \tfrac{1}{2}(\mathrm{J_k + iK_k})] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \tfrac{1}{2}(\mathrm{J_m + iK_m}) \\
&[\tfrac{1}{2}(\mathrm{J_j - iK_j}), \tfrac{1}{2}(\mathrm{J_k - iK_k})] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \tfrac{1}{2}(\mathrm{J_m - iK_m}) \\
&[\tfrac{1}{2}(\mathrm{J_j + iK_j}), \tfrac{1}{2}(\mathrm{J_k - iK_k})] = 0
\end{aligned} \tag{4}$$

where the indexes j, k, m = 1, 2, 3. This $SU(2) \times SU(2)$ algebra is associated with the description of spin angular momentum in quantum mechanics [2,5]. Please note that in the text that follows, an $SU(2) \times SU(2)$ algebra will often be written in a format like

$$\{\frac{1}{2}(\mathrm{J_1 \pm iK_1}), \frac{1}{2}(\mathrm{J_2 \pm iK_2}), \frac{1}{2}(\mathrm{J_3 \pm iK_3})\} \tag{5}$$

where the curly brackets are delimiters for a list of generators.

This article investigates $SU(2) \times SU(2)$ algebras in the context of the Lorentz group $O(3,3)$. This Lie group can be associated with transformations in a six-dimensional space containing three space dimensions and three time dimensions [3,4]. Another label for this group is the special orthogonal Lie group $SO(3,3)$, which has fifteen generators [3,6,7]. The group has three space rotation generators, here labelled $\mathrm{J_i}$ (i = 1, 2, 3), it has three time rotation generators, labelled $\mathrm{T_i}$ (i = 1, 2, 3), and it has nine boost generators, labelled $\mathrm{K_{ij}}$, where the i index denotes the time dimension (i = 1, 2, 3) and the j index denotes the space dimension (j = 1, 2, 3) (see Appendix A for a matrix representation of the generators). The commutation relations in this notation are,

$$\begin{aligned}
&[\mathrm{T_j, T_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{T_m} && [\mathrm{J_j, J_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} && [\mathrm{T_j, J_k}] = 0 \\
&[\mathrm{K_{jn}, K_{kn}}] = -\mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{T_m} && [\mathrm{K_{nj}, K_{nk}}] = -\mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} && \\
&[\mathrm{T_j, K_{kn}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{K_{mn}} && [\mathrm{J_j, K_{nk}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{K_{nm}} &&
\end{aligned} \tag{6}$$

where the indexes j, k, m, n = 1, 2, 3

The complexification of the Lie algebra of $SO(3,3)$ used in this article is one in which all the boost generators are multiplied by the imaginary unit, while the rotation generators are left unchanged. This is the same complexification commonly used on the Lie algebra of the Lorentz group $O(3,1)$ [5]. This results in the following commutation relations,

$$\begin{aligned}
&[\mathrm{T_j, T_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{T_m} && [\mathrm{J_j, J_k}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} && [\mathrm{T_j, J_k}] = 0 \\
&[\mathrm{iK_{jn}, iK_{kn}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{T_m} && [\mathrm{iK_{nj}, iK_{nk}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{J_m} && \\
&[\mathrm{T_j, iK_{kn}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{iK_{mn}} && [\mathrm{J_j, iK_{nk}}] = \mathrm{i}\, \epsilon_{\mathrm{jkm}}\, \mathrm{iK_{nm}} &&
\end{aligned} \tag{7}$$

where the indexes j, k, m, n = 1,2,3.

Complexified $SO(3,3)$ has three complexified $SO(3,1)$ subspaces which give rise to three $SU(2) \times SU(2)$ subalgebras containing spatial rotation generators:

$$\begin{aligned} \mathrm{e}_1 &= \{\tfrac{1}{2}(\mathrm{J}_1 \pm \mathrm{iK}_{11}), \tfrac{1}{2}(\mathrm{J}_2 \pm \mathrm{iK}_{12}), \tfrac{1}{2}(\mathrm{J}_3 \pm \mathrm{iK}_{13})\} \\ \mathrm{e}_2 &= \{\tfrac{1}{2}(\mathrm{J}_1 \pm \mathrm{iK}_{21}), \tfrac{1}{2}(\mathrm{J}_2 \pm \mathrm{iK}_{22}), \tfrac{1}{2}(\mathrm{J}_3 \pm \mathrm{iK}_{23})\} \\ \mathrm{e}_3 &= \{\tfrac{1}{2}(\mathrm{J}_1 \pm \mathrm{iK}_{31}), \tfrac{1}{2}(\mathrm{J}_2 \pm \mathrm{iK}_{32}), \tfrac{1}{2}(\mathrm{J}_3 \pm \mathrm{iK}_{33})\}. \end{aligned} \tag{8}$$

These have the standard form [2], and we are encouraged to think of them as a family, as they differ only by the value of the time index in the boost generators.

Complexified $SO(3,3)$ also has three complexified $SO(1,3)$ subspaces which give rise to a family of $SU(2) \times SU(2)$ subalgebras containing temporal rotation generators:

$$\begin{aligned} \mathrm{m}_1 &= \{\tfrac{1}{2}(\mathrm{T}_1 \pm \mathrm{iK}_{11}), \tfrac{1}{2}(\mathrm{T}_2 \pm \mathrm{iK}_{21}), \tfrac{1}{2}(\mathrm{T}_3 \pm \mathrm{iK}_{31})\} \\ \mathrm{m}_2 &= \{\tfrac{1}{2}(\mathrm{T}_1 \pm \mathrm{iK}_{12}), \tfrac{1}{2}(\mathrm{T}_2 \pm \mathrm{iK}_{22}), \tfrac{1}{2}(\mathrm{T}_3 \pm \mathrm{iK}_{32})\} \\ \mathrm{m}_3 &= \{\tfrac{1}{2}(\mathrm{T}_1 \pm \mathrm{iK}_{13}), \tfrac{1}{2}(\mathrm{T}_2 \pm \mathrm{iK}_{23}), \tfrac{1}{2}(\mathrm{T}_3 \pm \mathrm{iK}_{33})\}. \end{aligned} \tag{9}$$

These algebras differ only by the value of the space index in the boost generators.

3. Invariance under Temporal Rotations

We would like to determine the units of measurement for the conserved quantity due to invariance under temporal rotations. The field theory treatment of Noether's theorem that follows is adopted from Schwichtenberg [5] and applied to $O(3,3)$ space. We use the Einstein summation convention in this section.

For $O(3,3)$ space, a 6-vector is defined as having the form,

$$x_\mu = (x_1, x_2, x_3, x_4, x_5,\ x_6) \tag{10}$$

where the first three components are space dimensions and the last three components are time dimensions. In the following investigation we will restrict ourselves to the action, S_4, with respect to the time variable x_4. We define,

$$S_4 = \int dx_4 L_4 \quad L_4 = \int d^5x \mathcal{L}_4 \quad \mathcal{L}_4 = \mathcal{L}_4\Big(\Psi\big(x_\mu\big), \partial_\mu \Psi\big(x_\mu\big), x_\mu\Big) \tag{11}$$

where $\Psi\big(x_\mu\big)$ is a scalar field, L_4 is the Lagrangian, and the Lagrangian density, $\mathcal{L}_4$, is a density over an element $(\delta x_1, \delta x_2, \delta x_3, \delta x_5, \delta x_6)$. The equations of motion for this Lagrangian density are then given by the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}_4}{\partial\big(\partial_\mu \Psi\big)} \right) - \frac{\partial \mathcal{L}_4}{\partial \Psi} = 0. \tag{12}$$

3.1. *Infinitesimal Space-Time Translations for a Scalar Field*

For an infinitesimal space-time translation we have,

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu = x_\mu + a_\mu \tag{13}$$

where a_μ is an arbitrary infinitesimal change. If the transformation does not change the Lagrangian density we get,

$$\delta \mathcal{L} = -\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Psi)} \frac{\partial \Psi}{\partial x_\mu} - \delta^\nu_\mu \mathcal{L} \right) a^\mu = 0 \tag{14}$$

where δ is the Kronecker delta. If a^μ is arbitrary then we must have,

$$\partial_\nu T^\nu_\mu = 0 \text{ where } T^\nu_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Psi)} \frac{\partial \Psi}{\partial x_\mu} - \delta^\nu_\mu \mathcal{L} \right) \tag{15}$$

which gives us one continuity equation for each component μ. The elements T^ν_μ are said to define components of the energy-momentum tensor.

For $\mathcal{L}_4$, there are six continuity equations given by

$$\begin{aligned}
\partial_1 T^1_1 + \partial_2 T^2_1 + \partial_3 T^3_1 + \partial_4 T^4_1 + \partial_5 T^5_1 + \partial_6 T^6_1 = 0 \\
\partial_1 T^1_2 + \partial_2 T^2_2 + \partial_3 T^3_2 + \partial_4 T^4_2 + \partial_5 T^5_2 + \partial_6 T^6_2 = 0 \\
\partial_1 T^1_3 + \partial_2 T^2_3 + \partial_3 T^3_3 + \partial_4 T^4_3 + \partial_5 T^5_3 + \partial_6 T^6_3 = 0 \\
\partial_1 T^1_4 + \partial_2 T^2_4 + \partial_3 T^3_4 + \partial_4 T^4_4 + \partial_5 T^5_4 + \partial_6 T^6_4 = 0 \\
\partial_1 T^1_5 + \partial_2 T^2_5 + \partial_3 T^3_5 + \partial_4 T^4_5 + \partial_5 T^5_5 + \partial_6 T^6_5 = 0 \\
\partial_1 T^1_6 + \partial_2 T^2_6 + \partial_3 T^3_6 + \partial_4 T^4_6 + \partial_5 T^5_6 + \partial_6 T^6_6 = 0
\end{aligned} \tag{16}$$

Taking into consideration the fourth equation, we can rearrange it and integrate both sides over an infinite volume,

$$\begin{aligned}
\partial_1 T^1_4 + \partial_2 T^2_4 + \partial_3 T^3_4 + \partial_4 T^4_4 + \partial_5 T^5_4 + \partial_6 T^6_4 = 0 \\
-\partial_4 T^4_4 = \partial_1 T^1_4 + \partial_2 T^2_4 + \partial_3 T^3_4 + \partial_5 T^5_4 + \partial_6 T^6_4 \\
-\partial_4 \int_V d^5x T^4_4 = \int_V d^5x \left(\partial_1 T^1_4 + \partial_2 T^2_4 + \partial_3 T^3_4 + \partial_5 T^5_4 + \partial_6 T^6_4 \right) \\
-\partial_4 \int_V d^5x T^4_4 = \int_V d^5x \nabla \boldsymbol{T} \\
-\partial_4 \int_V d^5x T^4_4 = \oint_{\delta V} d^4x \boldsymbol{T}
\end{aligned} \tag{17}$$

where $\nabla \boldsymbol{T} = \partial_1 T^1_4 + \partial_2 T^2_4 + \partial_3 T^3_4 + \partial_5 T^5_4 + \partial_6 T^6_4$, δV is the boundary of volume V and we have used the divergence theorem in the last step. The surface integral on the right hand side of this equation vanishes because the field vanishes at infinity and we are left with,

$$\partial_4 \int_V d^5x T^4_4 = 0 \tag{18}$$

which implies that $\int d^5x T^4_4$ is conserved.

Using a similar method with the other equations gives us six conserved quantities. We know already that the conserved quantities for invariance under time and space translations in $O(3,1)$ are energy and momentum, respectively. We make the following assignments for the conserved quantities,

$$E_1 = \int d^5x T^4_4 \quad E_2 = \int d^5x T^4_5 \quad E_3 = \int d^5x T^4_6 \tag{19}$$

and

$$P_1 = \int d^5x T^4_1 \quad P_2 = \int d^5x T^4_2 \quad P_3 = \int d^5x T^4_3 \tag{20}$$

where $E_1,\ E_2, E_3$ are energies and $P_1,\ P_2, P_3$ are momentums.

3.2. *Infinitesimal Space-Time Rotations for a Scalar Field*

For an infinitesimal space-time rotation we have,

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu = x_\mu + M^\sigma_\mu x_\sigma \tag{21}$$

where the M^σ_μ are generators of rotations. Setting the change in the Lagrangian density to zero results in,

$$\begin{aligned} \delta\mathcal{L} &= -\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Psi)}\frac{\partial\Psi}{\partial x_\mu} - \delta^\nu_\mu\mathcal{L}\right)M^{\mu\sigma}x_\sigma = 0 \\ &\rightarrow \partial_\nu(T^{\mu\nu}x^\sigma - T^{\sigma\nu}x^\mu)M_{\mu\sigma} = 0 \end{aligned} \tag{22}$$

where there is one continuity equation for each rotation generator $M_{\mu\sigma}$.The values of μ and σ for the spatial rotation generators, J_i, are obtained from the relation,

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \tag{23}$$

where ϵ is again the Levi-Civita symbol. This gives:

$$J_1 = \frac{1}{2}M_{23} \quad J_2 = \frac{1}{2}M_{31} \quad J_3 = \frac{1}{2}M_{12}. \tag{24}$$

For $\mathcal{L}_4$, there are three equations:

$$\begin{aligned} &\partial_1\left(T^{21}x^3 - T^{31}x^2\right) + \partial_2\left(T^{22}x^3 - T^{32}x^2\right) + \partial_3\left(T^{23}x^3 - T^{33}x^2\right) \\ &+\partial_4\left(T^{24}x^3 - T^{34}x^2\right) + \partial_5\left(T^{25}x^3 - T^{35}x^2\right) + \partial_6\left(T^{26}x^3 - T^{36}x^2\right) = 0 \\ &\partial_1\left(T^{31}x^1 - T^{11}x^3\right) + \partial_2\left(T^{32}x^1 - T^{12}x^3\right) + \partial_3\left(T^{33}x^1 - T^{13}x^3\right) \\ &+\partial_4\left(T^{34}x^1 - T^{14}x^3\right) + \partial_5\left(T^{35}x^1 - T^{15}x^3\right) + \partial_6\left(T^{36}x^1 - T^{16}x^3\right) = 0 \\ &\partial_1\left(T^{11}x^2 - T^{21}x^1\right) + \partial_2\left(T^{12}x^2 - T^{22}x^1\right) + \partial_3\left(T^{13}x^2 - T^{23}x^1\right) \\ &+\partial_4\left(T^{14}x^2 - T^{24}x^1\right) + \partial_5\left(T^{15}x^2 - T^{25}x^1\right) + \partial_6\left(T^{16}x^2 - T^{26}x^1\right) = 0. \end{aligned} \tag{25}$$

We can again use the divergence theorem to obtain the three continuity equations corresponding to conserved quantities:

$$\begin{aligned} \partial_4 \int d^5x\left(T^{24}x^3 - T^{34}x^2\right) = 0 \\ \partial_4 \int d^5x\left(T^{34}x^1 - T^{14}x^3\right) = 0 \\ \partial_4 \int d^5x\left(T^{14}x^2 - T^{24}x^1\right) = 0. \end{aligned} \tag{26}$$

The terms in each integrand are a product of a momentum density (associated with one of P_1, P_2, P_3) and a space variable (one of x^1, x^2, x^3). We conclude that these have units of angular momentum, as required.

To determine the conserved quantities related to the temporal rotation generators, $\check{\mathrm{T}}_i$, we can get the values of μ and σ using the relation,

$$\check{\mathrm{T}}_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \tag{27}$$

This gives:

$$\check{\mathrm{T}}_4 = \frac{1}{2}M_{56} \quad \check{\mathrm{T}}_5 = \frac{1}{2}M_{64} \quad \check{\mathrm{T}}_6 = \frac{1}{2}M_{45}. \tag{28}$$

The resulting three continuity equations are,

$$\begin{aligned} &\partial_1\left(T^{51}x^6 - T^{61}x^5\right) + \partial_2\left(T^{52}x^6 - T^{62}x^5\right) + \partial_3\left(T^{53}x^6 - T^{63}x^5\right) \\ &+\partial_4\left(T^{54}x^6 - T^{64}x^5\right) + \partial_5\left(T^{55}x^6 - T^{65}x^5\right) + \partial_6\left(T^{56}x^6 - T^{66}x^5\right) = 0 \\ &\partial_1\left(T^{61}x^4 - T^{41}x^6\right) + \partial_2\left(T^{62}x^4 - T^{42}x^6\right) + \partial_3\left(T^{63}x^4 - T^{43}x^6\right) \\ &+\partial_4\left(T^{64}x^4 - T^{44}x^6\right) + \partial_5\left(T^{65}x^4 - T^{45}x^6\right) + \partial_6\left(T^{66}x^4 - T^{46}x^6\right) = 0 \\ &\partial_1\left(T^{41}x^5 - T^{51}x^4\right) + \partial_2\left(T^{42}x^5 - T^{52}x^4\right) + \partial_3\left(T^{43}x^5 - T^{53}x^4\right) \\ &+\partial_4\left(T^{44}x^5 - T^{54}x^4\right) + \partial_5\left(T^{45}x^5 - T^{55}x^4\right) + \partial_6\left(T^{46}x^5 - T^{56}x^4\right) = 0 \end{aligned} \tag{29}$$

which simplify to the equations,

$$\begin{aligned}\partial_4 \int\ d^5x\Big(T^{54}x^6 - T^{64}x^5\Big) = 0\\ \partial_4 \int\ d^5x\Big(T^{64}x^4 - T^{44}x^6\Big) = 0\\ \partial_4 \int\ d^5x\Big(T^{44}x^5 - T^{54}x^4\Big) = 0\end{aligned} \tag{30}$$

Here, the terms in each integrand are a product of an energy density (associated with one of E_1, E_2, E_3) and a time variable (one of x^4, x^5, x^6). If we consider the first equation then the units of measure for the first term are,

$$[d^5x]_{MKS} = m^5 \quad [T^{54}]_{MKS} = kg\ m^{-3}s^{-2} \quad [x^6]_{MKS} = s \tag{31}$$

giving

$$\Big[(d^5x\Big)(T^{54})\Big(x^6\Big)]_{MKS} = kg\ m^2s^{-1}. \tag{32}$$

We conclude that these have the same units of measure as the Planck constant.

We note that the units of measure for the conserved quantity due to invariance under spatial rotations are also the same units of measure as the Planck constant and that the conserved quantity, for some non-scalar fields, has been associated with spin angular momentum [5].

4. Breaking Time Reversal Symmetry

The spatial $SU(2) \times SU(2)$ algebras in complexified $SO(3,3)$ have the basic form

$$\begin{array}{ll}\text{left chirality:} & \{\frac{1}{2}(\mathrm{J}_1 + \mathrm{iK}_{a1}),\ \frac{1}{2}(\mathrm{J}_2 + \mathrm{iK}_{a2}),\ \frac{1}{2}(\mathrm{J}_3 + \mathrm{iK}_{a3})\}\\ \text{right chirality:} & \{\frac{1}{2}(\mathrm{J}_1 - \mathrm{iK}_{a1}),\ \frac{1}{2}(\mathrm{J}_2 - \mathrm{iK}_{a2}),\ \frac{1}{2}(\mathrm{J}_3 - \mathrm{iK}_{a3})\}\end{array} \tag{33}$$

where a = 1, 2, 3 and the two chiralities are related by a spatial parity transformation [2]. The temporal $SU(2) \times SU(2)$ algebras have the basic form

$$\begin{array}{ll}\text{first chirality:} & \{\frac{1}{2}(\mathrm{T}_1 + \mathrm{iK}_{1b}),\ \frac{1}{2}(\mathrm{T}_2 + \mathrm{iK}_{2b}),\ \frac{1}{2}(\mathrm{T}_3 + \mathrm{iK}_{3b})\}\\ \text{second chirality:} & \{\frac{1}{2}(\mathrm{T}_1 - \mathrm{iK}_{1b}),\ \frac{1}{2}(\mathrm{T}_2 - \mathrm{iK}_{2b}),\ \frac{1}{2}(\mathrm{T}_3 - \mathrm{iK}_{3b})\}\end{array} \tag{34}$$

where b = 1, 2, 3 and the two chiralities are related by a time reversal transformation.

The two chiral parts of a spatial $SU(2) \times SU(2)$ algebra are related by a spatial parity transformation and so appear to be unaffected by breaking time reversal symmetry. The two chiral parts of a temporal $SU(2) \times SU(2)$ algebra are related by a time reversal transformation. This suggests that breaking time reversal symmetry affects the chiral properties of a temporal $SU(2) \times SU(2)$ algebra.

5. Symmetry between Algebras

The special orthogonal Lie group $SO(4)$ can be associated with the group of rotations in a four-dimensional Euclidean space [4]. The group has six generators, here labelled $\mathrm{a_j}$,$\mathrm{b_j}$ (j = 1, 2, 3), and commutation relations:

$$\begin{aligned}[\mathrm{a_j}, \mathrm{a_k}] = \mathrm{i}\ \epsilon_{\mathrm{jkm}}\ \mathrm{a_m}\\ [\mathrm{b_j}, \mathrm{b_k}] = \mathrm{i}\ \epsilon_{\mathrm{jkm}}\ \mathrm{a_m}\\ [\mathrm{a_j}, \mathrm{b_k}] = \mathrm{i}\ \epsilon_{\mathrm{jkm}}\ \mathrm{b_m}\end{aligned} \tag{35}$$

where the indexes j, k, m = 1, 2, 3. The Lie group $SO(3)$, associated with the group of rotations in three dimensions, has three generators, here labelled $\mathrm{w_j}$ (j = 1, 2, 3), and commutation relations,

$$[\mathrm{w_j}, \mathrm{w_k}] = \mathrm{i}\ \epsilon_{\mathrm{jkm}}\ \mathrm{w_m} \tag{36}$$

where the indexes j, k, m = 1, 2, 3. The direct product $SO(3) \times SO(2)$ has four generators, here labelled w_j (j = 0, 1, 2, 3), and commutation relations,

$$[w_j, w_k] = i\, \epsilon_{jkm}\, w_m \qquad [w_0, w_k] = 0 \tag{37}$$

where the indexes j, k, m = 1, 2, 3. We also note that $SU(2)$ and $SO(3)$ have the same Lie algebra, and that $U(1)$ and $SO(2)$ are isomorphic [5].

5.1. SO(3) × SO(2) symmetry

The e_1 spatial $SU(2) \times SU(2)$ algebra might be represented in tabular form as,

$$\begin{array}{cccccc} \mathbf{\frac{1}{2}(a_1 + b_1)} & \mathbf{\frac{1}{2}(a_2 + b_2)} & \mathbf{\frac{1}{2}(a_3 + b_3)} & \mathbf{\frac{1}{2}(a_1 - b_1)} & \mathbf{\frac{1}{2}(a_2 - b_2)} & \mathbf{\frac{1}{2}(a_3 - b_3)} \\ \frac{1}{2}(J_1 + iK_{11}) & \frac{1}{2}(J_2 + iK_{12}) & \frac{1}{2}(J_3 + iK_{13}) & \frac{1}{2}(J_1 - iK_{11}) & \frac{1}{2}(J_2 - iK_{12}) & \frac{1}{2}(J_3 - iK_{13})\} \end{array} \tag{38}$$

where the a's and b's are the generic $SO(4)$ labels given in (35). With a change of basis this becomes:

$$\begin{array}{cccccc} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \\ J_1 & J_2 & J_3 & iK_{11} & iK_{12} & iK_{13} \end{array} \tag{39}$$

This $SO(4)$ contains four $SO(3)$ subalgebras. There is a spatial $SO(3)$ algebra:

$$\begin{array}{ccc} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \\ J_1 & J_2 & J_3 \end{array} \tag{40}$$

Here, the w's are the generic $SO(3)$ labels given in (36). There are also three other $SO(3)$ algebras:

$$\begin{array}{ccc} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \\ J_2 & iK_{13} & iK_{11} \\ J_3 & iK_{11} & iK_{12} \\ J_1 & iK_{12} & iK_{13}. \end{array} \tag{41}$$

Additionally, the $SO(4)$ commutes with a rotation generator, T_1, which will give us three $SO(3) \times SO(2)$ algebras,

$$\begin{array}{cccc} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} & \mathbf{w_0} \\ J_2 & iK_{13} & iK_{11} & T_1 \\ J_3 & iK_{11} & iK_{12} & T_1 \\ J_1 & iK_{12} & iK_{13} & T_1 \end{array} \tag{42}$$

where the w's are the generic $SO(3) \times SO(2)$ labels given in (37). Changing the basis to $\frac{1}{2}(w_1 \pm w_2)$ and $\frac{1}{2}(w_0 \pm w_3)$ yields

$$\begin{array}{cc} \mathbf{\frac{1}{2}(w_1 \pm w_2)} & \mathbf{\frac{1}{2}(w_0 \pm w_3)} \\ \frac{1}{2}(J_2 \pm iK_{13}) & \frac{1}{2}(T_1 \pm iK_{11}) \\ \frac{1}{2}(J_3 \pm iK_{11}) & \frac{1}{2}(T_1 \pm iK_{12}) \\ \frac{1}{2}(J_1 \pm iK_{12}) & \frac{1}{2}(T_1 \pm iK_{13}) \end{array} \tag{43}$$

If the columns are considered to be six component algebras then in horizontal form we have

$$\begin{aligned} \tfrac{1}{2}(w_1 \pm w_2) &= \{\tfrac{1}{2}(J_2 \pm iK_{13}), \tfrac{1}{2}(J_3 \pm iK_{11}), \tfrac{1}{2}(J_1 \pm iK_{12})\} \\ \tfrac{1}{2}(w_0 \pm w_3) &= \{\tfrac{1}{2}(T_1 \pm iK_{11}), \tfrac{1}{2}(T_1 \pm iK_{12}), \tfrac{1}{2}(T_1 \pm iK_{13})\}. \end{aligned} \tag{44}$$

Rotating $\frac{1}{2}(w_1 \pm w_2)$ within the vector space of the $SO(4)$ then gives

$$\frac{1}{2}(w_1 \pm w_2)' = \{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(J_3 \pm K_{13})\}. \tag{45}$$

We conclude that $\frac{1}{2}(w_1 \pm w_2)'$ and $\frac{1}{2}(w_0 \pm w_3)$ are related by $SO(3) \times SO(2)$ symmetry plus a rotation.

Inspection shows that the $\frac{1}{2}(w_1 \pm w_2)'$ algebra is the same as e_1 algebra. This suggests that the e-family is related to another family of algebras by $SO(3) \times SO(2)$ symmetry plus a rotation. This is the n-family:

$$\begin{aligned} n_1 &= \{\tfrac{1}{2}(T_1 \pm iK_{11}), \tfrac{1}{2}(T_1 \pm iK_{12}), \tfrac{1}{2}(T_1 \pm iK_{13})\} \\ n_2 &= \{\tfrac{1}{2}(T_2 \pm iK_{21}), \tfrac{1}{2}(T_2 \pm iK_{22}), \tfrac{1}{2}(T_2 \pm iK_{23})\} \\ n_3 &= \{\tfrac{1}{2}(T_3 \pm iK_{31}), \tfrac{1}{2}(T_3 \pm iK_{32}), \tfrac{1}{2}(T_3 \pm iK_{33})\}. \end{aligned} \tag{46}$$

These algebras are associated with three spatial dimensions, as indicated by the boost generators. The n-family members are not $SU(2) \times SU(2)$ algebras.

5.2. SO(4) Symmetry

The members of the n-family are related by $SO(4)$ symmetry. This can be illustrated by constructing an array of generators:

$$\begin{array}{cccc} & \mathbf{n_1} & \mathbf{n_2} & \mathbf{n_3} \\ \mathbf{m_1} & \tfrac{1}{2}(T_1 \pm iK_{11}) & \tfrac{1}{2}(T_2 \pm iK_{21}) & \tfrac{1}{2}(T_3 \pm iK_{31}) \\ \mathbf{m_2} & \tfrac{1}{2}(T_1 \pm iK_{12}) & \tfrac{1}{2}(T_2 \pm iK_{22}) & \tfrac{1}{2}(T_3 \pm iK_{32}) \\ \mathbf{m_3} & \tfrac{1}{2}(T_1 \pm iK_{13}) & \tfrac{1}{2}(T_2 \pm iK_{23}) & \tfrac{1}{2}(T_3 \pm iK_{33}). \end{array} \tag{47}$$

Here, the rows are the m-family algebras which have $SO(4) = SO(3) \times SO(3)$ symmetry, and the columns are the n-family. We also note that the n_1 algebra shares two of its components with each of m_1, m_2, and m_3. This suggests that an n-family algebra might be described as a mixture of m-family components.

6. Conclusions

This article has considered some of the mathematical properties and relationships associated with $SU(2) \times SU(2)$ subalgebras in an $O(3,3)$ space. In particular, we find the following:

1. The e-family members are the standard type of $SU(2) \times SU(2)$ algebra, associated with three space dimensions and one time dimension.
2. The e_1 algebra is related to the n_1 algebra by $SU(2) \times U(1)$ symmetry, plus a rotation.
3. We can describe the n_1 algebra as being a mixture of components from the three m-family algebras.
4. The m-family members are a second type of $SU(2) \times SU(2)$ algebra, associated with one space dimension and three time dimensions.
5. Breaking of time reversal symmetry affects the chiral properties of the m-family algebras.
6. The units of measure of the conserved quantity due to invariance under temporal rotations are the same as those of the Planck constant.

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Appendix A. *SO*(3,3) Generators (Referenced in Section 2)

Time rotation generators:

$$T1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} T2 = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} T3 = \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Space rotation generators:

$$J1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{bmatrix} J2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix} J3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Boost generators:

$$K11 = \begin{bmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K12 = \begin{bmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K13 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K21 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K22 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K23 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K31 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K32 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} K33 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{bmatrix}$$

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