On the Nature of γ -th Arithmetic Zeta Functions

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Abstract: Symmetry and elementary symmetric functions are main components of the proof of the celebrated Hermite–Lindemann theorem (about the transcendence of e^α , for algebraic values of α) which settled the ancient Greek problem of squaring the circle. In this paper, we are interested in similar results, but for powers such as $e^{\gamma \log n}$. This kind of problem can be posed in the context of arithmetic functions. More precisely, we study the arithmetic nature of the so-called γ -th arithmetic zeta function $\zeta_\gamma(n) := n^\gamma (= e^{\gamma \log n})$, for a positive integer n and a complex number γ . Moreover, we raise a conjecture about the exceptional set of ζ_γ , in the case in which γ is transcendental, and we connect it to the famous Schanuel’s conjecture.

Keywords: symmetry; zeta arithmetic function; transcendental numbers; Hermite–Lindemann theorem; Schanuel’s conjecture

MSC: Primary 11J81; Secondary 11A25

1. Introduction

An arithmetic function is a complex-valued function whose domain is the set of natural numbers $\mathbb{N}$. The set A of all arithmetic functions forms a ring with respect to addition (+) and the Dirichlet convolution (*), see [1,2] and some new results in [3–7]. Recall that the Dirichlet convolution of $f, g \in A$ is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d),$$

where the sum extends over all positive divisors d of n . A set of arithmetic functions $f_1, \dots, f_m$ is said to be $*$ -algebraically independent over $\mathbb{C}$ if there does not exist a non-zero polynomial P with complex coefficients such that

$$P(f_1, \dots, f_m) := \sum_{(i)} a_{(i)} f_1^{*i_1} * \dots * f_m^{*i_m} = 0,$$

where $a_{(i)} \in \mathbb{C}$ and $f^{*i} = f * \dots * f$ (i times).

Let γ be a complex number. The γ -th arithmetic zeta function is defined as

$$\zeta_\gamma(n) = n^\gamma.$$

Carlitz [8] showed that $\zeta_0, \dots, \zeta_r$ are $*$ -algebraically independent over $\mathbb{C}$, for any integer $r \geq 1$. A generalization for these Carlitz’s result can be found in [9]. Our interest starts with the following definition:

Definition 1. Let $f : D \rightarrow \mathbb{C}$ be a function, with $D \cap \overline{\mathbb{Q}} \neq \emptyset$. We define the exceptional set of f to be

$$S_f = \{\alpha \in D \cap \overline{\mathbb{Q}} : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Huang, Marques and Mereb [10] proved that for any $S \subseteq \overline{\mathbb{Q}}$, there exist uncountably many transcendental entire functions f for which $S_f = S$.

In this work, we study the possible exceptional sets of γ -th arithmetic zeta functions.

2. Algebraic Values of ζ_γ

We are interested in studying the set $S_{\zeta_\gamma} = \{\alpha \in \mathbb{N} : \zeta_\gamma(\alpha) \in \overline{\mathbb{Q}}\}$. For convenience, we write S_γ instead of S_{ζ_γ} . First, we point out to some simple facts (which hold for any $\gamma \in \mathbb{C}$):

- $S_\gamma \neq \emptyset$, since $1 \in S_\gamma$;
- If $n \in S_\gamma$, then $n^k \in S_\gamma$ for all $k \geq 1$;
- If $n \notin S_\gamma$, then $n^k \notin S_\gamma$ for all $k \geq 1$;
- If $n \in S_\gamma$ and $m \notin S_\gamma$, then $mn \notin S_\gamma$;
- If $Q \in \mathbb{Q} \setminus \{0\}$, then $S_\gamma = S_{Q\gamma}$.

Now, we shall split our study according to the arithmetic nature of γ .

2.1. The Number γ Is Algebraic

For the rational case we use the following lemma

Lemma 1. *Let r, s be any rational numbers. Then r^s is an algebraic number.*

Proof. Set $r = a/b$ and $s = m/n$. If $r^s = T \notin \overline{\mathbb{Q}}$, then we would have that $(a/b)^m = T^n \notin \overline{\mathbb{Q}}$. So $r^s \in \overline{\mathbb{Q}}$. $\square$

For the other case, we recall a well-known result

Theorem 1 (Gelfond–Schneider). *If $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then $\alpha^\beta \notin \overline{\mathbb{Q}}$.*

The proof of this theorem can be found in [11,12].

We state our first result on the exceptional sets of ζ_γ whose proof follows from Lemma 1 and by the Gelfond–Schneider theorem.

Proposition 1. *Let γ be an algebraic number.*

- (i) *If $\gamma \in \mathbb{Q}$, then $S_\gamma = \mathbb{N}$;*
- (ii) *If $\gamma \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then $S_\gamma = \{1\}$.*

2.2. The Number γ Is Transcendental

For this case, we need to recall some concepts:

Definition 2. *The complex numbers $x_1, \dots, x_m$ are said to be multiplicatively independent, when the relation $x_1^{a_1} \cdots x_m^{a_m} = 1$ with integral exponents $a_1, \dots, a_m$ implies that $a_1 = \dots = a_m = 0$. Otherwise, they are called multiplicatively dependent.*

Example 1. *The set of all prime numbers is multiplicatively independent (i.e., any finite subset of primes is multiplicatively independent). Furthermore, any set algebraically independent is multiplicatively independent.*

Remark 1. *The numbers $x_1, \dots, x_m$ are multiplicatively independent if and only if $\log x_1, \dots, \log x_m$ are $\mathbb{Q}$ -linearly independent.*

Now, recall the Lindemann theorem: if $t \in \overline{\mathbb{Q}} \setminus \{0\}$, then e^t is transcendental. This important result implies, in particular, in the transcendence of π . Since any transcendental number cannot be

constructed by using ruler and compass, the transcendence of π (and so $\sqrt{\pi}$) settled the old Greek problem of the quadrature of the circle by showing that this task is impossible. For the case in which t is transcendental, not much is known (e.g., is e^e a transcendental number? So far, this problem remains open). However, a remarkable advance in this direction was provided in the next result.

Theorem 2 (Six Exponential Theorem). *Let β_1, β_2 be complex numbers, linearly independent over $\mathbb{Q}$, and let z_1, z_2, z_3 be complex numbers, also linearly independent over $\mathbb{Q}$. Then at least one of the numbers*

$$e^{\beta_1 z_1}, e^{\beta_1 z_2}, e^{\beta_1 z_3}, e^{\beta_2 z_1}, e^{\beta_2 z_2}, e^{\beta_2 z_3}$$

is transcendental.

This result was firstly explicitly stated and proved independently by Lang [13] and Ramachandra [14,15].

Now, we are able to state our next result:

Proposition 2. *Let γ be a transcendental number. Then any three different numbers in S_γ must be multiplicatively dependent.*

Proof. Let $n_1, n_2, n_3 \in \mathbb{N}$ be multiplicatively independent numbers, so $\log n_1, \log n_2, \log n_3$ are linearly independent over $\mathbb{Q}$. Furthermore, $1, \gamma$ are linearly independent over $\mathbb{Q}$. Therefore, by Theorem 2, there exists at least one transcendental number among

$$e^{\log n_1}, e^{\log n_2}, e^{\log n_3}, e^{\gamma \log n_1}, e^{\gamma \log n_2}, e^{\gamma \log n_3}.$$

Hence at least one of the numbers $n_1^\gamma, n_2^\gamma, n_3^\gamma$ is transcendental. In other words, at least one of the numbers n_1, n_2, n_3 is not in S_γ . $\square$

As consequences, we have:

Corollary 1. *If γ is a transcendental number, then $\mathbb{N} \setminus S_\gamma$ is an infinite set.*

Proof. Let p be a prime number, since p, p^2 and p^3 are multiplicatively dependent, then, by Proposition 2, at least one of the numbers p, p^2, p^3 must belong to $\mathbb{N} \setminus S_\gamma$. So, this set is infinite, because the existence of infinitely many primes. $\square$

Corollary 2. *We have $S_\gamma = \mathbb{N}$ if and only if $\gamma \in \mathbb{Q}$.*

Proof. The ‘if’ part is the item (i) of Proposition 1. The hypothesis in the ‘only if’ part, is that $S_\gamma = \mathbb{N}$. Then γ cannot be transcendental (by Corollary 1). Furthermore, if $\gamma \in \mathbb{Q} \setminus \mathbb{Q}$, we would have $S_\gamma = \{1\}$, by Proposition 1 (ii). In conclusion, γ is a rational number. $\square$

Another consequence of Theorem 2 is the following result:

Proposition 3. *If γ is an irrational number and $A \subseteq \mathbb{N}$ is a set of multiplicatively independent numbers, then*

$$\#(S_\gamma \cap A) \leq 2.$$

Proof. Suppose, towards a contradiction, that $\#(S_\gamma \cap A) \geq 3$. If $a_1, a_2, a_3 \in S_\gamma \cap A$, then $\log a_1, \log a_2, \log a_3$ are linearly independent over $\mathbb{Q}$ (since every finite subset of A is multiplicatively independent).

Furthermore, $1, \gamma$ are linearly independent over $\mathbb{Q}$ (since γ is irrational). Therefore, by Theorem 2, there exists at least one transcendental number among

$$e^{\log a_1}, e^{\log a_2}, e^{\log a_3}, e^{\gamma \log a_1}, e^{\gamma \log a_2}, e^{\gamma \log a_3}.$$

However, these numbers are

$$a_1, a_2, a_3, a_1^\gamma, a_2^\gamma, a_3^\gamma,$$

which are all algebraic (since $a_i \in S_\gamma$). This contradicts Theorem 2 and then the proof is complete. $\square$

As an immediate consequence, we have:

Corollary 3. *There are at most two distinct prime p and q such that both p^π and q^π are algebraic numbers.*

Remark 2. *We remark that it is really expected that any number of the form n^π , with $n > 1$ integer, is transcendental. This is a consequence of Schanuel's conjecture (see Proposition 9 in the next section).*

The conclusion of Proposition 3 is the best possible that we can prove in an unconditional way. However, there is a conjecture which allows us to prove the sharpest related result.

Conjecture 1 (Four exponential conjecture). *Let β_1, β_2 be complex numbers, linearly independent over $\mathbb{Q}$, and let z_1, z_2 be complex numbers, also linearly independent over $\mathbb{Q}$. Then at least one of the numbers*

$$e^{\beta_1 z_1}, e^{\beta_1 z_2}, e^{\beta_2 z_1}, e^{\beta_2 z_2}$$

is transcendental.

So, with the same ideas of the proof of Proposition 3, we can obtain:

Proposition 4. *Suppose that Conjecture 1 holds. If γ is an irrational number and $A \subseteq \mathbb{N}$ is a set of multiplicatively independent numbers, then*

$$\#(S_\gamma \cap A) \leq 1.$$

Remark 3. *In fact, the previous result is sharp in the sense that we can not obtain that $\#(S_\gamma \cap A) = 0$ in the general case. For example, if $\gamma = \log 3 / \log 2$ and $A = \{2\}$, then $\#(S_\gamma \cap A) = 1$.*

Now, note that when $\gamma \notin \overline{\mathbb{Q}}$, then we can have that S_γ is an infinite set. For instance, $\{2^k : k \geq 0\} \subseteq S_{\frac{\log 3}{\log 2}}$. This simple example encourage us to making a related conjecture, which, in particular, implies that $S_{\frac{\log 3}{\log 2}} = \{2^k : k \geq 0\}$.

Conjecture 2. *If $\gamma \notin \overline{\mathbb{Q}}$, then there exists $B \in \mathbb{N}$ such that $S_\gamma = \{B^k : k \geq 0\}$.*

When $\gamma \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then $B = 1$. For the general case we have a hard problem, because until now we do not know the arithmetic nature of at least one of numbers of the form n^π (surely for $n \neq 1$). So, it would be great to prove this conjecture, but unfortunately this problem seems to be out of reach for the current level of mathematics. Thus, instead of that, in next section we relate the well-known Schanuel's conjecture with the previous conjecture.

3. Schanuel's Conjecture Versus Exceptional Set of ζ_γ

We recall the statement of Schanuel's conjecture which is one of the most important open problem in transcendental number theory. For ease of notation, we denote $\#\{x_1, \dots, x_m\}$ as the

transcendence degree (over $\mathbb{Q}$) of the field $\mathbb{Q}(x_1, \dots, x_m)$ (in this case, the transcendence degree is the largest cardinality of an algebraically independent subset of $\{x_1, \dots, x_m\}$).

Conjecture 3 (Schanuel). *Let $x_1, \dots, x_n$ be $\mathbb{Q}$ -linearly independent complex numbers. Then*

$$\#\{x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}\} \geq n.$$

Schanuel's conjecture implies many famous theorems and conjectures about transcendental numbers. For an interesting consequence, see [16]. From now on, we use this conjecture to prove some interesting facts on the exceptional set of the functions ζ_γ .

Proposition 5. *Suppose that Schanuel's conjecture holds. If $n \geq 2$ and $\gamma_1, \dots, \gamma_k$ are algebraic numbers such that $1, \gamma_1, \dots, \gamma_k$ are linearly independent over $\mathbb{Q}$, then the numbers $\log n, \zeta_{\gamma_1}(n), \dots, \zeta_{\gamma_k}(n)$ are algebraically independent.*

Proof. Since $\log n, \gamma_1 \log n, \dots, \gamma_k \log n$ are linearly independent over $\mathbb{Q}$, then, by Schanuel's conjecture,

$$\#\{\log n, \gamma_1 \log n, \dots, \gamma_k \log n, \zeta_{\gamma_1}(n), \dots, \zeta_{\gamma_k}(n)\} \geq k + 1.$$

However $n, \gamma_1, \dots, \gamma_k \in \overline{\mathbb{Q}}$ and then $\log n, \zeta_{\gamma_1}(n), \dots, \zeta_{\gamma_k}(n)$ must be algebraically independent. $\square$

Proposition 6. *Suppose that Schanuel's conjecture holds. If $\gamma \notin \overline{\mathbb{Q}}$ and $n_1, \dots, n_k \in \mathbb{N}$ are multiplicatively independent, then*

$$\#\{\zeta_\gamma(n_1), \dots, \zeta_\gamma(n_k)\} \geq k - 1.$$

Proof. Since $n_1, \dots, n_k$ are multiplicatively independent, then $\log n_1, \dots, \log n_k$ are linearly independent over $\mathbb{Q}$. Next, let us order the numbers $\log n_1, \dots, \log n_k$ in a such way that $\{\log n_1, \dots, \log n_k, \gamma \log n_1, \dots, \gamma \log n_t\}$ is a basis of the vector space generated by $\{\log n_1, \dots, \log n_k, \gamma \log n_1, \dots, \gamma \log n_k\}$ over $\mathbb{Q}$. Since γ is a transcendental number, then there exists a transcendence basis $\{\log n_{i_1}, \dots, \log n_{i_s}, \gamma\}$ of $\mathbb{Q}(\log n_1, \dots, \log n_k, \gamma) | \mathbb{Q}$. Therefore $\log n_1, \dots, \log n_k, \gamma \log n_{i_1}, \dots, \gamma \log n_{i_s}$ are linearly independent over $\mathbb{Q}$ and then $s \leq t$. On the other hand, it follows from Schanuel's conjecture that

$$\#\{\log n_1, \dots, \log n_k, \gamma \log n_1, \dots, \gamma \log n_t, n_1, \dots, n_k, n_1^\gamma, \dots, n_t^\gamma\} \geq t + k.$$

So, $\#\{\log n_1, \dots, \log n_k, \gamma \log n_1, \dots, \gamma \log n_k, n_1^\gamma, \dots, n_k^\gamma\} \geq t + k$. However

$$\#\{\log n_1, \dots, \log n_k, \gamma \log n_1, \dots, \gamma \log n_k\} = \#\{\log n_1, \dots, \log n_k, \gamma\} = s + 1.$$

Hence, we obtain that $\#\{\zeta_\gamma(n_1), \dots, \zeta_\gamma(n_k)\} \geq t + k - (s + 1) \geq k - 1$. $\square$

By considering that $\log n$ is a transcendental number for all $n \geq 2$ (by the Hermite–Lindemann theorem) and using a similar argument as in the previous proof, we get the next result:

Proposition 7. *Suppose that Schanuel's conjecture holds. If $n \geq 2$ and $\gamma_1, \dots, \gamma_k$ are linearly independent, then*

$$\#\{e^{\gamma_1}, \dots, e^{\gamma_k}, \zeta_{\gamma_1}(n), \dots, \zeta_{\gamma_k}(n)\} \geq k - 1.$$

Now we prove that Conjecture 2 is also connected with the powerful Schanuel's conjecture.

Proposition 8. *Schanuel's conjecture implies Conjecture 2.*

Proof. If γ is an algebraic irrational number, then $S_\gamma = \{1\}$ and we take $B = 1$. In the case in which γ is transcendental, then we have two possibilities. The first one is $S_\gamma = \{1\}$ for which we choose $B = 1$. Now, let us suppose that $S_\gamma \neq \{1\}$. In this case, we claim that there exists an integer $B \in S_\gamma \setminus \{1\}$ which is not a perfect power (i.e., $B \neq m^t$, for all integers $m \geq 1$ and $t > 1$). Indeed, this follows because if $m^t \in S_\gamma \setminus \{1\}$, then m also belongs to S_γ (since $(m^t)^\gamma = \alpha \in \overline{\mathbb{Q}}$ implies $m^\gamma = \alpha^{1/t} \in \overline{\mathbb{Q}}$). After taking this $B \in S_\gamma$ which is not a perfect power, it suffices to prove that $S_\gamma = \{B^k : k \geq 0\}$. Clearly, $\{B^k : k \geq 0\} \subseteq S_\gamma$. So, it remains to show that $S_\gamma \subseteq \{B^k : k \geq 0\}$. For that, take $n \in S_\gamma$. Note that the numbers B, n are multiplicatively dependent (in fact, on the contrary, the Proposition 6 implies that at least one of the numbers n^γ, B^γ is transcendental, which contradicts that fact that both B and n belong to S_γ). By the definition of multiplicative dependence, there are coprime integers a, b such that $n = B^{a/b}$. Since n is an integer and $\gcd(a, b) = 1$, this forces B to satisfy $B = m^b$, for some integer $m > 1$. However, B is not a perfect power and then $b = 1$ yielding $n = B^a \in \{B^k : k \geq 0\}$. In conclusion $S_\gamma \subseteq \{B^k : k \geq 0\}$ which completes the proof. $\square$

Let γ be a non-rational complex number. We call *exceptional representative* of γ as the number $B = B(\gamma)$ in Conjecture 2, if any exists. Assuming that the Schanuel's conjecture holds, we just proved the existence of exceptional representative for any non-rational complex number. It is easy to see that if the exceptional representative of α and β are multiplicatively independent, then $S_\alpha \cap S_\beta = \{1\}$. We finish our work by expliciting S_γ , for some well-known γ 's.

Proposition 9. If Schanuel's conjecture holds, then the numbers $\zeta_\pi(n), \zeta_e(n), \zeta_i(n)$ are algebraically independent over $\mathbb{Q}(e, \pi)$, for all $n \geq 2$. In particular, $S_\pi = S_i = S_e = \{1\}$.

Proof. If $n \geq 2$, then $i\pi, \log n$ are $\mathbb{Q}$ -linearly independent, so by Schanuel's conjecture $\#\{i\pi, \log n, -1, n\} \geq 2$. It follows that $\pi, \log n$ are algebraically independent. Thus $i\pi, \log n, \log \pi, \log \log n$ are $\mathbb{Q}$ -linearly independent. By Schanuel's conjecture $\#\{i\pi, \log n, \log \pi, \log \log n, -1, n, \pi, \log n\} \geq 4$ and then $1, i\pi, \log n, \log \pi, \log \log n$ are $\mathbb{Q}$ -linearly independent. Again, by Schanuel's conjecture the numbers $i\pi, \log n, \log \pi, \log \log n$ are algebraically independent. Thus, for finishing, the numbers $i\pi, \pi \log n, e \log n, i \log n, \log \pi, \log \log n, 1, \log \pi, \log n$ are $\mathbb{Q}$ -linearly independent. By Schanuel's conjecture the numbers n^π, n^e, n^i are algebraically independent over $\mathbb{Q}(e, \pi)$. $\square$

4. Conclusions

In this paper we have been interested in the arithmetic nature of the arithmetic zeta functions $\zeta_\gamma(n) = n^\gamma$, where n is a positive integer and γ is a complex number. In particular, we study this nature for the case in which γ is algebraic or transcendental. The main ingredients in the proofs are classical results on transcendence (such as, Lindemann, Gelfond–Schneider and the six exponential theorem) together with symmetry movements related to decrease (in size) a finite set in order to be able to apply some of our tools. We also pose a general conjecture and we show that it is a consequence of the important Schanuel's conjecture.

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