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Statistical Inference for the Inverted Scale Family under General Progressive Type-II Censoring

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Abstract: Two estimation problems are studied based on the general progressively censored samples, and the distributions from the inverted scale family (ISF) are considered as prospective life distributions. One is the exact interval estimation for the unknown parameter θ, which is achieved by constructing the pivotal quantity. Through Monte Carlo simulations, the average 90% and 95% confidence intervals are obtained, and the validity of the above interval estimation is illustrated with a numerical example. The other is the estimation of $R = P(Y < X)$ in the case of ISF. The maximum likelihood estimator (MLE) as well as approximate maximum likelihood estimator (AMLE) is obtained, together with the corresponding R-symmetric asymptotic confidence intervals. With Bootstrap methods, we also propose two R-asymmetric confidence intervals, which have a good performance for small samples. Furthermore, assuming the scale parameters follow independent gamma priors, the Bayesian estimator as well as the HPD credible interval of R is thus acquired. Finally, we make an evaluation on the effectiveness of the proposed estimations through Monte Carlo simulations and provide an illustrative example of two real datasets.

Keywords: Monte Carlo simulation; inverted scale family; interval estimation; maximum likelihood estimation; Bayesian estimation; approximate maximum likelihood estimator; Bootstrap method; general progressive Type-II censoring

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1. Introduction

General progressive censoring plays a key role in the field of lifetime analysis. It generalizes the progressive censoring scheme, which allows removing surviving units from a life test stage-by-stage and to some extent, resolves the time and money constraints (Sen, [1]). Additionally, it contains two censoring schemes: left censoring and right censoring. Here considers the situation of general progressive Type-II right censoring (GPTIIC).

In a life-test, first randomly select N units as the test sample. Suppose that the first r failures are unobserved; for $i = r+1, \ldots, n-1$, while the ith failure is observed, R_i surviving units are withdrawn from the test randomly; while the nth failure is observed, all remaining units are withdrawn and the life-test is terminated, that is, $R_n = N - n - \sum_{i=r+1}^{n-1} R_i$. Denote the totally observed failure times as $X_{r+1:n:N} \leq \ldots \leq X_{n:n:N}$, which are also called the general progressively Type-II censored order statistics (GPTIICOS). For notational convenience, represent it with $X_{r+1} \leq \ldots \leq X_n$. Here, denote the corresponding censoring scheme as $(r, n, N, R_{r+1}, \ldots, R_n)$. Note that: (i) GPTIIC with $(0, n, N, 0, \ldots, 0, N-n)$ represents the conventional Type-II right censoring scheme. (ii) GPTIIC with $(0, n, N, R_1, \ldots, R_n)$ represents the progressive Type-II right censoring scheme. The reader can find more information by consulting the book by Balakrishnan and Aggarwala [2], which has discussed some mathematical properties and inferential methods based on this general progressive censoring scheme.

Consider a class of distributions that have the inverted form of the standard probability density functions (PDF) and distribution functions (CDF) for the scale family. We call it the inverted scale family (ISF). It is said that a positive and continuous random variable X with CDF (Equation (1)) and PDF (Equation (2)) belongs to the inverted scale family (i.e., ISF(θ)) with the scale parameter θ.

$$F(x,\theta) = 1 - G(\frac{\theta}{x}), \theta > 0, x > 0 \tag{1}$$

$$f(x,\theta) = \frac{\theta}{x^2} g(\frac{\theta}{x}), \theta > 0, x > 0 \tag{2}$$

Note that a positive and continuous random variable Y with CDF $G(y)$ and PDF $g(y)$ is said to come from a standard scale distribution.

Distributions from ISF are essential for the reliability life data analysis. For example, the generalized inverse exponential distribution (GIED) from ISF, the case of the inverse exponential distribution (IED) included, is able to model the aging criteria of various shapes. In addition, a reliability model based on the inverse Weibull distribution (IWD) from ISF may be suitable when the empirical studies on a given data set indicate the probability of a unimodal hazard function. In light of the importance of ISF, statistical inference on the related distributions has appealed to many authors. For the inverted gamma distribution, Lin [3] acquired the MLE as well as the estimation of the corresponding reliability function. In the case of Type-II censoring, Kundu [4] dealt with the Bayesian estimation for IWD. Prakash [5] studied some properties for IED from a Bayesian point of view. Moreover, articles by Krishna [6], Dey [7], and Essam [8] are several recent works on the estimation for GIED models under the progressive censoring.

Although the scheme of GPTIIC and distributions from ISF both play a vital role in life researches, not much attention has been paid to the situation when the samples are from ISF under GPTIIC. So, in this case, we consider the following two estimation problems.

One is the interval estimation, which, as a classical estimation problem, has engaged the attention of many authors. Article by Viveros and Balakrishnan [9] derived the interval estimates for some life-time parameters. Balakrishnan et al. [10] obtained interval estimation for Normal distribution in the case of progressive Type-II censoring. Wang [11] conducted a study on the scale family under GPTIIC and proposed the exact interval estimation. Dey [7] derived interval estimations for GIED under progressive first censoring.

The other is the estimation of $R = P(Y < X)$, which denotes the stress-strength parameter. Suppose a random stress acts on a system; for constructing a stress-strength modeling, we denote the system strength as X and the value of the random stress as Y. The system will fail only when its strength is less than the received stress. Thus, $R = P(Y < X)$ can reflect the system reliability. As R provides a wide application in many areas, it is necessary to discuss the related estimation problems in the statistical literature. Articles by Church and Harris [12], Downtown [13], Govidarajulu [14], Woodward and Kelley [15], and Owen et al. [16] dealt with the situation of X and Y coming from normal distributions. Raqab and Kundu [17], Kundu and Gupta, [18] and Kundu and Raqad [19] conducted studies on this estimation problem for the generalized Rayleigh distributions, Weibull distributions, and three-parameter Weibull distributions, respectively.

Here we conduct a study on the exact interval estimation of θ as well as the estimation of R for ISF under GPTIIC. First, the exact interval estimation of θ is achieved in Section 2 by constructing the pivotal quantity. Then we make an assessment on the interval estimation performance through Monte Carlo simulations and further illustrate the estimation process with a remission time dataset. Moreover, the estimation of $R = P(Y < X)$ is achieved in Section 3. The MLE, AMLE, and Bayesian estimator of R are acquired, together with the corresponding interval estimations. In addition, with Bootstrap methods, we derive two interval estimates, suitable for small samples. Finally, we evaluate the effectiveness of the proposed estimations through Monte Carlo simulations and provide an illustrative example of two real datasets.

2. Exact Interval Estimation for θ

2.1. Interval Estimation with Pivotal Quantity Method

Here, a theorem (proposed by Aggarwala and Balakrishnan [2,20]) is introduced to construct the pivotal quantity.

Theorem 1. *Consider $U_{r+1}, U_{r+2}, \ldots, U_n$ are the GPTIICOS from the Uniform (0,1) distribution. Then*

$$L_{r+i} = \frac{1 - U_{n-i+1}}{1 - U_{n-i}}, i = 1, \ldots, n-r-1, L_n = 1 - U_{r+1}.$$

follow independent beta distributions given by Equation (3).

$$\begin{aligned} &L_{r+i} \sim Beta(i + \sum_{j=n-i+1}^{n} R_j, 1), i = 1, \ldots, n-r-1, \\ &L_n \sim Beta(N - r, r+1). \end{aligned} \tag{3}$$

Lemma 1. *Consider $X_{r+1} \leq \ldots \leq X_n$ are the GPTIICOS from the inverted scale family. Then*

$$L_{r+i} = \frac{G(\frac{\theta}{X_{n-i+1}})}{G(\frac{\theta}{X_{n-i}})}, i = 1, \ldots, n-r-1, L_n = G(\frac{\theta}{X_{r+1}}) \tag{4}$$

follow independent beta distributions given by Equation (3).

Proof. Since $F(X, \theta)$, ranging from 0 to 1, increases as X increases, $F(X_{r+1}, \theta) \leq \ldots \leq F(X_n, \theta)$ are the GPTIICOS from the Uniform (0,1) distribution. For $i = r+1, \ldots, n$, represent U_i in the Theorem 1 with $F(X_i, \theta)$. The proof is thus completed. □

Denote the CDF of $Beta(N - r, r+1)$ as $H(x)$. Then independent random variables

$$Q_{r+i} = L_{r+i}^{i+\sum_{j=n-i+1}^{n} R_j}, i = 1, \ldots, n-r-1, Q_n = H(L_n)$$

are from the Uniform (0,1) distribution. Here, L_{r+i} is described as Equation (4). Thus

$$-2\log(Q_{r+1}), -2\log(Q_{r+2}), \ldots, -2\log(Q_n)$$

are independent $\chi^2(2)$ random variables. Here, $\chi^2(df)$ represents the χ^2 distribution with df degrees of freedom. Then we construct the pivotal quantity as:

$$\begin{aligned} T(\theta) &= \sum_{i=1}^{n-r} [-2\log(Q_{r+i})] \\ &= -2 \sum_{i=r+2}^{n} (R_i + 1) \log[1 - F(X_i, \theta)] - 2\log[H(1 - F(X_{r+1}, \theta))] \\ &+ 2 \sum_{i=r+2}^{n} (R_i + 1) \log[1 - F(X_{r+1}, \theta)]. \end{aligned} \tag{5}$$

It is clear that $T(\theta)$ follows $\chi^2(2n - 2r)$. For the inverted scale family, when $r = 0$, $T(\theta)$ reduces to

$$T(\theta) = -2 \sum_{i=1}^{n} (R_i + 1) \log[1 - F(X_i, \theta)] = -2 \sum_{i=1}^{n} (R_i + 1) \log[G(\frac{\theta}{X_i})]. \tag{6}$$

As can be seen from Equation (6), for $r = 0$, $T(\theta)$ strictly decreases as θ increases. We can also find that for any positive integer $r < n$, the exact confidence interval can be derived from Equation (5) easily when $T(\theta)$ is strictly monotonic as θ increases. Here we consider the failure rate function $\lambda(x,\theta)$ for the inverted scale family to determine the monotonicity of $T(\theta)$. The following is a brief overview of $\lambda(x,\theta)$.

Suppose a positive continuous random variable X from ISF(θ) as the lifetime of some units in a life-test. Then, the failure rate function represents the probability intensity that an alive unit will fail at the time x, which is defined as:

$$\lambda(x,\theta) = \frac{f(x,\theta)}{[1 - F(x,\theta)]}, \theta > 0, x > 0.$$

For the inverted scale family, it becomes

$$\lambda(x,\theta) = \frac{\frac{\theta}{x^2} g(\frac{\theta}{x})}{G(\frac{\theta}{x})}. \tag{7}$$

Here, $\lambda(x,\theta)$ can be seen as a function of θ.

According to $\lambda(x,\theta)$, a theorem is given below. Note that $T(\theta)$ (as shown in Equation (5)) is strictly decreasing on $\theta \in (0,+\infty)$ if the condition given in Theorem 2 is satisfied.

Theorem 2. *Consider $X_{r+1}, \ldots, X_n$ are the GPTIICOS from ISF(θ) with censoring scheme $(r, n, N, R_{r+1}, \ldots, R_n)$. The pivotal quantity $T(\theta)$ (as shown in Equation (5)) is strictly decreasing on $(0,+\infty)$ if the failure rate function $\lambda(x,\theta)$ is strictly decreasing on $\theta \in (0,+\infty)$.*

Proof. Let $h(t) = \frac{tg(t)}{G(t)}$ and for $i = r+2, \ldots, n$, let $y_i(\theta) = -[log(1 - F(X_i,\theta)) - log(1 - F(X_{r+1},\theta))]$. The pivotal quantity $T(\theta)$ becomes

$$2\sum_{i=r+2}^{n}(R_i + 1)y_i(\theta) - 2log[H(G(\frac{\theta}{X_{r+1}}))].$$

The failure rate function from Equation (7) can be rewritten as:

$$\lambda(x,\theta) = \frac{1}{x} h(\frac{\theta}{x}). \tag{8}$$

According to Equation (8), since $\lambda(x,\theta)$ strictly decreases as θ increases, we can infer that $h(t)$ is strictly decreasing on $t \in (0,+\infty)$. Thus the monotonicity of $y_i(\theta)$ can be determined. The derivative of $y_i(\theta)$ is:

$$y_i'(\theta) = \frac{g(\frac{\theta}{x_{r+1}})}{x_{r+1}G(\frac{\theta}{x_{r+1}})} - \frac{g(\frac{\theta}{x_i})}{x_i G(\frac{\theta}{x_i})} = -\frac{1}{\theta}[h(\frac{\theta}{x_i}) - h(\frac{\theta}{x_{r+1}})] < 0. \tag{9}$$

From the above equation, it is clear that $2\sum_{i=r+2}^{n}(R_i + 1)y_i(\theta)$ is strictly decreasing on $\theta \in (0,+\infty)$. Considering the monotonicity of the CDF, we can find $-2log[H(G(\frac{\theta}{X_{r+1}}))]$ is also strictly decreasing on $\theta \in (0,+\infty)$. Thus, we complete the proof. □

The exact confidence interval of θ is given below.

Theorem 3. *For any $\xi \in (0,1)$, if $T(\theta)$ is strictly decreasing on $(0,+\infty)$, thus*

$$[T^{-1}(\chi^2_{1-\xi/2}(2n-2r)), T^{-1}(\chi^2_{\xi/2}(2n-2r))]. \tag{10}$$

is an exact $1-\xi$ confidence interval for θ. Here, $T^{-1}(x)$ denotes the solution to $T(\theta)=x$ for any $x \in (0,\infty)$ and $\chi^2_{\xi}(df)$ denotes the ξ percentile of $\chi^2(df)$.

Proof. Since $T(\theta)$ is a strictly decreasing function on $(0,+\infty)$, the probability that θ belongs to the interval of Equation (10) is:

$$P(T^{-1}(\chi^2_{1-\xi/2}(2n-2r)) \le \theta \le T^{-1}(\chi^2_{\xi/2}(2n-2r)) = P(\chi^2_{\xi/2}(2n-2r) \le T(\theta) \le \chi^2_{1-\xi/2}(2n-2r)) = 1-\xi.$$

□

Remark 1. *For any $\xi \in (0,1)$, if $T(\theta)$ is strictly increasing on $(0,+\infty)$, thus*

$$[T^{-1}(\chi^2_{\xi/2}(2n-2r)), T^{-1}(\chi^2_{1-\xi/2}(2n-2r))]. \tag{11}$$

is an exact $1-\xi$ confidence interval for θ. Here, $T^{-1}(x)$ denotes the solution to $T(\theta)=x$ for $x \in (0,\infty)$ and $\chi^2_{\xi}(df)$ denotes the ξ percentile of $\chi^2(df)$.

Example 1. *The generalized inverse exponential distribution (i.e., GIED(θ,α)), which includes the inverse exponential distribution (i.e., IED(θ)) as a special case when $\alpha=1$, is from the inverted scale family with*

$$\begin{aligned} F(x;\theta,\alpha) &= 1-[1-exp(-\frac{\theta}{x})]^{\alpha}, \theta,\alpha>0, x>0, \\ f(x;\theta,\alpha) &= \frac{\alpha\theta}{x^2}exp(-\frac{\theta}{x})[1-exp(-\frac{\theta}{x})]^{\alpha-1}, \theta,\alpha>0, x>0. \end{aligned}$$

By Theorem 2, we have $h(t)=\frac{\alpha t}{exp(t)-1}$ with the derivative $h'(t)=\frac{\alpha[exp(t)-texp(t)-1]}{[exp(t)-1]^2}$. Let $h_1(t)=exp(t)-texp(t)-1, t>0$, then, a system of equations is given by

$$\begin{cases} h_1(t) = exp(t)-texp(t)-1 \\ h_1^{'}(t) = -texp(t) < 0 \\ h_1(0) = 0 \end{cases} \tag{12}$$

From Equation (12), when $t \in (0,+\infty)$, we have $h_1(t)<0$ and thus $h'(t)<0$, which proves that $h(t)$ is strictly decreasing on $t \in (0,+\infty)$. So $\lambda(x,\theta)=\frac{1}{x}h(\frac{\theta}{x})$ is strictly decreasing on $\theta \in (0,+\infty)$.

For the generalized inverse exponential distribution, from Equation (5), the pivotal quantity is developed as:

$$T(\theta) = 2\alpha \sum_{i=r+2}^{n} (R_i+1)\log[\frac{1-exp(-\frac{\theta}{X_{r+1}})}{1-exp(-\frac{\theta}{X_i})}] - 2\log[H(1-exp(-\frac{\theta}{X_{r+1}}))^{\alpha}].$$

When $r=0$, $T(\theta)$ reduces to

$$T(\theta) = -2\alpha\sum_{i=1}^{n}(R_i+1)\log[1-exp(-\frac{\theta}{X_i})].$$

Then applying Theorem 3, we achieve the exact interval estimation.

Example 2. *The inverse Weibull distribution (i.e., IWD(θ,α)), including the inverse exponential distribution (when $\alpha = 1$) and the inverse Rayleigh distribution (IRD(θ), when $\alpha = 2$) as special cases, is from the inverted scale family with*

$$F(x;\theta,\alpha) = exp[-(\frac{\theta}{x})^{\alpha}], \theta,\alpha > 0, x > 0,$$
$$f(x;\theta,\alpha) = \frac{\alpha}{\theta}(\frac{\theta}{x})^{\alpha+1} exp[-(\frac{\theta}{x})^{\alpha}], \theta,\alpha > 0, x > 0.$$

By Theorem 2, we have $h(t) = \frac{\alpha t^{\alpha}}{exp(t^{\alpha})-1}$ *with the derivative* $h'(t) = \frac{\alpha^2 t^{\alpha-1}[exp(t^{\alpha}) - t^{\alpha} exp(t^{\alpha}) - 1]}{[exp(t^{\alpha})-1]^2}$. *Let* $h_2(t) = exp(t^{\alpha}) - t^{\alpha} exp(t^{\alpha}) - 1, t > 0$, *then, a system of equations is given by:*

$$\begin{cases} h_2(t) = exp(t^{\alpha}) - t^{\alpha} exp(t^{\alpha}) - 1 \\ h_2'(t) = -\alpha t^{2\alpha-1} exp(t^{\alpha}) < 0 \\ h_2(0) = 0 \end{cases} \quad (13)$$

Clearly, when $t \in (0,+\infty)$, *we have* $h_2(t) < 0$ *and thus* $h'(t) < 0$, *which proves that* $h(t)$ *is strictly decreasing on* $t \in (0,+\infty)$. *So* $\lambda(x,\theta) = \frac{1}{x} h(\frac{\theta}{x})$ *is strictly decreasing on* $\theta \in (0,+\infty)$.

For the inverse Weibull distribution, from Equation (5), the pivotal quantity is developed as:

$$T(\theta) = 2\sum_{i=r+2}^{n}(R_i+1)\log[\frac{1-exp(-(\frac{\theta}{X_{r+1}})^{\alpha})}{1-exp(-(\frac{\theta}{X_i})^{\alpha})}] - 2\log[H(1-exp(-(\frac{\theta}{X_{r+1}})^{\alpha}))].$$

When $r = 0$, $T(\theta)$ *reduces to*

$$T(\theta) = -2\sum_{i=1}^{n}(R_i+1)\log[1-exp(-(\frac{\theta}{X_i})^{\alpha})].$$

Then applying Theorem 3, we achieve the exact interval estimation.

2.2. Simulation Study

Here, the exact interval estimation for ISF under GPTIIC is implemented through Monte Carlo simulations. Take IED as a prospective life distribution (the CDF and PDF are given in Example 1). With interval coverage percentages, we evaluate the validity of the exact confidence interval derived in Theorem 3.

According to Balakrishnan [2], GPTIICOS from IED(θ) can be generated by the following algorithm.

Step 1: Generate a random variable L_n from $Beta(N-r, r+1)$.
Step 2: Generate $n-r-1$ independent Uniform (0,1) random variables $Z_{r+1},\ldots,Z_{n-1}$.
Step 3: Set $L_{r+i} = Z^{\frac{1}{a_{r+i}}}$ where $a_{r+i} = i + \sum_{j=n-i+1}^{n} R_j, i = 1,\ldots,n-r$.
Step 4: Set $U_{r+i} = 1 - L_{n-i+1}L_{n-i+2}\ldots L_n$ for $i = 1,\ldots,n-r$.
Step 5: Set $X_{r+i} = -\frac{\theta}{\log(U_{r+i})}$ for $i = 1,\ldots,n-r$.

$X_{r+1},\ldots,X_n$ thus denote the GPTIICOS from IED(θ) with the censoring scheme $(r,n,N,R_{r+1},\ldots,R_n)$.

Here, suppose $X_{r+1},\ldots,X_n$ are from IED(5). For evaluating the performance of the exact confidence interval estimation, we consider the cases of $N = 20$, $N = 30$ and $N = 40$ as well as four different (r,n), including $(1,10)$, $(3,12)$, $(1,16)$, and $(3,18)$. In this case, we give the average confidence intervals for θ and the corresponding interval lengths and interval coverage percentages with 10,000 simulations. The results are shown in Table 1. Here, the considered confidence levels are 90% and 95%.

As can be seen from Table 1, the obtained interval coverage percentage is close to the corresponding confidence level. This indicates the proposed interval estimation has a good performance. Additionally, the simulation results show the relationship between the confidence interval length and the values of N, r, n:

(a) When $n - r$ (i.e., the number of completely observed failures) and N (i.e., the sample size) are fixed, the confidence interval length is, to some extent, related to r (that is, the number of the first unobserved failures), as shown in Table 1 that the average confidence interval is wider when r is larger.

(b) When the values of r and n are fixed, the interval length is related to N as it is larger when N is smaller.

(c) When the values of N and r are fixed, the interval length is related to n as it is smaller when n is larger.

Obviously, a narrow confidence interval can provide more information about the unknown parameter. We conclude that the estimation is more accurate when more samples are completely observed for the confidence interval length decreases as the number of the completely observed samples increases.

Table 1. Average confidence intervals (CI), interval lengths (IL), and interval coverage percentages (CP).

	$\mathbf{1-\xi=0.90}$			$\mathbf{1-\xi=0.95}$		
(N, r, n)	CI	IL	CP	CI	IL	CP
$(20, 1, 10)$	$(2.97, 13.18)$	10.21	0.904	$(2.63, 15.62)$	12.99	0.953
$(20, 3, 12)$	$(2.87, 13.85)$	10.98	0.900	$(2.52, 16.39)$	13.87	0.951
$(20, 1, 16)$	$(2.93, 10.84)$	7.91	0.901	$(2.60, 12.33)$	9.73	0.951
$(20, 3, 18)$	$(2.76, 11.97)$	9.21	0.902	$(2.42, 13.77)$	11.35	0.948
$(30, 1, 10)$	$(3.37, 8.40)$	5.03	0.897	$(3.08, 9.12)$	6.04	0.952
$(30, 3, 12)$	$(3.27, 9.00)$	5.73	0.896	$(2.99, 9.91)$	6.92	0.952
$(30, 1, 16)$	$(3.28, 9.13)$	5.85	0.897	$(2.99, 10.21)$	7.22	0.947
$(30, 3, 18)$	$(3.13, 10.46)$	7.33	0.895	$(2.80, 11.78)$	8.98	0.951
$(40, 1, 10)$	$(3.53, 7.83)$	4.30	0.900	$(3.39, 8.48)$	5.09	0.949
$(40, 3, 12)$	$(3.46, 8.39)$	4.93	0.903	$(3.19, 9.17)$	5.98	0.950
$(40, 1, 16)$	$(3.54, 7.71)$	4.17	0.900	$(3.29, 8.32)$	5.03	0.950
$(40, 3, 18)$	$(3.41, 8.57)$	5.16	0.897	$(3.14, 9.37)$	6.23	0.950

2.3. *An Illustrated Example*

A real dataset, reported by Gross and Declark [21], is provided below for illustrating the proposed estimation process in the previous subsections. It shows the remission time of 20 patients receiving analgesics (in minutes):

1.1	1.2	1.3	1.4	1.4	1.5	1.6	1.6	1.7	1.7
1.7	1.8	1.8	1.9	2.0	2.2	2.3	2.7	3.0	4.1

Here, we consider three reliability models based on IWD (as shown in Example 2), GIED (as shown in Example 1), and the Weibull distribution (i.e., WD, the CDF and PDF are given below), respectively.

$$F(x; \theta, \alpha) = 1 - e^{-\theta x^{\alpha}}, \theta, \alpha > 0, x > 0$$

$$f(x; \theta, \alpha) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}, \theta, \alpha > 0, x > 0$$

We use maximum likelihood estimation to construct the above three parametric models, and for each model, five measures are used to test the goodness-of-fit: (1) $-\ln$, (2) $K - S$ distance, (3) p-value, (4) AIC, and (5) BIC. Here, for each model, $-\ln$ can be calculated with the data and MLEs of the corresponding parameters. The following briefly introduces AIC and BIC.

(i) AIC, proposed by Akaike [22], can reflect the fitting effect of a reliability model to the data. The equation is given below:

$$AIC = 2m - 2ln.$$

Here, for an estimated parametric model, m denotes the number of parameters, and ln denotes the maximum value of the log-likelihood function. For a given data set, the best fit order of several competing models can be given from their AIC, and the one with lower AIC ranks higher.

(ii) BIC, proposed by Schwarz [23], is a selection criterion for choosing between several estimated statistical models with different numbers of parameters. The equation is given by:

$$BIC = m\log(n) - 2ln.$$

Here, for an estimated parametric model, m denotes the number of parameters, n denotes the number of observed failures, and ln denotes the maximum value of the log-likelihood function. BIC has a close relationship with the AIC. When a data set is given, same as in AIC, the best fit order of several competing models can be given from their BIC, and the one with lower BIC ranks higher. However, compared with AIC, BIC has a greater penalty for additional parameters.

In the case of MLE, the values of the above five measures based on the three reliability models are listed in Table 2. Clearly, the best fit order for the given data set is:

$$BEST \quad IWD \quad \rightarrow \quad GIED \quad \rightarrow \quad WD \quad WORST$$

Since IWD has the lowest values of $-\ln$, $K-S$ distance, AIC, BIC, and the highest p-value, it is considered the best of the three reliability models. Furthermore, the fitting effects of the three reliability models are intuitively shown in Figure 1. Hence, we fit the IWD model to the data set. It appears from Table 2 that the values of $-\ln$, $K-S$ distance, p-value, AIC, and BIC in the case of MLE are 15.19896, 0.1306, 0.9187, 34.39793, and 36.17867, respectively, indicating the IWD model fits well to the data when α takes the value of 3.777054. Hence, for estimating the exact confidence interval of θ, we can assume that the sample follows IWD(θ,3.777054). Set $r = 2$, $n = 16$, and then Table 3 shows the censoring scheme and the generated censored data.

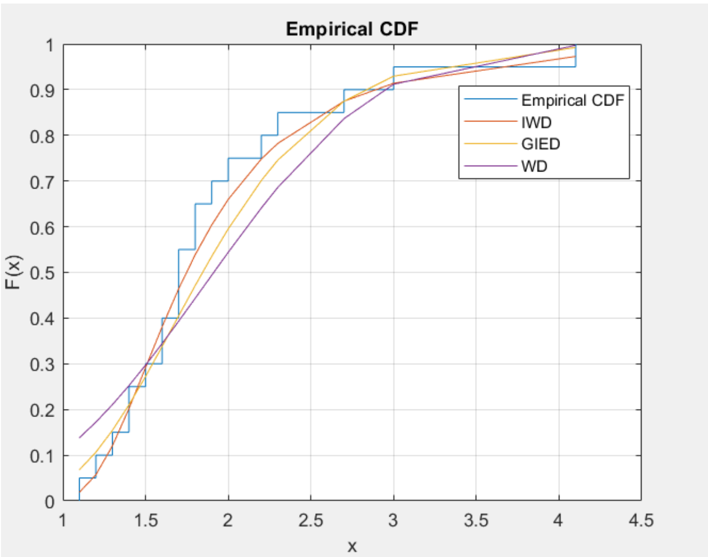

Figure 1. An empirical distribution function (CDF) and other CDFs based on different statistical models.

Table 2. Fitting results for the three reliability models.

Model	α	θ	$-\ln$	$K-S$	p-Value	AIC	BIC
IWD	3.777054	1.58526	15.19896	0.1306	0.9187	34.39793	36.17867
GIED	19.519704	6.189745	16.22355	0.14001	0.8721	36.4471	38.22784
WD	2.785545	8.779702	19.06114	0.17819	0.6171	42.12228	43.90302

Table 3. Generated dataset and censoring scheme.

i, j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X_i	–	–	1.3	1.4	1.6	1.7	1.7	1.8	1.8	1.9	2.0	2.2	2.3	2.7	3.0	4.1
R_i	–	–	1	1	1	1	0	0	0	0	0	0	0	0	0	0

We compute the exact confidence interval for θ with Equation (5). Here, the 90% and 95% confidence intervals are obtained to be $(1.251123, 1.817043)$ and $(1.208818, 1.874985)$, respectively.

3. Estimation of $R = P(Y < X)$

Consider the situation when the distributions of independent random variables X and Y are ISF(θ_1) and ISF(θ_2) (the CDF and PDF are shown in Section 1), respectively.

Then, the stress-strength parameter is described as:

$$\begin{aligned} R = P(Y < X) &= \int_0^{+\infty} dx \int_0^x \frac{\theta_1}{x^2} \frac{\theta_2}{y^2} g(\frac{\theta_1}{x}) g(\frac{\theta_2}{y}) dy \\ &= \int_0^{+\infty} \frac{\theta_1}{x^2} g(\frac{\theta_1}{x})[1 - G(\frac{\theta_2}{x})]dx \\ &= R(\theta_1, \theta_2). \end{aligned} \tag{14}$$

Suppose $X_{r+1} \leq \ldots \leq X_n$ are the GPTIICOS from X with censoring scheme $(r, n, N, R_{r+1}, \ldots, R_n)$; $Y_{r'+1} \leq \ldots \leq Y_m$ are the GPTIICOS from Y with censoring scheme $(r', m, M, R'_{r'+1}, \ldots, R'_m)$. Based on the above samples, we discuss the estimation of R.

3.1. Maximum Likelihood Estimator (MLE)

In order to get the MLE of R, the MLEs of θ_1 and θ_2 need to be derived. Here, denote the MLEs of θ_1, θ_2, R as $\widehat{\theta_1}$, $\widehat{\theta_2}$, $\widehat{R}$ for convenience.

The likelihood function is described as Equation (15) based on the above samples.

$$L(\theta_1, \theta_2) = c_1 [F(x_{r+1})]^r \prod_{i=r+1}^{n} f(x_i) [1 - F(x_i)]^{R_i} \times c_2 [F(y_{r'+1})]^{r'} \prod_{j=r'+1}^{m} f(y_j) \left[1 - F(y_j)\right]^{R'_j}, \tag{15}$$

where

$$c_1 = \binom{N}{r} (N - r) \prod_{j=r+2}^{n} \left(N - \sum_{i=r+1}^{j-1} R_i - j + 1\right),$$

$$c_2 = \binom{M}{r'} (M - r') \prod_{j=r'+2}^{m} \left(M - \sum_{i=r'+1}^{j-1} R'_i - j + 1\right).$$

It seems impossible to get the MLEs of θ_1 and θ_2 explicitly from the likelihood Equations (16) and (17). However, we can solve Equations (16) and (17) with the R programming language to obtain $\widehat{\theta_1}$ and $\widehat{\theta_2}$. Then $\widehat{R}$ can be expressed as $\widehat{R} = R(\widehat{\theta_1}, \widehat{\theta_2})$. In the following subsection, explicit solutions of θ_1 and θ_2 will be given with an approximative method.

$$\frac{\partial \ln L(\theta_1, \theta_2)}{\partial \theta_1} = \frac{n - r}{\theta_1} - \frac{r g(\frac{\theta_1}{x_{r+1}})}{1 - G(\frac{\theta_1}{x_{r+1}})} \frac{1}{x_{r+1}} + \sum_{i=r+1}^{n} [\frac{g'(\frac{\theta_1}{x_i})}{g(\frac{\theta_1}{x_i})} \frac{1}{x_i} + \frac{R_i g(\frac{\theta_1}{x_i})}{G(\frac{\theta_1}{x_i})} \frac{1}{x_i}] = 0 \tag{16}$$

$$\frac{\partial \ln L(\theta_1, \theta_2)}{\partial \theta_2} = \frac{m - r'}{\theta_2} - \frac{r' g(\frac{\theta_2}{y_{r'+1}})}{1 - G(\frac{\theta_2}{y_{r'+1}})} \frac{1}{y_{r'+1}} + \sum_{j=r'+1}^{m} [\frac{g'(\frac{\theta_2}{y_j})}{g(\frac{\theta_2}{y_j})} \frac{1}{y_j} + \frac{R'_j g(\frac{\theta_2}{y_j})}{G(\frac{\theta_2}{y_j})} \frac{1}{y_j}] = 0 \tag{17}$$

3.2. Approximate Maximum Likelihood Estimator (AMLE)

From the above analysis, getting explicit solutions for θ_1 and θ_2 from the nonlinear Equations (16) and (17) is quite difficult. So we use Taylor expansions to acquire approximate estimators. Denote the AMLEs of θ_1, θ_2, R as $\widetilde{\theta_1}$, $\widetilde{\theta_2}$, $\widetilde{R}$, respectively.

Considering the transformation $Z = \frac{\theta}{X}$, then its CDF and PDF are respectively given by:

$$M(z) = 1 - G(z), z > 0, \tag{18}$$
$$m(z) = -g(z), z > 0. \tag{19}$$

Remark 2. *$M(z)$ is decreasing on* $(0, +\infty)$.

Proof. According to the monotonicity of $G(x)$, the proof is thus completed. □

Let $z_i = \frac{\theta_1}{x_i}$ and $w_j = \frac{\theta_2}{y_j}$. The likelihood equations of θ_1 and θ_2 can be rewritten by:

$$\frac{\partial \ln L^*(\theta_1, \theta_2)}{\partial \theta_1} = \frac{n-r}{\theta_1} + \frac{r}{\theta_1}\frac{m(z_{r+1})}{M(z_{r+1})}z_{r+1} + \sum_{i=r+1}^{n}[\frac{1}{\theta_1}\frac{m'(z_i)}{m(z_i)}z_i - \frac{R_i}{\theta_1}\frac{m(z_i)}{1-M(z_i)}z_i] = 0, \tag{20}$$

$$\frac{\partial \ln L^*(\theta_1, \theta_2)}{\partial \theta_2} = \frac{m-r'}{\theta_2} + \frac{r'}{\theta_2}\frac{m(w_{r'+1})}{M(w_{r'+1})}w_{r'+1} + \sum_{j=r'+1}^{m}[\frac{1}{\theta_2}\frac{m'(w_j)}{m(w_j)}w_j - \frac{R_j}{\theta_2}\frac{m(w_i)}{1-M(w_j)}w_j] = 0. \tag{21}$$

Consider the following Taylor expansions that keep only the first two terms. Let $p_1(z_i) = \frac{m(z_i)}{M(z_i)}$, $p_2(z_i) = \frac{m'(z_i)}{m(z_i)}$, $p_3(z_i) = \frac{m(z_i)}{1-M(z_i)}$ and expand them around the point $e_i = E(Z_i)$; let $q_1(w_j) = \frac{m(w_j)}{M(w_j)}$, $q_2(w_j) = \frac{m'(w_j)}{m(w_j)}$, $q_3(w_j) = \frac{m(w_j)}{1-M(w_j)}$ and expand them around the point $e'_j = E(W_j)$. Note that for $k \in \{1,2,3\}$, $p'_k(z_i)$ and $q'_k(w_j)$ are the derivatives of $p_k(z_i)$ and $q_k(w_j)$, respectively.

$$\begin{aligned}
p_1(z_i) &\approx p_1(e_i) + p'_1(e_i)(z_i - e_i) = A_i - B_i z_i, && \textit{where} \quad A_i = p_1(e_i) - e_i p'_1(e_i), B_i = -p'_1(e_i),\\
p_2(z_i) &\approx p_2(e_i) + p'_2(e_i)(z_i - e_i) = C_i - D_i z_i, && \textit{where} \quad C_i = p_2(e_i) - e_i p'_2(e_i), D_i = -p'_2(e_i),\\
p_3(z_i) &\approx p_3(e_i) + p'_3(e_i)(z_i - e_i) = E_i - F_i z_i, && \textit{where} \quad E_i = p_3(e_i) - e_i p'_3(e_i), F_i = -p'_3(e_i),\\
q_1(w_j) &\approx q_1(e'_j) + q'_1(e'_j)(w_j - e'_j) = A'_j - B'_j w_j, && \textit{where} \quad A'_j = q_1(e'_j) - e'_j q'_1(e'_j), B'_j = -q'_1(e'_j),\\
q_1(w_j) &\approx q_2(e'_j) + q'_2(e'_j)(w_j - e'_j) = C'_j - D'_j w_j, && \textit{where} \quad C'_j = q_2(e'_j) - e'_j q'_2(e'_j), D'_j = -q'_2(e'_j),\\
q_3(w_j) &\approx q_3(e'_j) + q'_3(e'_j)(w_j - e'_j) = E'_j - F'_j w_j, && \textit{where} \quad E'_j = q_3(e'_j) - e'_j q'_3(e'_j), F'_j = -q'_3(e'_j).
\end{aligned}$$

Then we consider the expression for e_i and e'_j. According to the transformation of $Z = \frac{\theta}{X}$ and the monotonicity of $M(z)$, we have $Z_{r+1} \geq \ldots \geq Z_n$ and $M(Z_{r+1}, \theta) \leq \ldots \leq M(Z_n, \theta)$. For $i = r+1, \ldots, n$, let $M(Z_i) = U_i$. Here $U_{r+1} \leq \ldots \leq U_n$ denote the GPTIICOS from the Uniform (0,1) distribution. Then e_i can be approximated as $e_i = E(Z_i) \approx M^{-1}(E(U_i))$, where $E(U_i)$ can be obtained from the following equations (proposed by Balakrishnan [2]).

$$E(U_{r+i}) = 1 - \prod_{j=n-i+1}^{n} \alpha_j, i = 1, \ldots, n-r,$$

where

$$\begin{aligned}
\alpha_i &= \frac{a_i}{1+a_i}, i = r+1, \ldots, n-1,\\
\alpha_n &= \frac{a_n}{1+r+a_n},\\
a_{r+i} &= i + \sum_{j=n-i+1}^{n} R_j, i = 1, \ldots, n-r.
\end{aligned}$$

Clearly, e'_j can be approximated in a similar way.

Combining the above Taylor expansions, the likelihood Equations (20) and (21) then reduce to

$$\begin{aligned}\frac{\partial \ln L^*(\theta_1,\theta_2)}{\partial\theta_1} &= \frac{1}{\theta_1}\{(n-r)+r(A_{r+1}-B_{r+1}z_{r+1})z_{r+1}+\sum_{i=r+1}^{n}[(C_i-D_iz_i)z_i-R_i(E_i-F_iz_i)z_i]\}=0,\\ \frac{\partial \ln L^*(\theta_1,\theta_2)}{\partial\theta_2} &= \frac{1}{\theta_2}\{(m-r')+r'(A'_{r'+1}-B'_{r'+1}w_{r'+1})w_{r'+1}+\sum_{j=r'+1}^{m}[(C'_j-D'_jw_j)w_j-R'_j(E'_j-F'_jw_j)w_j]\}\\ &=0.\end{aligned}$$

Here, we show two theorems to give the explicit solutions for θ_1 and θ_2.

Theorem 4. *[24] If $\ln g(x)$ is concave on (a,b), then it is said that the function $g(x)$ is log-concave, and the following conditions are satisfied: (i) $\frac{g'(x)}{g(x)}$ is monotone decreasing on (a,b). (ii) $(\ln g(x))'' < 0$.*

Remark 3. *It is easy to prove Theorem 4 holds in the case of $a=0$ and $b=+\infty$.*

Theorem 5. *[25] Suppose $g(x)$ is the density function on $(0,+\infty)$ and $G(x)$ is the distribution function. If $g(x)$ is log-concave on $(0,+\infty)$, then (i) $M(x)=1-G(x)$ is log-concave, (ii) $G(x)$ is also log-concave on $(0,+\infty)$.*

Lemma 2. *If the standard scale density function $g(x)$ is log-concave on $(0,+\infty)$, then the AMLEs of θ_1 and θ_2 are given in Equation (22) and $\widetilde{R}$ can be obtained with Equation (14).*

$$\widetilde{\theta_1}=\frac{-\alpha_2-\sqrt{{\alpha_2}^2-4(n-r)\alpha_1}}{2\alpha_1},\widetilde{\theta_2}=\frac{-\beta_2-\sqrt{{\beta_2}^2-4(m-r')\beta_1}}{2\beta_1}, \tag{22}$$

where

$$\begin{aligned}\alpha_1&=\sum_{i=r+1}^{n}(\frac{R_iF_i-D_i}{{x_i}^2})-\frac{rB_{r+1}}{x_{r+1}^2}, &\alpha_2&=\frac{rA_{r+1}}{x_{r+1}}+\sum_{i=r+1}^{n}(\frac{C_i-R_iE_i}{x_i}),\\ \beta_1&=\sum_{j=r'+1}^{m}(\frac{R'_jF'_j-D'_j}{{y_j}^2})-\frac{r'B'_{r'+1}}{y_{r'+1}^2}, &\beta_2&=\frac{r'A'_{r'+1}}{y_{r'+1}}+\sum_{j=r'+1}^{m}(\frac{C'_j-R'_jE'_j}{y_j}).\end{aligned}$$

Proof. Substitute z_i by $\frac{\theta_1}{x_i}$ and w_j by $\frac{\theta_2}{y_j}$ and then Equation (22) are the solutions to the likelihood equations. We need to prove that $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ are positive. By Theorem 5, $M(x)$ and $G(x)$ are log-concave on $(0,+\infty)$ when $g(x)$ is log-concave and we have $p'_1(e_i), p'_2(e_i)\le 0$ and $p'_3(e_i)\ge 0$ applying Theorem 4. Thus, $B_i, D_1 \le 0$ and $F_i \ge 0$. It is impossible that $B_i, D_1, F_i \equiv 0$ for $i=r+1,\dots,n$. So, α_1 is negative. Thus, from Equation (22), $\widetilde{\theta_1}$ is positive. Similarly, we can prove $\widetilde{\theta_2}$ is also positive.

$$p_1(e_i)=\frac{m(e_i)}{M(e_i)},\quad p_2(e_i)=\frac{g'(e_i)}{g(e_i)},\quad p_3(e_i)=-\frac{g(e_i)}{G(e_i)}.$$

□

Note that

(i) in the cases of GIED and IWD (given in Section 2.1, when $\alpha>1$), $g(x)$ is log-concave on $(0,+\infty)$;

(ii) If $g(x)$ is not log-concave on $(0,+\infty)$, we can take the positive roots of the likelihood equations to get the AMLEs $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$.

3.3. R-Symmetric and R-Asymmetric Approximate Confidence Intervals

First, we obtain an asymptotic confidence interval, which is symmetric about $\widehat{R}$, with the asymptotic distributions of $\widehat{\theta_1}$, $\widehat{\theta_2}$ as well as $\widehat{R}$. Then considering the obtained confidence interval may not work well for small samples, with Bootstrap methods, we further acquire two R-asymmetric interval estimates.

3.3.1. R-Symmetric Interval Estimation Based on Asymptotic Distributions

The asymptotic distribution of $\widehat{\theta} = (\widehat{\theta_1}, \widehat{\theta_2})$ can be derived with the Fisher information matrix of $\theta = (\theta_1, \theta_2)$, which is described as follows.

$$I(\theta) = -\begin{bmatrix} E(\frac{\partial^2 l}{\partial \theta_1^2}) & E(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}) \\ E(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}) & E(\frac{\partial^2 l}{\partial \theta_2^2}) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} I_{11} &= -E(\frac{\partial^2 l}{\partial \theta_1^2}), \;\; I_{22} = -E(\frac{\partial^2 l}{\partial \theta_2^2}), \;\; I_{12} = I_{21} = 0, \\ \frac{\partial^2 l}{\partial \theta_1^2} &= -\frac{n-r}{\theta_1^2} - \frac{r}{x_{r+1}^2} \frac{g'(\frac{\theta_1}{x_{r+1}})[1 - G(\frac{\theta_1}{x_{r+1}})] + (g(\frac{\theta_1}{x_{r+1}}))^2}{[1 - G(\frac{\theta_1}{x_{r+1}})]^2} \\ &+ \sum_{i=r+1}^{n} [\frac{g''(\frac{\theta_1}{x_i}) g(\frac{\theta_1}{x_i}) - (g'(\frac{\theta_1}{x_i}))^2}{x_i^2 (g(\frac{\theta_1}{x_i}))^2} + \frac{R_i}{x_i^2} \frac{g'(\frac{\theta_1}{x_i}) G(\frac{\theta_1}{x_i}) - (g(\frac{\theta_1}{x_i}))^2}{(G(\frac{\theta_1}{x_i}))^2}], \\ \frac{\partial^2 l}{\partial \theta_2^2} &= -\frac{m-r'}{\theta_2^2} - \frac{r'}{y_{r'+1}^2} \frac{g'(\frac{\theta_2}{y_{r'+1}})[1 - G(\frac{\theta_2}{y_{r'+1}})] + (g(\frac{\theta_2}{y_{r'+1}}))^2}{[1 - G(\frac{\theta_2}{y_{r'+1}})]^2} \\ &+ \sum_{j=r'+1}^{m} [\frac{g''(\frac{\theta_2}{y_j}) g(\frac{\theta_2}{y_j}) - (g'(\frac{\theta_2}{y_j}))^2}{y_j^2 (g(\frac{\theta_2}{y_j}))^2} + \frac{R_j'}{y_j^2} \frac{g'(\frac{\theta_2}{y_j}) G(\frac{\theta_2}{y_j}) - (g(\frac{\theta_2}{y_j}))^2}{(G(\frac{\theta_2}{y_j}))^2}]. \end{aligned}$$

$I(\theta)$ can further be used to derive the approximate variance-covariance matrix. If all the regularity conditions are satisfied for the distribution from ISF, we give the following asymptotic distributions for $\widehat{\theta} = (\widehat{\theta_1}, \widehat{\theta_2})$ and $\widehat{R}$ based on the asymptotic properties of MLE.

Theorem 6. *For $n - r \to \infty$, $m - r' \to \infty$, we have $(\sqrt{n-r}(\widehat{\theta_1} - \theta_1), \sqrt{m-r'}(\widehat{\theta_2} - \theta_2)) \to N(0, C^{-1}(\theta_1, \theta_2))$ in distribution where $C(\theta_1, \theta_2) = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}$ with $c_{11} = \frac{I_{11}}{n-r}$, $c_{22} = \frac{I_{22}}{m-r'}$.*

Theorem 7. *For $n - r \to \infty$, $m - r' \to \infty$, we have $\sqrt{n-r}(\widehat{R} - R) \to N(0, A)$ where*

$$\begin{aligned} A &= b^T C^{-1} b, C^{-1}(\theta_1, \theta_2) = \begin{bmatrix} \frac{n-r}{I_{11}} & 0 \\ 0 & \frac{m-r'}{I_{22}} \end{bmatrix}, b = \begin{bmatrix} b_1(\theta_1, \theta_2) \\ b_2(\theta_1, \theta_2) \end{bmatrix}, \\ b_1(\theta_1, \theta_2) &= \int_0^{+\infty} [\frac{1}{x^2} g(\frac{\theta_1}{x}) + \frac{\theta_1}{x^3} g'(\frac{\theta_1}{x})][1 - G(\frac{\theta_2}{x})] dx, \\ b_2(\theta_1, \theta_2) &= -\int_0^{+\infty} \frac{\theta_1}{x^3} g(\frac{\theta_1}{x}) g(\frac{\theta_2}{x}) dx. \end{aligned}$$

Proof. Applying the Delta method and Theorem 6, we can describe the asymptotic distribution of $\widehat{R} = R(\widehat{\theta_1}, \widehat{\theta_2})$ as $\sqrt{n-r}(\widehat{R} - R) \to N(0, A)$, where

$$A = \begin{pmatrix} \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_1} & \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_2} \end{pmatrix} C^{-1}(\theta_1, \theta_2) \begin{pmatrix} \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_1} \\ \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_2} \end{pmatrix}.$$

It is easy to prove that $b_1(\theta_1, \theta_2) = \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_1}$ and $b_2(\theta_1, \theta_2) = \frac{\partial R(\theta_1,\theta_2)}{\partial \theta_2}$. Thus the proof is completed. □

From the results of Theorem 7, a $100(1-\xi)\%$ asymptotic confidence interval can be determined as: $[\widehat{R} - \frac{\sqrt{\widehat{A}}}{\sqrt{n-r}} z_{1-\frac{\xi}{2}}, \widehat{R} + \frac{\sqrt{\widehat{A}}}{\sqrt{n-r}} z_{1-\frac{\xi}{2}}]$, which is symmetric about $\widehat{R}$. Here, A is estimated with $\widehat{\theta_1}$ and $\widehat{\theta_2}$. z_ξ is 100ξth percentile of $N(0,1)$. Similarly, by estimating matrix A with $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$, the R-symmetric asymptotic confidence interval based on $\widetilde{R}$ is thus acquired. Note that $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ is given in Section 3.2.

3.3.2. R-Asymmetric Interval Estimation Based on Bootstrap Methods

Since the obtained asymptotic distributions may not hold for small samples, two Bootstrap confidence intervals are thereby proposed, which are R-asymmetric and considered to have a better performance.

(a) Bootstrap-p method

Step 1: Generate $(x^*_{r+1}, \ldots, x^*_n)$ from $(x_{r+1}, \ldots, x_n)$ and $(y^*_{r'+1}, \ldots, y^*_m)$ from $(y_{r'+1}, \ldots, y_m)$, respectively. Based on the generated samples, calculate $\hat{R}^*$ with Equation (14);
Step 2: Repeat step 1, *NBOOT* times;
Step 3: Define the CDF of $\hat{R}^*$ as $\Phi_1(z) = P(\hat{R}^* \leq z)$. Let $\hat{R}_{BP}(z) = \Phi_1^{-1}(z)$ for a given z. Thus an approximate $100(1-\xi)\%$ confidence interval of R is $(\hat{R}_{BP}(\frac{\xi}{2}), \hat{R}_{BP}(1-\frac{\xi}{2}))$.

(b) Bootstrap-t method

Step 1: Calculate $\hat{\theta_1}$, $\hat{\theta_2}$ and $\hat{R}$ with $(x_{r+1}, \ldots, x_n)$ and $(y_{r'+1}, \ldots, y_m)$;
Step 2: Generate $(x^*_{r+1}, \ldots, x^*_n)$ from $(x_{r+1}, \ldots, x_n)$ and $(y^*_{r'+1}, \ldots, y^*_m)$ from $(y_{r'+1}, \ldots, y_m)$, respectively;
Step 3: Calculate $\hat{R}^*$ with Equation (14) and calculate B^* with $B^* = \frac{\sqrt{n-r}(\hat{R}^* - \hat{R})}{\sqrt{Var(\hat{R}^*)}}$ (Note that $Var(\hat{R}^*)$ can be derived from Theorem 7);
Step 4: Repeat steps 2 and 3, *NBOOT* times;
Step 5: Define the CDF of B^* as $\Phi_2(z) = P(B^* \leq z)$. Let $\hat{R}_{BT}(z) = \hat{R} + \Phi_2^{-1}(z)\sqrt{\frac{Var(\hat{R})}{n-r}}$ for a given z. Thus an approximate $100(1-\xi)\%$ confidence interval of R is $(\hat{R}_{BT}(\frac{\xi}{2}), \hat{R}_{BT}(1-\frac{\xi}{2}))$.

3.4. Bayesian Estimation

Since gamma distributions are frequently used as prior distributions for parameters from the inverted scale family according to the related articles [4,7,26–28], we assume independent gamma priors on θ_1 and θ_2, that is, $\theta_i \sim Gamma(\alpha_i, \beta_i)$ for $i \in \{1,2\}$. The PDFs are given in Equation (24). Note that R can be derived in a similar way when θ follows other prior distributions.

$$\pi(\theta_i|\alpha_i, \beta_i) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \theta_i^{\alpha_i - 1} e^{-\beta_i \theta_i}, i = 1, 2, \alpha_i, \beta_i > 0 \tag{24}$$

Then, the joint posterior density function of θ_1 and θ_2 is:

$$L(\theta_1, \theta_2|data) = \frac{L(data|\theta_1, \theta_2)\pi(\theta_1)\pi(\theta_2)}{\int_0^\infty \int_0^\infty L(data|\theta_1, \theta_2)\pi(\theta_1)\pi(\theta_2) d\theta_1 d\theta_2}. \tag{25}$$

Equation (25) is found to be intractable, and therefore, getting the respective Bayesian estimator analytically is quite difficult. However, samples can be generated from the posterior distribution with

Gibbs sampling so that the Bayesian estimator of R and HPD credible interval, proposed by Chen [29], can be obtained easily. Here give the posterior density functions of θ_1 and θ_2:

$$f(\theta_1|\theta_2, data) \propto [1 - G(\frac{\theta_1}{x_{r+1}})]^r \prod_{i=r+1}^{n} \frac{\theta_1}{x_i^2} g(\frac{\theta_1}{x_i}) G(\frac{\theta_1}{x_i})^{R_j} \times \theta_1^{\alpha_1 - 1} e^{-\beta_1 \theta_1},$$
$$f(\theta_2|\theta_1, data) \propto [1 - G(\frac{\theta_2}{y_{r'+1}})]^{r'} \prod_{j=r'+1}^{m} \frac{\theta_2}{y_j^2} g(\frac{\theta_2}{y_j}) G(\frac{\theta_2}{y_j})^{R'_j} \times \theta_2^{\alpha_2 - 1} e^{-\beta_2 \theta_2}.$$

Both posterior density functions are found to be unknown. So the Metropolis-Hastings ($M - H$) method is used to generate $\theta_1^{(k)}$ and $\theta_2^{(k)}$ from $f(\theta_1|\theta_2, data)$ and $f(\theta_2|\theta_1, data)$, respectively. Gibbs sampling algorithm is as follows:

Step 1: Determine the initial value $(\theta_1^{(0)}, \theta_2^{(0)})$;
Step 2: Set $k = 1$;
Step 3: Using the $M - H$ method, generate $\theta_1^{(k)}$ from $f(\theta_1|\theta_2, data)$. Here, the proposal distribution is $exp(\theta_1^{(k-1)})$;
Step 4: Using the $M - H$ method, generate $\theta_2^{(k)}$ from $f(\theta_2|\theta_1, data)$. Here, the proposal distribution is $exp(\theta_2^{(k-1)})$;
Step 5: Compute $R^{(k)}$ by Equation (14);
Step 6: Set $k = k + 1$;
Step 7: Repeat steps 3 to 6 K times.

Then the approximate posterior mean and posterior variance of R is:

$$\widehat{E}(R|data) = \frac{1}{K} \sum_{k=1}^{K} R^{(k)},$$
$$\widehat{Var}(R|data) = \frac{1}{K} \sum_{k=1}^{K} [R^{(k)} - \hat{E}(R|data)]^2.$$

The HPD credible interval at $100(1 - \xi)\%$ credible level can be given by:

$$(R_{[\frac{\xi}{2}K]}, R_{[(1-\frac{\xi}{2})K]}).$$

Here, $R_{[i]}$ denotes the ith smallest of $\{R^{(k)}(k = 1, 2, \ldots, K)\}$.

3.5. Simulation Study

Supposing both X and Y are from IED (as shown in Example 1), the obtained estimations for R can be achieved through Monte Carlo simulations. The GPTIICOS from X and Y are generated with the algorithm in Section 2.2.

Here three censoring schemes are considered, which are listed and numbered in Table 4, and six sets of parameter values (θ_1, θ_2) are used for simulations, including $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$, $(2, 3)$, and $(3, 3)$.

Table 4. Censoring schemes ($C.S.$).

$C.S.$	$(r, n, N, R_{r+1}, R_{r+2}, \ldots, R_n)$
R_1	(2, 10, 29, 3, 1, 2, 3, 1, 2, 3, 4)
R_2	(2, 10, 26, 2, 2, 2, 2, 2, 2, 2, 2)
R_3	(2, 10, 26, 4, 2, 4, 2, 1, 1, 1, 1)

We make an evaluation on the validity of estimations obtained from each censoring scheme and each value of (θ_1, θ_2). The evaluation process is as follows. First, the accuracy of the point estimates is compared in light of their mean squared errors (MSE) and biases. Then, with the average interval lengths and interval coverage percentages, we make an assessment on the different interval estimates.

The average point estimates for each parameter value and censoring scheme are obtained through 1000 simulations, which are shown in Table 5 together with the corresponding average biases and MSEs. Further, we give the heat maps of Table 5 in Figures 2 and 3. The proposed intervals at 95% confidence/credible level are also obtained. Table 6 gives the obtained average interval lengths and interval coverage percentages for different methods. In our simulations, the HPD credible intervals are acquired after 2000 samples while the Bootstrap confidence intervals are acquired after 250 resamples.

From Table 5, all the point estimates are quite effective as the corresponding MSEs and biases are pretty small. Additionally, we speculate from Figures 2 and 3 that the accuracy of the estimators is somewhat related to the true value of (θ_1, θ_2) because the MSEs and biases of the Bayesian estimator show a great difference between the case of $\theta_1 = \theta_2$ and the others. We can find in the cases that (θ_1, θ_2) takes $(1, 3)$ and $(2, 3)$, the MSEs and biases of the Bayesian estimator are much higher than those in the other cases. However, in the cases of $\theta_1 = \theta_2$, Bayes estimates are extremely accurate.

For the interval estimations, we refer to Table 6. It can be seen that the interval length of the Bootstrap-t interval is wider than others for most cases and according to the interval coverage percentages, the performance of Bootstrap-t intervals is slightly better than the performance of asymptotic confidence intervals obtained from $\widehat{R}$ and $\widetilde{R}$. Moreover, the interval coverage percentages of the R-symmetric asymptotic confidence intervals are more deviated from 95%, which indicates that their performances are not good when compared with others. That is probably because small samples may affect the asymptotic distribution of R, which thereby may affect the accuracy of interval estimation. However, it is obvious that the HPD credible interval performs better when compared to the asymptotic intervals and Bootstrap intervals since the corresponding interval coverage percentages are closer to the nominal level. So, Bayes estimation is considered as the best method, and according to the interval coverage percentages, the ranking of the three methods is given below.

$$BEST \quad Bayes \quad \rightarrow \quad Bootstrap \quad \rightarrow \quad Asymptotics \quad WORST$$

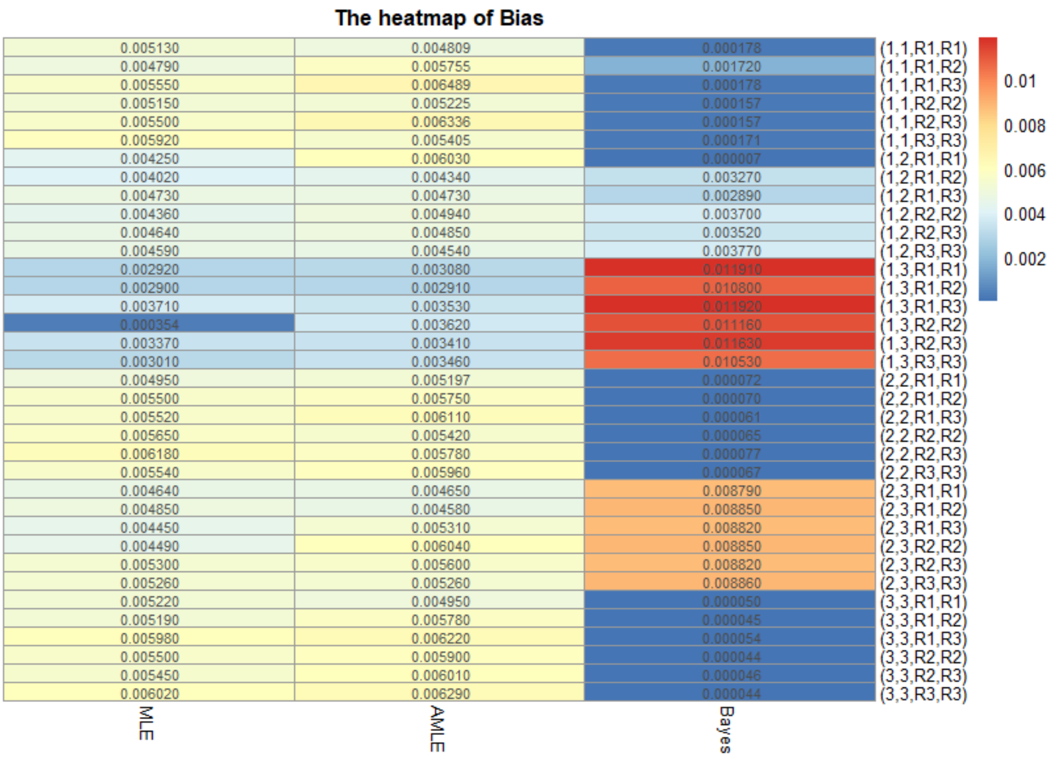

Figure 2. The bias heat map.

Table 5. Average point estimates of R and the corresponding Biases (MSEs).

		MLE		$AMLE$		$Bayes$	
(θ_1,θ_2)	$C.S.$	R	$Bias(Mse)$	R	$Bias(Mse)$	R	$Bias(Mse)$
$(1,1)$	(R_1,R_1)	0.50085	0.00085(0.00513)	0.50291	0.00291(0.00481)	0.50016	0.00016(0.00018)
	(R_1,R_2)	0.50468	0.01468(0.00479)	0.50699	0.00699(0.00576)	0.50049	0.00049(0.00017)
	(R_1,R_3)	0.51370	0.01370(0.00555)	0.52850	0.02850(0.00649)	0.50050	0.00050(0.00018)
	(R_2,R_2)	0.50230	0.00230(0.00515)	0.50231	0.00231(0.00523)	0.50009	0.00009(0.00016)
	(R_2,R_3)	0.50833	0.00833(0.00550)	0.51731	0.01731(0.00634)	0.50078	0.00078(0.00016)
	(R_3,R_3)	0.50295	0.00295(0.00592)	0.50262	0.00262(0.00541)	0.50066	0.00066(0.00017)
$(1,2)$	(R_1,R_1)	0.33615	0.00315(0.00425)	0.34790	0.01490(0.00603)	0.33341	0.00041(0.00007)
	(R_1,R_2)	0.34654	0.01354(0.00402)	0.34425	0.01125(0.00434)	0.35512	0.02212(0.00327)
	(R_1,R_3)	0.34605	0.01305(0.00473)	0.34831	0.01531(0.00473)	0.35051	0.01751(0.00289)
	(R_2,R_2)	0.33812	0.00512(0.00436)	0.33780	0.00480(0.00494)	0.35882	0.02582(0.00370)
	(R_2,R_3)	0.33332	0.00032(0.00464)	0.34141	0.00841(0.00485)	0.35614	0.02314(0.00352)
	(R_3,R_3)	0.33670	0.00370(0.00459)	0.33432	0.00132(0.00454)	0.36062	0.02762(0.00377)
$(1,3)$	(R_1,R_1)	0.25153	0.00153(0.00292)	0.25651	0.00651(0.00308)	0.32660	0.07660(0.01191)
	(R_1,R_2)	0.25812	0.00812(0.00290)	0.25621	0.00621(0.00291)	0.32476	0.07476(0.01080)
	(R_1,R_3)	0.25847	0.00847(0.00371)	0.27132	0.02132(0.00353)	0.32228	0.07228(0.01192)
	(R_2,R_2)	0.25594	0.00594(0.00354)	0.25403	0.00403(0.00362)	0.32491	0.07491(0.01116)
	(R_2,R_3)	0.25292	0.00292(0.00337)	0.25022	0.00022(0.00341)	0.32323	0.07323(0.01163)
	(R_3,R_3)	0.25245	0.00245(0.00301)	0.25808	0.00808(0.00346)	0.32516	0.07516(0.01053)
$(2,2)$	(R_1,R_1)	0.50245	0.00245(0.00495)	0.50019	0.00019(0.00520)	0.50005	0.00005(0.00007)
	(R_1,R_2)	0.50341	0.00341(0.00550)	0.51469	0.01469(0.00575)	0.50044	0.00044(0.00007)
	(R_1,R_3)	0.50896	0.00896(0.00552)	0.52607	0.02607(0.00611)	0.50027	0.00027(0.00006)
	(R_2,R_2)	0.50033	0.00033(0.00565)	0.50055	0.00055(0.00542)	0.50013	0.00013(0.00007)
	(R_2,R_3)	0.50055	0.00055(0.00618)	0.51098	0.01098(0.00578)	0.50010	0.00010(0.00008)
	(R_3,R_3)	0.50365	0.00365(0.00554)	0.50225	0.00255(0.00596)	0.50079	0.00079(0.00007)
$(2,3)$	(R_1,R_1)	0.40467	0.00467(0.00464)	0.40345	0.00345(0.00465)	0.49350	0.09350(0.00879)
	(R_1,R_2)	0.41019	0.01019(0.00485)	0.40838	0.00838(0.00458)	0.49376	0.09376(0.00885)
	(R_1,R_3)	0.40929	0.00929(0.00445)	0.42179	0.02179(0.00531)	0.49361	0.09361(0.00882)
	(R_2,R_2)	0.40152	0.00152(0.00449)	0.40775	0.00775(0.00604)	0.49382	0.09382(0.00885)
	(R_2,R_3)	0.40044	0.00044(0.00530)	0.41298	0.01298(0.00560)	0.49362	0.09362(0.00882)
	(R_3,R_3)	0.40132	0.00132(0.00526)	0.40099	0.00099(0.00526)	0.49382	0.09382(0.00886)
$(3,3)$	(R_1,R_1)	0.50172	0.00172(0.00522)	0.50209	0.00209(0.00495)	0.50009	0.00009(0.00005)
	(R_1,R_2)	0.50577	0.00577(0.00519)	0.51180	0.01180(0.00581)	0.50010	0.00010(0.00005)
	(R_1,R_3)	0.50220	0.00220(0.00598)	0.52334	0.02334(0.00622)	0.50086	0.00086(0.00005)
	(R_2,R_2)	0.50062	0.00062(0.00550)	0.50583	0.00583(0.00590)	0.50003	0.00003(0.00004)
	(R_2,R_3)	0.50077	0.00077(0.00545)	0.50499	0.00499(0.00601)	0.50043	0.00043(0.00005)
	(R_3,R_3)	0.50363	0.00363(0.00602)	0.50470	0.00470(0.00629)	0.50020	0.00020(0.00004)

Table 6. Average interval length (interval coverage percentage) for different interval estimations.

(θ_1, θ_2)	$C.S.$	MLE	$AMLE$	$Bayes$	$Boot - p$	$Boot - t$
$(1,1)$	(R_1, R_1)	0.268(0.943)	0.272(0.925)	0.250(0.954)	0.268(0.922)	0.299(0.948)
	(R_1, R_2)	0.272(0.899)	0.273(0.911)	0.250(0.938)	0.275(0.924)	0.302(0.943)
	(R_1, R_3)	0.278(0.914)	0.276(0.875)	0.250(0.930)	0.276(0.931)	0.305(0.954)
	(R_2, R_2)	0.282(0.912)	0.284(0.924)	0.250(0.947)	0.282(0.914)	0.305(0.926)
	(R_2, R_3)	0.286(0.919)	0.280(0.907)	0.247(0.946)	0.287(0.930)	0.309(0.935)
	(R_3, R_3)	0.284(0.935)	0.284(0.925)	0.247(0.955)	0.311(0.954)	0.311(0.946)
$(1,2)$	(R_1, R_1)	0.237(0.914)	0.239(0.929)	0.249(0.935)	0.241(0.929)	0.266(0.915)
	(R_1, R_2)	0.248(0.927)	0.245(0.911)	0.249(0.953)	0.253(0.937)	0.270(0.929)
	(R_1, R_3)	0.255(0.912)	0.247(0.909)	0.249(0.943)	0.258(0.928)	0.272(0.941)
	(R_2, R_2)	0.250(0.905)	0.251(0.914)	0.250(0.931)	0.255(0.943)	0.276(0.920)
	(R_2, R_3)	0.256(0.922)	0.253(0.910)	0.249(0.947)	0.261(0.944)	0.277(0.930)
	(R_3, R_3)	0.257(0.933)	0.257(0.906)	0.250(0.949)	0.261(0.946)	0.279(0.931)
$(1,3)$	(R_1, R_1)	0.205(0.924)	0.205(0.934)	0.249(0.930)	0.210(0.939)	0.228(0.928)
	(R_1, R_2)	0.215(0.910)	0.208(0.913)	0.250(0.951)	0.219(0.936)	0.231(0.919)
	(R_1, R_3)	0.221(0.924)	0.211(0.896)	0.249(0.947)	0.226(0.917)	0.232(0.902)
	(R_2, R_2)	0.214(0.914)	0.214(0.919)	0.249(0.940)	0.221(0.938)	0.236(0.926)
	(R_2, R_3)	0.221(0.909)	0.217(0.905)	0.249(0.943)	0.226(0.937)	0.236(0.910)
	(R_3, R_3)	0.217(0.931)	0.217(0.909)	0.250(0.957)	0.223(0.957)	0.236(0.927)
$(2,2)$	(R_1, R_1)	0.269(0.920)	0.265(0.925)	0.250(0.937)	0.268(0.949)	0.298(0.924)
	(R_1, R_2)	0.274(0.912)	0.275(0.921)	0.250(0.936)	0.276(0.932)	0.303(0.924)
	(R_1, R_3)	0.274(0.900)	0.276(0.911)	0.249(0.946)	0.277(0.926)	0.306(0.859)
	(R_2, R_2)	0.281(0.911)	0.279(0.911)	0.249(0.962)	0.282(0.942)	0.307(0.920)
	(R_2, R_3)	0.281(0.907)	0.281(0.908)	0.250(0.955)	0.283(0.933)	0.308(0.934)
	(R_3, R_3)	0.286(0.930)	0.284(0.914)	0.251(0.945)	0.287(0.949)	0.309(0.931)
$(2,3)$	(R_1, R_1)	0.257(0.909)	0.256(0.900)	0.250(0.943)	0.260(0.925)	0.285(0.911)
	(R_1, R_2)	0.267(0.926)	0.262(0.911)	0.249(0.956)	0.269(0.952)	0.293(0.934)
	(R_1, R_3)	0.270(0.912)	0.264(0.896)	0.250(0.934)	0.272(0.929)	0.293(0.922)
	(R_2, R_2)	0.267(0.907)	0.270(0.916)	0.250(0.947)	0.272(0.939)	0.296(0.918)
	(R_2, R_3)	0.274(0.921)	0.271(0.910)	0.250(0.940)	0.275(0.943)	0.297(0.929)
	(R_3, R_3)	0.272(0.912)	0.271(0.907)	0.249(0.952)	0.275(0.944)	0.296(0.933)
$(3,3)$	(R_1, R_1)	0.267(0.908)	0.267(0.923)	0.249(0.959)	0.269(0.939)	0.296(0.923)
	(R_1, R_2)	0.274(0.914)	0.275(0.895)	0.249(0.947)	0.275(0.935)	0.302(0.939)
	(R_1, R_3)	0.276(0.888)	0.276(0.899)	0.250(0.934)	0.276(0.913)	0.305(0.921)
	(R_2, R_2)	0.281(0.915)	0.282(0.927)	0.250(0.929)	0.283(0.934)	0.308(0.925)
	(R_2, R_3)	0.283(0.917)	0.280(0.908)	0.250(0.925)	0.284(0.950)	0.308(0.909)
	(R_3, R_3)	0.285(0.907)	0.283(0.911)	0.251(0.945)	0.285(0.939)	0.307(0.911)

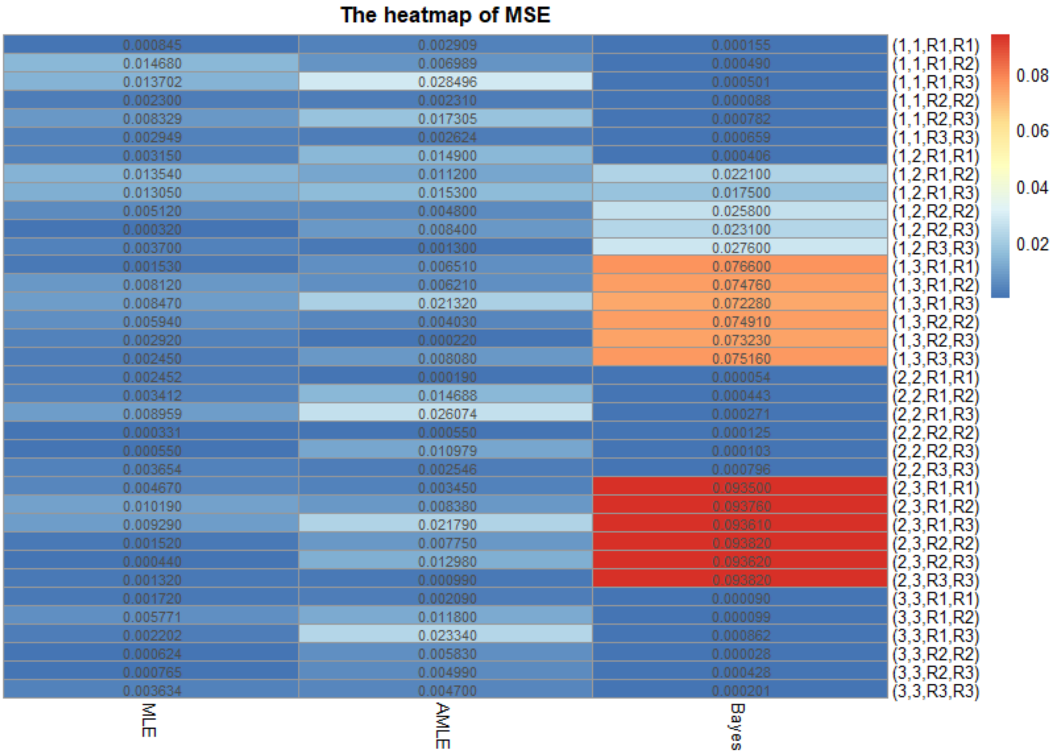

Figure 3. The MSE heat map.

3.6. An Illustrated Example

Here, an analysis of two real datasets, reported by Badar and Priest [30], is given below. Many reliability models, for example, the models based on the generalized Rayleigh distributions, Weibull distributions and three-parameter Weibull distributions, have been applied to the two datasets (see Raqab and Kundu [17], Kundu and Gupta [18], and Kundu and Raqad [19]).

Data Set 1. *The data below shows the breaking strength for 63 carbon fibers of 10 mm gauge length in a tension test (unit: GPA).*

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Data Set 2. *The data below shows the breaking strength for 69 carbon fibers of 20 mm gauge length in a tension test (unit: GPA).*

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Here, the fitting of three reliability models based on GIED (as shown in Example 1), WD (as shown in Section 2.3), IWD (as shown in Example 2) is considered. We use maximum likelihood estimation to construct the above three parametric models. Same as in Section 2.3, for each model, five measures are considered to test the goodness-of-fit: (1) $-\ln$, (2) $K-S$ distance, (3) p-value, (4) AIC, and (5) BIC. In the case of MLE, these values based on the three reliability models are listed in Table 7. Clearly, the best fit order for both datasets is:

$$BEST \quad GIED \;\to\; WD \;\to\; IWD \quad WORST$$

Table 7. Fitting results for the three reliability models.

Data Set	Model	α	θ	$-\ln$	$K-S$	p-Value	AIC	BIC
1	GIED	175.2879	16.8110	57.7251	0.8601	0.7399	119.4504	123.7367
	WD	5.0494	424.5597	61.9570	0.0876	0.7192	127.914	132.2002
	IWD	5.433788	2.721432	58.9022	0.1001	0.5528	121.8043	126.0906
2	GIED	205.87851	13.88254	49.16796	0.041419	0.9998	102.3359	106.8041
	WD	5.504847	214.130524	49.59614	0.056132	0.9816	103.1923	107.6605
	IWD	4.126731	2.143723	63.62361	0.13363	0.1700	131.2472	135.7154

Since GIED has not only the lowest values of $-\ln$, AIC, and BIC but also the highest p-value, it is considered as the best model. Furthermore, the fitting effects of the three reliability models are intuitively shown in Figure 4. So GIED models are used to fit the two datasets separately. It appears from Table 7 that for Data Set 1, the values of $-\ln$, $K-S$ distance, p-value, AIC, and BIC in the case of MLE are 57.7251, 0.8601, 0.7399, 123.7367, and 119.4504, respectively, indicating the GIED model fits the Data Set 1 very well when α takes 175.2879. Same as for Data Set 1, the GIED model fits Data Set 2 very well when α takes 205.87851. Hence, it is reasonable to assume X follows GIED(θ_1,175.2879) and Y follows GIED(θ_2,205.87851). Set $r=3$. Table 8 shows the respective censoring schemes and the generated censored data.

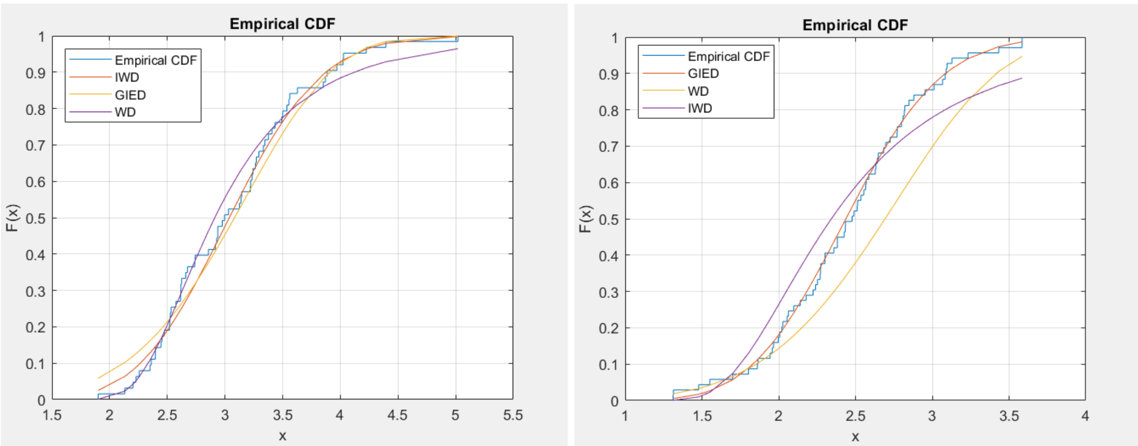

Figure 4. An empirical CDF and other CDFs based on different statistical models for the datasets.

Table 8. Generated datasets and censoring schemes.

i,j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X_i	-	-	-	2.228	2.397	2.522	2.616	2.738	2.937	3.125	3.235	3.332	3.493	3.628	4.024
R_i	-	-	-	4	4	4	4	4	4	4	4	4	4	4	4
Y_j	-	-	-	1.552	1.944	2.021	2.140	2.270	3.359	2.435	2.535	2.629	2.697	2.807	2.954
R'_j	-	-	-	4	4	4	4	4	4	4	4	4	4	4	10

With the R programming language, the proposed estimates of R are derived. As shown in Table 9, the MLE, AMLE, Bayesian estimator are 0.782, 0.794, and 0.782, respectively. The corresponding 95% confidence/credible intervals are (0.680,0.885), (0.690,0.898), and (0.651,0.871). Additionally, The Bootstrap-p and Bootstrap-t confidence intervals are obtained to be (0.665,0.878) and (0.694,0.913), respectively.

Table 9. Estimation results for the proposed estimators and confidence/credible intervals.

Estimation	MLE	AMLE	Bayes	Bootstrap-p	Bootstrap-t
Point Estimation	0.782	0.794	0.782	–	–
Interval Estimation	(0.680, 0.885)	(0.690, 0.898)	(0.651, 0.871)	(0.665, 0.878)	(0.694, 0.913)

4. Conclusions

Two estimation problems are solved in this paper. Under GPTIIC, we first achieve the exact interval estimation for ISF, and the Monte Carlo simulations indicate that it has a good performance. Then we derive the point and interval estimates of R in the case of ISF. The three point estimates are all accurate according to their biases and MSEs. However, according to the results of Bayes estimators, the accuracy of the estimators is thought to be related to the true values of the scale parameters. For interval estimates, the simulation results show that the HPD credible interval performs better since the corresponding interval coverage percentage is closer to the nominal level. But the R-symmetric asymptotic intervals based on $\widehat{R}$ and $\widetilde{R}$ are not good due to the small sample size. Moreover, the Bootstrap-t interval has the widest interval length in most cases. According to the interval percentages, the ranking of the three methods is: Bayes $>$ Bootstrap $>$ Asymptotics. For each estimation problem, we also demonstrate the estimation process using real datasets to show the feasibility of the proposed methods. Further work should focus on the two estimation problems for other distributions based on GPTIICS. In addition, under other censoring schemes other than GPTIIC, we can further consider the estimation problems for ISF.

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