

On (ψ, ϕ) -Rational Contractions

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Abstract: In this paper, we examine the notion of (ψ, ϕ) -contractions by involving rational forms in the context of complete metric spaces. We note that some well-known fixed point theorems for rational forms can be deduced from our main results. We also consider some examples to indicate the validity of the presented results.

Keywords: fixed point theorems; metric space; contraction mapping

1. Introduction and Preliminaries

Thousands of results have been published since Banach [1] proved the first fixed point theorem. Some of these results are equivalent to the results published previously, while others were understood to be a sub-result of the previous results. Therefore, recently, publications that collect and consolidate the results in the literature have started to appear.

Very recently, Proinov (2020) [2], to extend and unify many earlier results, proved that the fixed point theorem of Skof (1977) [3], in the setting of metric spaces, covers several existing results, including the attractive results of Wardowski (2012) [4] and Jleli-Samet (2014) [5]. He also proved that the analog of this observation holds true in the context of dislocated metric spaces.

On the other hand, starting from Das-Gupta (1975) [6] and Jaggi (1977) [7], rational expressions were used to prove fixed point theorems. Later, these approaches were modified for Boyd and Wong contractions, ϕ -contractions, Geraghty contractions, Wardowski contractions, etc. We observe that the concerns of Proinov [2] are valid for fixed point theorems involving rational expression; that is, some published results are equivalent to earlier results or consequences.

In this paper, we prove that the analog of the fixed theorem of Skof [3] with rational expression unifies and extends several fixed point theorems in the literature.

To begin with, we recall the first main result of Proinov [2].

Theorem 1. [2] Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping such that:

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)),$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$, where the functions $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ are such that the following conditions are satisfied:

- (1) ψ is nondecreasing;
- (2) $\phi(s) < \psi(s)$ for any $s > 0$;
- (3) $\limsup_{s \rightarrow s_0+} \phi(s) < \psi(s_0+)$ for any $s_0 > 0$.

Then, T admits a unique fixed point.

We also recall the main results in which some rational expressions were studied in a contraction condition.

Theorem 2 ([6]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping such that there exist $k_1, k_2 \in [0, 1)$, with $k_1 + k_2 < 1$ such that:

$$d(Tx, Ty) \leq k_1 \cdot d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + k_2 \cdot d(x, y) \quad (1)$$

for all $x, y \in X$. Then, T has a unique fixed point $v \in X$, and the sequence $\{T^n x\}$ converges to the fixed point v for all $x \in X$.

Theorem 3 ([7]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a continuous mapping. If there exist $k_1, k_2 \in [0, 1)$, with $k_1 + k_2 < 1$ such that:

$$d(Tx, Ty) \leq k_1 \cdot \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + k_2 \cdot d(x, y), \quad (2)$$

for all distinct $x, y \in X$, then T possesses a unique fixed point in X .

We mention that over the last few years, many interesting and different generalizations for rational contractions have been provided (see, for example, [8–12]).

Finally, let us consider the next lemma (which can be found in many papers; see, e.g., [2]), which will be useful in the sequel.

Lemma 1 ([2]). Let $\{x_n\}$ be a sequence in a metric space (X, d) such that $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If the sequence $\{x_n\}$ is not Cauchy, then there exist $\epsilon > 0$ and the subsequences $\{q_k\}$ and $\{r_k\}$ of positive integers such that:

$$\lim_{k \rightarrow \infty} d(x_{q_k}, x_{r_k}) = \epsilon, \quad \lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) = \epsilon +, \quad (3)$$

2. Main Results

Throughout this section, we will consider that $\phi, \psi : (0, \infty) \rightarrow \mathbb{R}$ are two functions such that:

(f₀) $\phi(s) < \psi(s)$, for all $s > 0$.

Definition 1. Let (X, d) be a complete metric space. A mapping $T : X \rightarrow X$ is a (ψ, ϕ) -rational contraction of Type 1 if for every distinct $x, y \in X$ such that $d(Tx, Ty) > 0$, the following inequality:

$$\psi(d(Tx, Ty)) \leq \phi(\mathcal{M}_1(x, y)), \quad (4)$$

holds, where \mathcal{M}_1 is defined by:

$$\mathcal{M}_1(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \right\} \quad (5)$$

Theorem 4. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a continuous (ψ, ϕ) -rational contraction of Type 1. Assume that:

- (f₁) $\lim_{s \rightarrow s_0} \psi(s) > -\infty$, for any $s_0 > 0$;
- (f₂) $\limsup_{s \rightarrow s_0} \phi(s) < \liminf_{s \rightarrow s_0+} \psi(s)$, for any $s_0 > 0$;
- (f₃) T is continuous.

Then, T admits exactly one fixed point.

Proof. Starting with a point $x \in X$, we define the sequence $\{\lambda_n\}$ by:

$$\lambda_1 = Tx, \lambda_2 = T^2x, \dots, \lambda_n = T^n x, \quad (6)$$

with $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ (indeed, on the contrary, if there exists $j_n \in \mathbb{N}$ such that $x_{j_n} = x_{j_n+1} = Tx_{j_n}$, we get that x_{j_n} is a fixed point of T). Under this consideration, for $x = x_{n-1}$ and $y = x_n$, we have:

$$\begin{aligned}\mathcal{M}_1(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \right. \\ &\quad \left. \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)} \right\} \\ &= \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned}$$

and by (4) (since $d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) > 0$), we get:

$$\psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) \leq \phi(\mathcal{M}_1(x_{n-1}, x_n)),$$

which is equivalent, denoting by $\varsigma_n = d(x_{n-1}, x_n)$, to:

$$\psi(\varsigma_{n+1}) \leq \phi(\max \{\varsigma_n, \varsigma_{n+1}\}). \quad (7)$$

(Of course, we can assume that $\varsigma_n > 0$, since on the contrary, we can find $l \in \mathbb{N}$ such that $d(x_{l-1}, x_l) = \varsigma_l = 0$. Thus, $x_l = x_{l+1} = Tx_l$ and x_l is the fixed point of T .) If there exists $n \in \mathbb{N}$ such that $\max \{\varsigma_n, \varsigma_{n+1}\} = \varsigma_{n+1}$, then $\psi(\varsigma_{n+1}) \leq \phi(\varsigma_{n+1})$, which contradicts the assumption (f_0) . Therefore, for all $n > 0$, we have $\varsigma_n > \varsigma_{n+1}$, so that the sequence $\{\varsigma_n\}$ is decreasing, and since it is strictly positive, there exists $\varsigma \geq 0$ such that $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma$ and $\varsigma_n > \varsigma$ for all $n > 0$. Supposing that $\varsigma > 0$, because $\mathcal{M}_1(x_{n-1}, x_n) = \varsigma_n$, replacing in (4) and taking into account (f_0) , we have,

$$\psi(\varsigma_{n+1}) \leq \phi(\varsigma_n) < \psi(\varsigma_n).$$

It follows that the sequence $\{\psi(\varsigma_n)\}$ is strictly decreasing, and since it is bounded (below) (because $\varsigma_n > \varsigma$ and due to the assumption (f_1)), we can conclude that $\{\psi(\varsigma_n)\}$ is a convergent sequence. Moreover, from the above inequality, the sequence $\{\phi(\varsigma_n)\}$ is also convergent as the same limit. Thus, keeping in mind (f_2) ,

$$\liminf_{s \rightarrow \varsigma_+} \psi(s) \leq \lim_{n \rightarrow \infty} \psi(\varsigma_n) = \lim_{n \rightarrow \infty} \phi(\varsigma_n) \leq \limsup_{s \rightarrow \varsigma} \phi(s) < \liminf_{s \rightarrow \varsigma_+} \psi(s),$$

which is a contradiction. Therefore, $\varsigma = 0$ and:

$$\lim_{n \rightarrow \infty} \varsigma_n = \lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0. \quad (8)$$

We claim that $\{x_n\}$ is a Cauchy sequence. Let us suppose by contradiction that the sequence $\{x_n\}$ defined by (6) is not Cauchy. Then, by Lemma 1, there exist $\varepsilon > 0$ and two sequences of positive real numbers (q_k) and (r_k) such that:

$$\lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) = \varepsilon, \quad \lim_{k \rightarrow \infty} d(x_{q_k}, x_{r_k}) = \varepsilon. \quad (9)$$

Furthermore, for all $k \geq 1$, we have $d(x_{q_k+1}, x_{r_k+1}) > \varepsilon$. Replacing $x = x_{q_k+1}$ and $y = x_{r_k+1}$ in (4) and taking into account (f_0) , we have:

$$\psi(d(x_{q_k+1}, x_{r_k+1})) \leq \phi(\mathcal{M}_1(x_{q_k}, x_{r_k})) \leq \psi(\mathcal{M}_1(x_{q_k}, x_{r_k})),$$

where:

$$\begin{aligned}\mathcal{M}_1(x_{q_k}, x_{r_k}) &= \max \left\{ d(x_{q_k}, x_{r_k}), d(x_{q_k}, Tx_{q_k}), d(x_{r_k}, Tx_{r_k}), \right. \\ &\quad \left. \frac{d(x_{q_k}, Tx_{q_k})d(x_{r_k}, Tx_{r_k})}{d(x_{q_k}, x_{r_k})} \right\} \\ &= \max \left\{ d(x_{q_k}, x_{r_k}), d(x_{q_k}, x_{q_k+1}), d(x_{r_k}, x_{r_k+1}), \right. \\ &\quad \left. \frac{d(x_{q_k}, x_{q_k+1})d(x_{r_k}, x_{r_k+1})}{d(x_{q_k}, x_{r_k})} \right\}.\end{aligned}$$

Now, by (8) and (9), we have $\lim_{k \rightarrow \infty} \mathcal{M}_1(x_{q_k}, x_{r_k}) = e$, and it follows by (4) that:

$$\liminf_{s \rightarrow e+} \psi(s) \leq \liminf_{k \rightarrow \infty} \psi(d(x_{q_k+1}, x_{r_k+1})) \leq \limsup_{k \rightarrow \infty} \phi(\mathcal{M}_1(x_{q_k}, x_{r_k})) \leq \limsup_{s \rightarrow e} \phi(s).$$

This contradicts (f_2) , and then, $\{x_n\}$ is a Cauchy sequence on a complete metric space. Thus, the sequence converges to a point $v \in X$, that is:

$$\lim_{n \rightarrow \infty} d(x_n, v) = 0 \quad (10)$$

and since the mapping \mathcal{T} is continuous, we have:

$$v = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Tx_n = \mathcal{T}(\lim_{n \rightarrow \infty} x_n) = \mathcal{T}v$$

which shows that v is a fixed point of \mathcal{T} .

If there exists another fixed point of \mathcal{T} , $\tilde{v} \in X$, such that $\tilde{v} \neq v$, since $d(\mathcal{T}\tilde{v}, \mathcal{T}v) > 0$, from (4), we have:

$$\begin{aligned}\psi(d(\tilde{v}, v)) &= \psi(d(\mathcal{T}\tilde{v}, \mathcal{T}v)) \leq \phi(\mathcal{M}_1(\tilde{v}, v)) \\ &= \phi(\max \left\{ d(\tilde{v}, v), d(\tilde{v}, \mathcal{T}\tilde{v}), d(v, \mathcal{T}v), \frac{d(\tilde{v}, \mathcal{T}\tilde{v})d(v, \mathcal{T}v)}{d(\tilde{v}, v)} \right\}) \\ &= \phi(d(\tilde{v}, v)).\end{aligned}$$

Therefore, from the above inequality together with (f_0) , we get:

$$\psi(d(\tilde{v}, v)) \leq \phi(d(\tilde{v}, v)) < \psi(d(\tilde{v}, v))$$

which is a contradiction. This closes the proof. \square

Example 1. Let the set $X = [0, 2]$ and $d : X \rightarrow X$ be the distance defined as $d(x, y) = |x - y|$ for every $x, y \in X$. Let also $\mathcal{T} : X \rightarrow X$ be a self-mapping with $\mathcal{T}x = \frac{-x^2 + 2x + 4}{8}$ and two functions $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, $\psi(s) = \frac{s}{2}$ and $\phi(s) = \frac{s}{4}$. Since the assumptions (f_1) – (f_3) are satisfied, it remains to check that \mathcal{T} is a (ψ, ϕ) -rational contraction of Type 1. We have:

$$d(\mathcal{T}x, \mathcal{T}y) = \left| \frac{-x^2 + 2x + 4}{8} - \frac{-y^2 + 2y + 4}{8} \right| = \frac{1}{8} |(x - y)(-x - y + 2)| = \frac{1}{8} |(x - y)| |(-x - y + 2)|$$

and since $|(-x - y + 2)| < 4$ for every $x, y \in [0, 1]$, we have:

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) = \frac{1}{16} |(x - y)| |(-x - y + 2)| \leq \frac{1}{4} |x - y| = \frac{1}{4} d(x, y) \leq \frac{1}{4} \mathcal{M}_1(x, y),$$

which shows us that \mathcal{T} is a (ψ, ϕ) -rational contraction of Type 1. Furthermore, by Theorem 4, we get that \mathcal{T} has a unique fixed point in X , that is $x = 0.605551$.

Next, we show that the continuity condition of the operator \mathcal{T} can be replaced by the assumption of the continuity of only some iterations of \mathcal{T} .

Theorem 5. If in Theorem 4 the statement (f_3) is replaced by:

(f'_3) \mathcal{T}^m is continuous for some integer $m > 1$,

then \mathcal{T} has a unique fixed point.

Proof. Let $\{\chi_n\}$ be the sequence defined by (6). By the proof of Theorem 4, we know that this sequence is convergent to some point $v \in X$, which means that $d(\chi_n, v) = 0$. Let $\{c_{n(j)}\}$ be a subsequence of $\{\chi_n\}$, where $n(j) = j \cdot m$ for all $j \in \mathbb{N}_0$ and $m > 1$ fixed. Moreover, assuming that \mathcal{T}^0 is the identity map on x , we have $\chi_{n(j)} = \mathcal{T}^m \chi_{n(j)-m}$. Then, since \mathcal{T}^m is continuous,

$$d(v, \mathcal{T}^m v) = \lim_{j \rightarrow \infty} d(v, \mathcal{T}^m \chi_{n(j)-m}) = \lim_{j \rightarrow \infty} d(v, \chi_{n(j)}) = d(v, v) = 0.$$

This means that v is a fixed point of \mathcal{T}^m .

If we assume that $v \neq \mathcal{T}v$, we have for any $j = 0, 1, \dots, m-1$ that $\mathcal{T}^{m-j-1}v \neq \mathcal{T}^{m-j}v$. By replacing x by $\mathcal{T}^{m-j-1}v$ and y by $\mathcal{T}^{m-j}v$, we have:

$$\begin{aligned} \mathcal{M}_1(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v) &= \max \left\{ d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v), \right. \\ &\quad \left. \frac{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v)d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)}{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j+1}v)} \right\} \\ &= \max \{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)\} \end{aligned} \quad (11)$$

and (4) becomes,

$$\begin{aligned} \psi(d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)) &\leq \phi(\mathcal{M}_1(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v)) \\ &= \phi(\max \{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)\}). \end{aligned} \quad (12)$$

Taking into account (f_0) , it follows that:

$$\begin{aligned} \psi(d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)) &\leq \phi(\max \{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)\}) \\ &< \psi(\max \{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)\}). \end{aligned}$$

Now, since the function ψ is nondecreasing, we get:

$$d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v) < \max \{d(\mathcal{T}^{m-j-1}v, \mathcal{T}^{m-j}v), d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v)\}$$

This leads us to:

$$d(\mathcal{T}^{m-j}v, \mathcal{T}^{m-j+1}v) < d(\mathcal{T}^{m-k-1}v, \mathcal{T}^{m-k}v),$$

for every $k = j, j+1, \dots, m-1$. Taking in the above inequality $j = 0$ and $k = m-1$, we get:

$$d(v, \mathcal{T}v) = d(\mathcal{T}^m v, \mathcal{T}^{m+1}v) < d(v, \mathcal{T}v).$$

This is a contradiction. Consequently, $\mathcal{T}v = v$. \square

Example 2. Let the set $X = [0, 2]$ be endowed with the usual distance $d(x, y) = |x - y|$ for every $x, y \in X$.

Let the mapping $\mathcal{T} : X \rightarrow X$ be defined by $\mathcal{T}x = \begin{cases} 0, & \text{if } x \in [0, 1] \\ \frac{1}{4}, & \text{if } x \in (1, 2]. \end{cases}$ It is clear that the mapping \mathcal{T}

is not continuous and that Theorem 4 cannot be applied. However, we have that $\mathcal{T}^2x = 0$ for any $x \in X$, so the assumption (f'_3) holds. Choosing, for example, the functions $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, where $\psi(s) = e^s$ and $\phi(s) = s + \frac{3}{4}$, we have that the assumptions $(f_0) - (f_2)$ are also satisfied, and we need to check if the inequality (4) holds for all distinct $x, y \in X$ with $d(\mathcal{T}x, \mathcal{T}y) > 0$.

Of course, since $\phi(s) = s + 1$ is an increasing function, for $x \in [0, 1]$ and $y \in (1, 2]$, we have:

$$\begin{aligned}\psi(d(\mathcal{T}x, \mathcal{T}y)) &= \psi\left(\frac{1}{4}\right) = \sqrt[4]{e} < 1 + \frac{1}{2} < y + \frac{1}{2} = \phi\left(\left|y - \frac{1}{4}\right|\right) = \phi(d(y, \mathcal{T}y)) \\ &\leq \phi(\mathcal{M}_1(x, y))\end{aligned}$$

so that all the assumptions of Theorem 5 are satisfied.

Definition 2. Let (X, d) be a complete metric space. The mapping $\mathcal{T} : X \rightarrow X$ is said to be a (ψ, ϕ) -rational contraction of Type 2 if for all $x, y \in X$ with $d(\mathcal{T}x, \mathcal{T}y) > 0$, the following condition is satisfied:

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \phi(\mathcal{M}_2(x, y)), \quad (13)$$

where \mathcal{M}_2 is defined by:

$$\mathcal{M}_2(x, y) = \max \left\{ d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(y, \mathcal{T}y)(d(x, \mathcal{T}x) + 1)}{1 + d(x, y)} \right\}. \quad (14)$$

Theorem 6. Let (X, d) be a complete metric space and $\mathcal{T} : X \rightarrow X$ be a (ψ, ϕ) -rational contraction of Type 2. Assume that:

(f₁') ψ is non-decreasing and lower semi-continuous;

(f₄) $\limsup_{s \rightarrow s_0+} \phi(s) < \psi(s_0+)$;

Then, \mathcal{T} admits exactly one fixed point.

Proof. Let $\{\chi_n\}$ be the sequence defined by (6). Thus, by similar reasoning, we have that $\varsigma_n = d(\chi_{n-1}, \chi_n) > 0$ for every $n \in \mathbb{N}$. Therefore, since $d(\mathcal{T}\chi_{n-1}, \mathcal{T}\chi_n) > 0$, for every $n \in \mathbb{N}$, for $x = \chi_{n-1}$ and $y = \chi_n$, we have:

$$\begin{aligned}\mathcal{M}_2(\chi_{n-1}, \chi_n) &= \max \left\{ d(\chi_{n-1}, \chi_n), d(\chi_{n-1}, \mathcal{T}\chi_{n-1}), d(\chi_n, \mathcal{T}\chi_n), \right. \\ &\quad \left. \frac{d(\chi_n, \mathcal{T}\chi_n)(1 + d(\chi_{n-1}, \mathcal{T}\chi_{n-1}))}{1 + d(\chi_{n-1}, \chi_n)} \right\} \\ &= \max \left\{ d(\chi_{n-1}, \chi_n), d(\chi_n, \chi_{n+1}), \frac{d(\chi_n, \chi_{n+1})(d(\chi_{n-1}, \chi_n) + 1)}{1 + d(\chi_{n-1}, \chi_n)} \right\} \\ &= \max \{d(\chi_{n-1}, \chi_n), d(\chi_n, \chi_{n+1})\} \\ &= \max \{\varsigma_n, \varsigma_{n+1}\}.\end{aligned}$$

Consequently, by (13), we have:

$$\psi(d(\mathcal{T}\chi_{n-1}, \mathcal{T}\chi_n)) \leq \phi(\mathcal{M}_2(\chi_{n-1}, \chi_n)) = \phi(\max \{\varsigma_n, \varsigma_{n+1}\}),$$

which, keeping in mind (f₀), is equivalent to:

$$\psi(\varsigma_{n+1}) \leq \phi(\max \{\varsigma_n, \varsigma_{n+1}\}) < \psi(\max \{\varsigma_n, \varsigma_{n+1}\}). \quad (15)$$

Thus, due to the monotony of the function ψ , $\varsigma_{n+1} < \max \{\varsigma_n, \varsigma_{n+1}\}$, so that $0 < \varsigma_{n+1} < \varsigma_n$, for each $n \in \mathbb{N}$, then there exists $\varsigma \geq 0$ such that $\varsigma_n \searrow \varsigma$. We claim that $\varsigma = 0$. If we assume by contradiction that $\varsigma > 0$, we have:

$$\psi(\varsigma) \leq \psi(\varsigma_{n+1}) \leq \phi(\varsigma_n) < \psi(\varsigma_n).$$

Taking the superior limit in the above inequality and keeping in mind (f_4) , we get:

$$\psi(\zeta+) = \lim_{n \rightarrow \infty} \psi(\zeta_{n+1}) \leq \limsup_{n \rightarrow \infty} \phi(\zeta_n) < \limsup_{n \rightarrow \infty} \psi(\zeta_n) < \psi(\zeta+)$$

which is a contradiction. Thus, we have:

$$\zeta = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (16)$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Again, arguing by contradiction, by Lemma (1), we have that there exist $e > 0$ and the sequences of positive real numbers (q_k) and (r_k) such that:

$$\lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) = e + \text{ and } \lim_{k \rightarrow \infty} d(x_{q_k}, x_{r_k}) = e. \quad (17)$$

Thus, $d(x_{q_k+1}, x_{r_k+1}) = d(Tx_{q_k}, Tx_{r_k}) > e > 0$ for all $k \geq 1$, and from (13), together with (f_0) we have:

$$\psi(d(x_{q_k+1}, x_{r_k+1})) \leq \phi(\mathcal{M}_2(x_{q_k}, x_{r_k})) < \psi(\mathcal{M}_2(x_{q_k}, x_{r_k})). \quad (18)$$

Since ψ is non-decreasing we get $d(x_{q_k+1}, x_{r_k+1}) < \mathcal{M}_2(x_{q_k}, x_{r_k})$, for each $k \geq 1$, where:

$$\mathcal{M}_2(x_{q_k}, x_{r_k}) = \max \left\{ \begin{array}{l} d(x_{q_k}, x_{r_k}), d(x_{q_k}, x_{q_k+1}), d(x_{r_k}, x_{r_k+1}), \\ \frac{d(x_{r_k}, x_{r_k+1})(1+d(x_{q_k}, x_{q_k+1}))}{1+d(x_{q_k}, x_{r_k})} \end{array} \right\} \quad (19)$$

and taking into account (16) and (17):

$$\lim_{k \rightarrow \infty} \mathcal{M}_2(x_{q_k}, x_{r_k}) = e + .$$

In this case, letting $k \rightarrow \infty$ in (18), we have:

$$\psi(e+) = \lim_{k \rightarrow \infty} \psi(d(x_{q_k+1}, x_{r_k+1})) \leq \limsup_{k \rightarrow \infty} \phi(\mathcal{M}_2(x_{q_k}, x_{r_k})) \leq \limsup_{s \rightarrow e+} \phi(s) < \psi(e+),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence. By the completeness of the space (X, d) , the sequence $\{x_n\}$ converges to a point v in X , that is:

$$\lim_{n \rightarrow \infty} d(x_n, v) = 0. \quad (20)$$

We claim that v is a fixed point of T . Supposing by contradiction that $d(Tv, v) > 0$ and using the same arguments as in the previous theorem, we have that there exists $n_0 \in \mathbb{N}$ such that $d(Tv, x_{n+1}) = d(Tv, Tx_n) > 0$ for any $n \geq n_0$. Now, by (13) we have:

$$\psi(d(Tv, Tx_n)) \leq \phi(\mathcal{M}_2(v, x_n)), \quad (21)$$

where:

$$\mathcal{M}_2(v, x_n) = \max \left\{ d(v, x_n), d(v, Tv), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})(1+d(v, Tv))}{1+d(v, x_n)} \right\}.$$

On the one hand, from (16) and (20), we get:

$$\mathcal{M}_2(v, x_n) = d(v, Tv), \quad \text{for } n \text{ sufficiently large} \quad (22)$$

and then:

$$\psi(d(Tv, Tx_n)) \leq \phi(d(v, Tv)) < \psi(d(v, Tv)).$$

On the other hand, $\lim_{n \rightarrow \infty} d(\mathcal{T}v, \mathcal{T}\mathcal{X}_n) = \lim_{n \rightarrow \infty} d(\mathcal{T}v, \mathcal{X}_{n+1}) = d(\mathcal{T}v, v)$. Therefore, taking the inferior limit in (21) when $n \rightarrow \infty$ and taking into account the lower semi-continuity of ψ , we have:

$$\liminf_{s \rightarrow d(\mathcal{T}v, v)} \psi(s) \leq \lim_{n \rightarrow \infty} \psi(\mathcal{T}v, \mathcal{T}\mathcal{X}_n) \leq \phi(d(v, \mathcal{T}v)) < \psi(d(v, \mathcal{T}v)) < \liminf_{s \rightarrow d(\mathcal{T}v, v)} \psi(s),$$

which is a contradiction. Therefore, we have $\mathcal{T}v = v$, and we claim that this is the unique fixed point of \mathcal{T} . If we suppose that \tilde{v} is also a fixed point of \mathcal{T} such that $d(\mathcal{T}v, \mathcal{T}\tilde{v}) = d(v, \tilde{v}) > 0$ and from (13), we have:

$$\psi(d(\mathcal{T}v, \mathcal{T}\tilde{v})) \leq \phi(\mathcal{M}_2(v, \tilde{v})), \quad (23)$$

with:

$$\begin{aligned} \mathcal{M}_2(v, \tilde{v}) &= \max \left\{ d(v, \tilde{v}), d(v, \mathcal{T}v), d(\tilde{v}, \mathcal{T}\tilde{v}), \frac{d(\tilde{v}, \mathcal{T}\tilde{v})(1+d(v, \mathcal{T}v))}{1+d(v, \tilde{v})} \right\} \\ &= d(v, \tilde{v}). \end{aligned}$$

Thus, by (23),

$$\psi(d(v, \tilde{v})) = \psi(d(\mathcal{T}v, \mathcal{T}\tilde{v})) \leq \phi(\mathcal{M}_2(v, \tilde{v})) = \phi(d(v, \tilde{v})) < \psi(d(v, \tilde{v})),$$

which is a contradiction. \square

Example 3. Let $X = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $d : X \times X \rightarrow [0, \infty)$ be a distance defined as follows:

$$\begin{aligned} d(x, y) &= d(y, x), \text{ for every } x, y \in X; \\ d(\omega_1, \omega_2) &= 2, \quad d(\omega_1, \omega_3) = 6, \quad d(\omega_1, \omega_4) = 7; \\ d(\omega_2, \omega_3) &= 4, \quad d(\omega_2, \omega_4) = 5, \quad d(\omega_3, \omega_4) = 1. \end{aligned}$$

Let the mapping $\mathcal{T} : X \rightarrow X$, with $\mathcal{T}\omega_1 = \omega_4$, $\mathcal{T}\omega_2 = \mathcal{T}\omega_3 = \mathcal{T}\omega_4 = \omega_3$. Letting $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, where $\psi(s) = e^s$ and $\phi(s) = 1 + \ln(1 + s)$, we have that the assumptions (f_0) , (f'_1) , (f_4) are satisfied. We have to consider the following cases:

- a. If $x = \omega_1$, $y = \omega_2$, then $d(\mathcal{T}\omega_1, \mathcal{T}\omega_2) = d(\omega_4, \omega_3) = 1$, $d(\omega_1, \omega_2) = 2$, $d(\omega_1, \mathcal{T}\omega_1) = 7$, $d(\omega_2, \mathcal{T}\omega_2) = 4$, and $\mathcal{M}_2(\omega_1, \omega_2) = \max \left\{ 2, 7, 4, \frac{32}{3} \right\} = \frac{32}{3}$:

$$\psi(d(\mathcal{T}\omega_1, \mathcal{T}\omega_2)) = \psi(1) = e < 1 + \ln \frac{32}{3} < \phi(\mathcal{M}_2(\omega_1, \omega_2)).$$

- b. If $x = \omega_1$, $y = \omega_3$, then $d(\mathcal{T}\omega_1, \mathcal{T}\omega_3) = d(\omega_4, \omega_3) = 1$, $d(\omega_1, \omega_3) = 6$, $d(\omega_1, \mathcal{T}\omega_1) = 7$, $d(\omega_3, \mathcal{T}\omega_3) = 0$, and $\mathcal{M}_2(\omega_1, \omega_3) = \max \left\{ 6, 7, 0, \frac{7}{3} \right\} = 7$:

$$\psi(d(\mathcal{T}\omega_1, \mathcal{T}\omega_3)) = \psi(1) = e < 1 + \ln 7 < \phi(\mathcal{M}_2(\omega_1, \omega_3)).$$

- c. If $x = \omega_1$, $y = \omega_4$, then $d(\mathcal{T}\omega_1, \mathcal{T}\omega_4) = d(\omega_4, \omega_3) = 1$, $d(\omega_1, \omega_4) = 7$, $d(\omega_1, \mathcal{T}\omega_1) = 7$, $d(\omega_4, \mathcal{T}\omega_4) = 1$, and $\mathcal{M}_2(\omega_1, \omega_4) = \max \left\{ 7, 7, 1, \frac{14}{3} \right\} = 7$:

$$\psi(d(\mathcal{T}\omega_1, \mathcal{T}\omega_4)) = \psi(1) = e < 1 + \ln 7 < \phi(\mathcal{M}_2(\omega_1, \omega_4)).$$

Thus, all the assumptions of Theorem 6 hold, so that \mathcal{T} has a unique fixed point.

Theorem 7. A (ψ, ϕ) -rational contraction of Type 2 on the complete metric space (X, d) has a unique fixed point presuming that the following conditions are satisfied:

- (f_1) $\lim_{s \rightarrow s_0} \psi(s) > -\infty$, for any $s_0 > 0$;
 (f'_2) $\limsup_{s \rightarrow s_0+} \phi(s) < \liminf_{s \rightarrow s_0} \psi(s)$;

$$(f_5) \quad \phi(s_0) < \liminf_{s \rightarrow s_0} \psi(s) \text{ for any } s_0 > 0.$$

Proof. Following the lines and using the same notations as in the proof of Theorem 6, by (15), we have that $\psi(\zeta_n) < \psi(\max\{\zeta_n, \zeta_{n+1}\})$. Since for $\max\{\zeta_n, \zeta_{n+1}\} = \zeta_{n+1}$, we get a contradiction. Therefore, we conclude that $\zeta_n > \zeta_{n+1}$. Consequently, on the one hand, we have that there exists a point $\varsigma \geq 0$ such that $\zeta_n \searrow \varsigma$. We claim that $\varsigma = 0$. On the contrary, if we suppose that $\varsigma > 0$, by (13) together with f_0 , we have:

$$\psi(\zeta_{n+1}) \leq \phi(\zeta_n) < \psi(\zeta_n),$$

for all $n \in \mathbb{N}$. Then, the sequence $\{\psi(\zeta_n)\}$ is decreasing and also bounded (because (f_1) and $\zeta_n > \varsigma$). Therefore, the sequence $\{\psi(\zeta_n)\}$ is convergent, and moreover, by the above inequality, the sequence $\{\phi(\zeta_n)\}$ is also convergent to the same limit. Thus, keeping in mind (f_2) , we have:

$$\liminf_{s \rightarrow \varsigma+} \psi(s) \leq \lim_{n \rightarrow \infty} \psi(\zeta_n) = \lim_{n \rightarrow \infty} \phi(\zeta_n) < \limsup_{s \rightarrow \varsigma+} \phi(s) < \liminf_{s \rightarrow \varsigma+} \psi(s),$$

which is a contradiction, so that,

$$\varsigma = \lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0. \quad (24)$$

We will show that $\{x_n\}$ is a Cauchy sequence. In order to prove that, arguing by contradiction, by Lemma 1, there exist $\epsilon > 0$ and $(q_k), (r_k)$ two sequences of positive integers such that (3) holds. Since $\lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) = \epsilon+$, we have that $d(x_{q_k+1}, x_{r_k+1}) = d(\mathcal{T}x_{q_k}, \mathcal{T}x_{r_k}) > 0$, and replacing in (13), we get:

$$\psi(d(x_{q_k+1}, x_{r_k+1})) = \psi(d(\mathcal{T}x_{q_k}, \mathcal{T}x_{r_k})) \leq \phi(\mathcal{M}_2(x_{q_k}, x_{r_k})).$$

On the other hand, from the above inequality and (f_2) , we have:

$$\liminf_{s \rightarrow \epsilon} \psi(s) \leq \liminf_{k \rightarrow \infty} \psi(d(x_{q_k+1}, x_{r_k+1})) \leq \limsup_{k \rightarrow \infty} \phi(\mathcal{M}_2) \leq \limsup_{s \rightarrow \epsilon+} \phi(s) < \liminf_{s \rightarrow \epsilon} \psi(s).$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence, so it is convergent to some point $v \in X$ (due to the completeness of the metric space (X, d)). If we suppose that $d(\mathcal{T}v, v) > 0$, because $d(\mathcal{T}v, \mathcal{T}x_n) \rightarrow d(\mathcal{T}v, v)$, we have that there exists $n_0 \in \mathbb{N}$ such that $d(\mathcal{T}v, \mathcal{T}x_n) > 0$, for $n \geq n_0$. Then, from (13),

$$\begin{aligned} \psi(d(\mathcal{T}v, \mathcal{T}x_n)) &\leq \phi(\mathcal{M}_2(v, x_n)) \\ &= \phi\left(\max\left\{d(v, x_n), d(v, \mathcal{T}v), d(x_n, \mathcal{T}x_n), \frac{d(x_n, \mathcal{T}x_{n+1})(1+d(v, \mathcal{T}v))}{1+d(v, x_n)}\right\}\right) \end{aligned}$$

and moreover, taking into account (22):

$$\psi(d(\mathcal{T}v, \mathcal{T}x_n)) \leq \phi(d(v, \mathcal{T}v)).$$

Taking the limit as inferior and using (f_5) , we obtain:

$$\liminf_{s \rightarrow d(\mathcal{T}v, v)} \psi(s) \leq \liminf_{n \rightarrow \infty} \psi(d(\mathcal{T}v, x_{n+1})) \leq \phi(d(v, \mathcal{T}v)) < \liminf_{s \rightarrow d(\mathcal{T}v, v)} \psi(s).$$

This is a contradiction. Therefore, $\mathcal{T}v = v$, that is v is a fixed point of \mathcal{T} , and using the same arguments as in Theorem 6, we have that, in fact, this fixed point is unique. \square

Example 4. Let $X = [0, \infty)$ and d be the usual distance on X . Let $\mathcal{T} : X \rightarrow X$, where $\mathcal{T}x = \frac{1}{2} \ln(x^2 + x + 2)$ and $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, where $\psi(s) = e^s$ and $\phi(s) = 1 + s$. We check that \mathcal{T} is a (ψ, ϕ) -rational contraction of Type 2. Indeed, if $x > y$ (and it is analogues for the case $x < y$), then:

$$d(\mathcal{T}x, \mathcal{T}y) = \left| \frac{\ln(x^2 + x + 2) - \ln(y^2 + y + 2)}{2} \right| = \frac{1}{2} \ln \frac{x^2 + x + 2}{y^2 + y + 2} = \ln \sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}}.$$

On the other hand, since:

$$\sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}} \leq 1 + x - y \Leftrightarrow \frac{x^2 + x + 2}{y^2 + y + 2} \leq (1 + x - y)^2 \Leftrightarrow (1 + y^2)(x - y) + y^2 + xy + 3 \geq 0,$$

we obtain:

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) = \sqrt{\frac{x^2 + x + 2}{y^2 + y + 2}} \leq 1 + x - y = 1 + d(x, y) \leq 1 + \mathcal{M}_2(x, y) = \phi(\mathcal{M}_2(x, y)).$$

Thus, (13) is satisfied, and by Theorem 7, we have that the mapping \mathcal{T} has a fixed point.

Definition 3. Let (X, d) be a complete metric space. The mapping $\mathcal{T} : X \rightarrow X$ is said to be a (ψ, ϕ) -rational contraction of Type 3 if for all $x, y \in X$, when $\max\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \neq 0$, then $d(\mathcal{T}x, \mathcal{T}y) > 0$, and the following condition is satisfied:

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \phi \left(\frac{d(x, \mathcal{T}x)d(x, \mathcal{T}y) + d(y, \mathcal{T}y)d(y, \mathcal{T}x)}{\max\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}} \right); \quad (25)$$

if $\max\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} = 0$, then $d(\mathcal{T}x, \mathcal{T}y) = 0$.

Theorem 8. Let (X, d) be a complete metric space and $\mathcal{T} : X \rightarrow X$ be a (ψ, ϕ) -rational contraction of Type 3. The mapping \mathcal{T} admits exactly one fixed point provided that:

(f_1'') ψ is non-decreasing and $\limsup_{s \rightarrow s_0+} \phi(s) < \psi(s_0+)$, for any $s_0 > 0$.

Proof. Let $\{x_n\}$ be the sequence defined by (6). Thus, by similar reasoning, we have that $x_n = d(x_{n-1}, x_n) > 0$ for every $n \in \mathbb{N}$. Therefore, since $d(\mathcal{T}x_{n-1}, \mathcal{T}x_n) > 0$, for every $n \in \mathbb{N}$, for $x = x_{n-1}$ and $y = x_n$, by (25), we have:

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)) \\ &\leq \phi \left(\frac{d(x_{n-1}, \mathcal{T}x_{n-1})d(x_{n-1}, \mathcal{T}x_n) + d(x_n, \mathcal{T}x_n)d(x_n, \mathcal{T}x_{n-1})}{\max\{d(x_{n-1}, \mathcal{T}x_n), d(x_n, \mathcal{T}x_{n-1})\}} \right) \\ &= \phi \left(\frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_{n-1})}{\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_{n-1})\}} \right) \\ &= \phi(d(x_{n-1}, x_n)), \end{aligned}$$

which, keeping in mind (f_0) , gives us:

$$\psi(d(x_n, x_{n+1})) \leq \phi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)). \quad (26)$$

Thus, from (f_1'') , $0 < d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$, so the sequence $(d(x_n, x_{n+1}))$ is convergent to some $\varsigma \geq 0$. We claim that $\varsigma = 0$. In the case that $\varsigma > 0$, from (25),

$$\psi(d(x_n, x_{n+1})) \leq \phi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n))$$

Taking the limit as superior in the above inequality and keeping in mind (f_1'') , we get:

$$\psi(\zeta+) = \lim_{n \rightarrow \infty} \psi(d(x_n, x_{n+1})) \leq \limsup_{n \rightarrow \infty} \phi(d(x_{n-1}, x_n)) < \limsup_{n \rightarrow \infty} \psi(d(x_{n-1}, x_n)) < \psi(\zeta+).$$

This is a contradiction, and then, we have:

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \zeta = 0. \quad (27)$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Again, arguing by contradiction, by Lemma (1), we have that there exist $e > 0$ and the sequences of positive real numbers (q_k) and (r_k) such that:

$$\lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) = e + \text{ and } \lim_{k \rightarrow \infty} d(x_{q_k}, x_{r_k}) = e. \quad (28)$$

Thus, it follows that $d(x_{q_k+1}, x_{r_k+1}) = d(\mathcal{T}x_{q_k}, \mathcal{T}x_{r_k}) > e > 0$ for all $k \geq 1$, and from (25), together with (f_0) , we have:

$$\begin{aligned} \psi(d(x_{q_k+1}, x_{r_k+1})) &\leq \phi\left(\frac{d(x_{q_k}, x_{q_k+1})d(x_{q_k}, x_{r_k+1}) + d(x_{r_k}, x_{r_k+1})d(x_{r_k}, x_{q_k+1})}{\max\{d(x_{q_k}, x_{r_k+1}), d(x_{r_k}, x_{q_k+1})\}}\right) \\ &< \phi\left(\frac{d(x_{q_k}, x_{q_k+1})d(x_{q_k}, x_{r_k+1}) + d(x_{r_k}, x_{r_k+1})d(x_{r_k}, x_{q_k+1})}{\max\{d(x_{q_k}, x_{r_k+1}), d(x_{r_k}, x_{q_k+1})\}}\right) \\ &= \phi\left(\frac{d(x_{q_k}, x_{q_k+1})d(x_{q_k}, x_{r_k+1})}{\max\{d(x_{q_k}, x_{r_k+1}), d(x_{r_k}, x_{q_k+1})\}} + \frac{d(x_{r_k}, x_{r_k+1})d(x_{r_k}, x_{q_k+1})}{\max\{d(x_{q_k}, x_{r_k+1}), d(x_{r_k}, x_{q_k+1})\}}\right) \\ &\leq \phi\left(\frac{d(x_{q_k}, x_{q_k+1})d(x_{q_k}, x_{r_k+1})}{d(x_{q_k}, x_{r_k+1})} + \frac{d(x_{r_k}, x_{r_k+1})d(x_{r_k}, x_{q_k+1})}{d(x_{r_k}, x_{q_k+1})}\right) \\ &= \phi(d(x_{q_k}, x_{q_k+1}) + d(x_{r_k}, x_{r_k+1})) \\ &< \psi(d(x_{q_k}, x_{q_k+1}) + d(x_{r_k}, x_{r_k+1})). \end{aligned} \quad (29)$$

Since ψ is non-decreasing, we get:

$$d(x_{q_k+1}, x_{r_k+1}) < d(x_{q_k}, x_{q_k+1}) + d(x_{r_k}, x_{r_k+1}),$$

for each $k \geq 1$.

Taking into account (27) and (28):

$$0 < e = \lim_{k \rightarrow \infty} d(x_{q_k+1}, x_{r_k+1}) < \lim_{k \rightarrow \infty} (d(x_{q_k}, x_{q_k+1}) + d(x_{r_k}, x_{r_k+1})) = 0.$$

In this case, we get $e = 0$, which shows us that $\{x_n\}$ is a Cauchy sequence, and by the completeness of the space (X, d) , (x_n) converges to a point v in X , that is:

$$\lim_{n \rightarrow \infty} d(x_n, v) = 0. \quad (30)$$

We claim that v is a fixed point of \mathcal{T} . Supposing by contradiction that $d(\mathcal{T}v, v) > 0$ and using the same arguments as in the previous theorem, we have that there exists $n_0 \in \mathbb{N}$ such that $d(\mathcal{T}v, x_{n+1}) = d(\mathcal{T}v, \mathcal{T}x_n) > 0$ for any $n \geq n_0$. Now, by (25), we have:

$$\begin{aligned} \psi(d(\mathcal{T}v, \mathcal{T}x_n)) &\leq \phi\left(\frac{d(v, \mathcal{T}v)d(v, x_{n+1}) + d(x_n, x_{n+1})d(x_n, \mathcal{T}v)}{\max\{d(v, \mathcal{T}x_n), d(x_n, \mathcal{T}v)\}}\right) \\ &< \psi\left(\frac{d(v, \mathcal{T}v)d(v, x_{n+1}) + d(x_n, x_{n+1})d(x_n, \mathcal{T}v)}{\max\{d(v, \mathcal{T}x_n), d(x_n, \mathcal{T}v)\}}\right) \end{aligned} \quad (31)$$

Now, from (f_1'') , we have:

$$0 < d(Tv, T\chi_n) < \frac{d(v, Tv)d(v, \chi_{n+1}) + d(\chi_n, \chi_{n+1})d(\chi_n, Tv)}{\max\{d(v, T\chi_n), d(\chi_n, Tv)\}}$$

and letting $n \rightarrow \infty$, we get $0 < \lim_{n \rightarrow \infty} d(Tv, T\chi_n) < 0$, which is a contradiction. Therefore, we have $Tv = v$. Finally, we claim that this is the unique fixed point of T . If we suppose that \tilde{v} is also a fixed point of T such that $d(Tv, T\tilde{v}) = d(v, \tilde{v}) > 0$ and from (25): we have:

$$\begin{aligned} \psi(d(Tv, T\tilde{v})) &\leq \phi\left(\frac{d(v, Tv)d(v, T\tilde{v}) + d(\tilde{v}, T\tilde{v})d(\tilde{v}, Tv)}{\max\{d(v, T\tilde{v}), d(\tilde{v}, Tv)\}}\right) \\ &< \psi\left(\frac{d(v, Tv)d(v, T\tilde{v}) + d(\tilde{v}, T\tilde{v})d(\tilde{v}, Tv)}{\max\{d(v, T\tilde{v}), d(\tilde{v}, Tv)\}}\right), \end{aligned} \quad (32)$$

Thus, by (f_1'') ,

$$0 < d(v, \tilde{v}) < \frac{d(v, Tv)d(v, T\tilde{v}) + d(\tilde{v}, T\tilde{v})d(\tilde{v}, Tv)}{\max\{d(v, T\tilde{v}), d(\tilde{v}, Tv)\}} = 0,$$

which is a contradiction. \square

We can state many corollaries from our main results. For example, choosing $\psi(s) = s$ and $\phi(s) = \beta(s)s$ in Theorem 4, we have:

Corollary 1. Let (X, d) be a complete metric space and $\beta : (0, \infty) \rightarrow (0, 1)$ be a function such that $\limsup_{s \rightarrow s_0+} \beta(s) < 1$ for every $s_0 > 0$. A continuous mapping $T : X \rightarrow X$ has a unique fixed point provided that:

$$d(Tx, Ty) \leq \beta(\mathcal{M}_1(x, y))\mathcal{M}_1(x, y), \text{ for all } x, y \in X \text{ with } d(Tx, Ty) > 0.$$

If in Theorem 7, we take $\phi(s) = \kappa\psi(s)$, we get the following corollary.

Corollary 2. Let (X, d) be a complete metric space and a self-mapping T on X such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\psi(d(Tx, Ty)) \leq \kappa\psi(\mathcal{M}_2(x, y)),$$

where $\kappa \in [0, 1)$, $\psi : (0, \infty) \rightarrow (0, \infty)$ is a nondecreasing and left-continuous function, and \mathcal{M}_2 is defined by (14). Then, T admits a unique fixed point.

Letting $\phi(s) = \psi(s) - \tau$ in Theorem 8, we obtain an improvement of Theorem 3.1 in [12].

Corollary 3. Let (X, d) be a complete metric space and a mapping $T : X \rightarrow X$ such that there exist $\tau > 0$ and a nondecreasing and left-continuous function $\psi : (0, \infty) \rightarrow \mathbb{R}$ such that for all $x, y \in X$ if $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, then $d(x, y) > 0$:

$$\tau + \psi(d(Tx, Ty)) \leq \psi\left(\frac{d(x, Tx)d(x, Ty) + d(y, Ty)(d(y, Tx))}{\max\{d(x, Ty), d(y, Tx)\}}\right) \quad (33)$$

and if $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, then $d(x, y) = 0$. Then, T has a unique fixed point.

3. Conclusions

In this paper, we were interested in finding some conditions on the functions ψ and ϕ that guarantee that T has a unique fixed point in terms of rational expression. Our main results offered improvements to known results by applying weaker conditions on the self-map of a complete metric space. Here we mentioned just one corollary for each type of (ψ, ϕ) -rational contraction by choosing

different functions ψ and ϕ , but it is clear that many similar consequences can be listed, consequences that actually represent independent results.

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