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Oscillation Criteria for First Order Differential Equations with Non-Monotone Delays

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Abstract: New sufficient criteria are obtained for the oscillation of a non-autonomous first order differential equation with non-monotone delays. Both recursive and *lower-upper* limit types criteria are given. The obtained results improve most recent published results. An example is given to illustrate the applicability and strength of our results.

Keywords: Oscillation; Differential equations; Non-monotone delays

1. Introduction

Consider the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1)$$

where $p, \tau \in C([t_0, \infty), [0, \infty))$ and $\tau(t) < t$ for $t \geq t_0$, such that $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

A solution of Equation (1) is a function $x(t)$ on $[\bar{t}, \infty)$, where $\bar{t} = \min_{t \geq t_0} \tau(t)$, which is continuously differentiable on $[t_0, \infty)$ and satisfies Equation (1) for all $t \geq t_0$. As customary, a solution of Equation (1) is called oscillatory if it has arbitrarily large zeros. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The oscillation of Equation (1) has been extensively studied for many decades; see [1–17]. As far as these authors know, the earliest systematic study of the oscillation of Equation (1) was due to Myshkis [14], who proved that Equation (1) is oscillatory when

$$\limsup_{t \rightarrow \infty} (t - \tau(t)) < \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} (t - \tau(t)) \liminf_{t \rightarrow \infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas et al. [13] proved that Equation (1) is oscillatory if

$$L := \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds > 1, \quad (2)$$

where the delay $\tau(t)$ is assumed to be a nondecreasing function.

In 1979, Ladas [12] (for Equation (1) with constant delay) and in 1982, Koplatadze and Chanturija [10] established the celebrated oscillation criterion

$$k := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds > \frac{1}{e}. \quad (3)$$

The oscillation of Equation (1) has been studied when $0 < k \leq \frac{1}{e}$, $L \leq 1$ and $\tau(t)$ is nondecreasing, see [8,9,15,16] and the references cited therein. In most of these works, the oscillation criteria have been formulated as relations between L and k . For example, Jaroš and Stavroulakis [8], Kon et al. [9], Philos and Sficas [15], and Sficas and Stavroulakis [16] obtained the following criteria, respectively:

$$L > \frac{\ln(\lambda(k)) + 1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2},$$

$$L > 2k + \frac{2}{\lambda(k)} - 1,$$

$$L > 1 - \frac{k^2}{2(1 - k)} - \frac{k^2}{2}\lambda(k),$$

and

$$L > \frac{\ln \lambda(k) - 1 + \sqrt{5 - 2\lambda(k) + 2k\lambda(k)}}{\lambda(k)}, \quad (4)$$

where $\lambda(k)$ is the smaller real root of the equation $\lambda = e^{\lambda k}$.

The same problem has been considered for Equation (1) with non-monotone delays, see [2,4,11,17–19]. The latter case is much more complicated than the monotone delays case. In fact, according to Braverman and Karpuz ([2], Theorem 1), condition (2) does not need to be sufficient for the oscillation of Equation (1) if $\tau(t)$ is non-monotone. To overcome this difficulty, many authors used a nondecreasing function $\delta(t)$ defined by:

$$\delta(t) = \max_{s \leq t} \tau(s), \quad t \geq t_0; \quad (5)$$

hence, many results were obtained by using techniques similar to those of the monotonic delays case. Most of these results were given by recursive formulas. Next, we give an overview of such results:

In 1994, Koplatadze and Kvinikadze [11] proved the following interesting result which requires the definition of the sequence of functions $\{\psi_i\}_{i=1}^{\infty}$ as follows:

$$\psi_1(t) = 0, \quad \psi_i(t) = e^{\int_{\tau(t)}^t p(s) \psi_{i-1}(s) ds}, \quad i = 2, 3, \dots \quad (6)$$

Theorem 1 ([11]). *Let $j \in \{1, 2, \dots\}$ exist such that*

$$\limsup_{t \rightarrow \infty} \int_{\delta(t)}^t p(s) e^{\int_{\delta(s)}^{\delta(t)} p(u) \psi_j(u) du} ds > 1 - c(k), \quad (7)$$

where k , δ , and ψ_j , are defined respectively by (3), (5), and (6) and

$$c(k) = \begin{cases} 0, & \text{if } k > \frac{1}{e}, \\ \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, & \text{if } 0 \leq k \leq \frac{1}{e}. \end{cases}$$

Then, Equation (1) is oscillatory.

In 2011, Braverman and Karpuz [2] obtained the following sufficient condition for the oscillation of Equation (1),

$$\limsup_{t \rightarrow \infty} \int_{\delta(t)}^t p(s) e^{\int_{\tau(s)}^{\delta(t)} p(u) du} ds > 1. \quad (8)$$

In 2014, Stavroulakis [17] improved condition (8) to

$$\limsup_{t \rightarrow \infty} \int_{\delta(t)}^t p(s) e^{\int_{\tau(s)}^{\delta(t)} p(u) du} ds > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}. \quad (9)$$

In 2015, Infante et al. [19] proved that Equation (1) is oscillatory if one of the following conditions is satisfied:

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) dv} du} ds > 1, \quad (10)$$

or

$$\limsup_{\epsilon \rightarrow 0^+} \left(\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{(\lambda(k) - \epsilon) \int_{\tau(s)}^{g(t)} p(u) du} ds \right) > 1, \quad (11)$$

where $g(t)$ is a nondecreasing function satisfying that $\tau(t) \leq g(t) \leq t$ for all $t \geq t_1$ and some $t_1 \geq t_0$.

In 2016, El-Morshedy and Attia [4] proved that Equation (1) is oscillatory if there exists a positive integer n such that

$$\limsup_{t \rightarrow \infty} \left(\int_{g(t)}^t q_n(s) ds + c(k^*) e^{\int_{g(t)}^t \sum_{i=0}^{n-1} q_i(s) ds} \right) > 1, \quad (12)$$

where $k^* := \liminf_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds$, c, g are defined as before, and $\{q_n(t)\}$ is given by

$$q_0(t) = p(t), \quad q_1(t) = q_0(t) \int_{\tau(t)}^t q_0(s) e^{\int_{\tau(s)}^t q_0(u) du} ds, \\ q_n(t) = q_{n-1}(t) \int_{g(t)}^t q_{n-1}(s) e^{\int_{g(s)}^t q_{n-1}(u) du} ds, \quad n = 2, 3, \dots$$

Very recently, Bereketoglu et al. [18] proved that Equation (1) oscillates if for some $\ell \in \mathbb{N}$ the following criterion holds

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} P_\ell(u) du} ds > 1 - c(k^*), \quad (13)$$

where

$$P_\ell(t) = p(t) \left[1 + \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^t P_{\ell-1}(u) du} ds \right], \quad P_0(t) = p(t).$$

In this work, we obtain new sufficient criteria of recursive type for the oscillation of Equation (1), when the delay is non-monotone and $k^* \leq \frac{1}{e} < \tilde{L} < 1$, where $\tilde{L} := \limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds$. In addition, new practical lower limit-upper limit type criteria similar to those in [8,9,15,16] are obtained. These new conditions improve some results in [2,5,8,9,11,13,16–19]. An illustrative example is given to show the strength and applicability of our results.

2. Main Results

Throughout this work, we assume that c, g, k^*, λ, t_1 are defined as above and $g^i(t)$ stands for the i th composition of g .

For fixed $n \in \mathbb{N}$, we define $\{R_{m,n}(t)\}$, $\{Q_{m,n}(t)\}$, eventually, as follows:

$$R_{m,n}(t) = 1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) Q_{m-1,n}(u) du} ds, \quad m = 1, 2, \dots,$$

$$Q_{i,j}(t) = e^{\int_{\tau(t)}^t p(s) Q_{i,j-1}(s) ds}, \quad i = 1, 2, \dots, m-1, j = 1, 2, \dots, n$$

where

$$Q_{0,0}(t) = (\lambda(k^*) - \epsilon) \left(1 + (\lambda(k^*) - \epsilon) \int_{\tau(t)}^{g(t)} p(s) ds \right),$$

$$Q_{0,r}(t) = e^{\int_{\tau(t)}^t p(s) Q_{0,r-1}(s) ds}, \quad r = 1, 2, \dots, n$$

$$Q_{i,0}(t) = R_{i,n}, \quad i = 1, 2, \dots, m-1$$

and $\epsilon \in (0, \lambda(k^*))$.

Lemma 1. Assume that $x(t)$ is an eventually positive solution of Equation (1). Then,

$$\frac{x(\tau(t))}{x(t)} \geq R_{m,n}(t),$$

for all sufficiently large t .

Proof. Since $x(t)$ is an eventually positive solution of Equation (1), there exists a sufficiently large $T > t_1$ such that $x(t)$ satisfies eventually

$$x'(t) + p(t)x(g(t)) \leq 0, \quad t > T.$$

Using ([5], Lemma 2.1.2), for sufficiently small $\epsilon > 0$ and sufficiently large t , we have

$$\frac{x(\tau(t))}{x(t)} \geq \frac{x(g(t))}{x(t)} > \lambda(k^*) - \epsilon. \quad (14)$$

On the other hand, dividing both sides of Equation (1) by $x(t)$ and integrating the resulting equation from s to t , $s \leq t$, we obtain

$$x(s) = x(t) e^{\int_s^t p(u) \frac{x(\tau(u))}{x(u)} du}. \quad (15)$$

Therefore,

$$\begin{aligned} x(\tau(t)) &= x(t) e^{\int_{\tau(t)}^t p(u) \frac{x(\tau(u))}{x(g(u))} \frac{x(g(u))}{x(u)} du} \\ &\geq x(t) e^{(\lambda(k^*) - \epsilon) \int_{\tau(t)}^t p(u) \frac{x(\tau(u))}{x(g(u))} du}. \end{aligned} \quad (16)$$

Integrating Equation (1) from $\tau(\xi)$ to $g(\xi)$,

$$x(g(\xi)) - x(\tau(\xi)) + \int_{\tau(\xi)}^{g(\xi)} p(r)x(\tau(r))dr = 0.$$

Using (14) as well as the nonincreasing nature of $x(t)$, it follows that

$$x(g(\xi)) - x(\tau(\xi)) + (\lambda(k^*) - \epsilon) x(g(\xi)) \int_{\tau(\xi)}^{g(\xi)} p(r)dr \leq 0.$$

Thus,

$$\frac{x(\tau(\xi))}{x(g(\xi))} \geq 1 + (\lambda(k^*) - \epsilon) \int_{\tau(\xi)}^{g(\xi)} p(r) dr.$$

This together with (16) gives

$$\begin{aligned} \frac{x(\tau(t))}{x(t)} &\geq e^{(\lambda(k^*) - \epsilon) \int_{\tau(t)}^t p(u) \left(1 + (\lambda(k^*) - \epsilon) \int_{\tau(u)}^{g(u)} p(r) dr \right) du} \\ &= e^{\int_{\tau(t)}^t p(u) Q_{0,0}(u) du} = Q_{0,1}(t). \end{aligned} \quad (17)$$

Since (15) implies that $\frac{x(\tau(t))}{x(t)} = e^{\int_{\tau(t)}^t p(s) \frac{x(\tau(s))}{x(s)} ds}$, (17) yields

$$\frac{x(\tau(t))}{x(t)} \geq e^{\int_{\tau(t)}^t p(s) Q_{0,1}(s) ds} = Q_{0,2}(t).$$

Repeating this process, we arrive at the following inequality

$$\frac{x(\tau(t))}{x(t)} \geq Q_{0,n}(t). \quad (18)$$

On the other hand, by integrating Equation (1) from $\tau(t)$ to t , we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s) x(\tau(s)) ds = 0. \quad (19)$$

Using (15), we obtain $x(\tau(s)) = x(t) e^{\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} du}$. Therefore, (19) implies that

$$\frac{x(\tau(t))}{x(t)} = 1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} du} ds = 0. \quad (20)$$

Now, substituting (18) into (20), we have

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) Q_{0,n}(u) du} ds = R_{1,n}(t).$$

From the last inequality and (15), we obtain

$$\frac{x(\tau(t))}{x(t)} \geq e^{\int_{\tau(t)}^t p(s) R_{1,n}(s) ds} = e^{\int_{\tau(t)}^t p(s) Q_{1,0}(s) ds} = Q_{1,1}(t).$$

It follows from this and (15) that

$$\frac{x(\tau(t))}{x(t)} \geq e^{\int_{\tau(t)}^t p(s) Q_{1,1}(s) ds} = Q_{1,2}(t).$$

A simple induction implies that

$$\frac{x(\tau(t))}{x(t)} \geq e^{\int_{\tau(t)}^t p(s) Q_{1,n-1}(s) ds} = Q_{1,n}(t).$$

Substituting the previous inequality into (20), we get

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) Q_{1,n}(u) du} ds = R_{2,n}(t).$$

Therefore, by using the same arguments, as before, we obtain

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) Q_{m-1,n}(u) du} ds = R_{m,n}(t).$$

□

Theorem 2. Assume that $k^* \leq \frac{1}{e}$ and $m, n \in \mathbb{N}$ such that

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du} ds > 1 - c(k^*). \quad (21)$$

Then, every solution of Equation (1) is oscillatory.

Proof. Assume the contrary, i.e., there exists a non-oscillatory solution $x(t)$. Due to the linearity of Equation (1), one can assume that $x(t)$ is eventually positive. Now, integrating Equation (1) from $g(t)$ to t , we obtain

$$x(t) - x(g(t)) + \int_{g(t)}^t p(s) x(\tau(s)) ds = 0. \quad (22)$$

By using (15), it follows that

$$\begin{aligned} x(\tau(s)) &= x(g(t)) e^{\int_{\tau(s)}^{g(t)} p(u) \frac{x(\tau(u))}{x(u)} du} \\ &= x(g(t)) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) \frac{x(\tau(v))}{x(v)} dv} du}. \end{aligned}$$

Therefore, Lemma 1 yields

$$x(\tau(s)) \geq x(g(t)) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du}.$$

Substituting into (22), we get

$$x(t) - x(g(t)) + x(g(t)) \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du} ds \leq 0,$$

that is,

$$\int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du} ds \leq 1 - \frac{x(t)}{x(g(t))},$$

for sufficiently large t . Therefore,

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du} ds \leq 1 - \liminf_{t \rightarrow \infty} \frac{x(t)}{x(g(t))}.$$

However, $\liminf_{t \rightarrow \infty} \frac{x(t)}{x(g(t))} \geq c(k^*)$ (see [5], Lemma 2.1.3). Consequently,

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^u p(v) R_{m,n}(v) dv} du} ds \leq 1 - c(k^*),$$

which contradicts to (21). \square

The proofs of the following two results are basically similar to that of Lemma 1 and Theorem 2.

Theorem 3. Assume that $k^* \leq \frac{1}{e}$ and

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{(\lambda(k^*) - \epsilon) \int_{\tau(s)}^{g(t)} p(u) du + (\lambda(k^*) - \epsilon)^2 \int_{\tau(s)}^{g(t)} p(u) \int_{\tau(u)}^{g(u)} p(v) dv du} ds > 1 - c(k^*), \quad (23)$$

where $\epsilon \in (0, \lambda(k^*))$. Then, all solutions of Equation (1) oscillate.

Theorem 4. Assume that $k^* \leq \frac{1}{e}$ and $m, n \in \mathbb{N}$ such that

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) e^{\int_{\tau(s)}^{g(t)} p(u) R_{m,n}(u) du} ds > 1 - c(k^*). \quad (24)$$

Then, all solutions of Equation (1) oscillate.

Lemma 2. Let $x(t)$ be an eventually positive solution of Equation (1). Then,

$$\limsup_{t \rightarrow \infty} \left(\int_{g(t)}^t p(s) ds + w(g(t)) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v) w(v) dv} du ds \right) = 1 - M,$$

where

$$M := \liminf_{t \rightarrow \infty} \frac{x(t)}{x(g(t))}, \quad \text{and} \quad w(t) := \frac{x(g(t))}{x(t)}.$$

Proof. The positivity of $x(t)$ implies that $x(t)$ is an eventually non-increasing function. Integrating Equation (1) from $g(t)$ to t , we obtain

$$x(t) - x(g(t)) + \int_{g(t)}^t p(s) x(\tau(s)) ds = 0. \quad (25)$$

Since $\tau(s) \leq g(t)$ for $s \leq t$, integrating Equation (1) from $\tau(s)$ to $g(t)$, we have

$$x(\tau(s)) = x(g(t)) + \int_{\tau(s)}^{g(t)} p(u) x(\tau(u)) du.$$

Substituting into (25), we get

$$x(t) - x(g(t)) + x(g(t)) \int_{g(t)}^t p(s) ds + \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) x(\tau(u)) du ds = 0. \quad (26)$$

It is clear that $\tau(u) \leq g^2(t)$, for $u \leq g(t)$. Therefore, (15) implies that

$$x(\tau(u)) = x(g^2(t)) e^{\int_{\tau(u)}^{g^2(t)} p(v)w(v)dv}.$$

From this and (26), it follows that

$$x(t) - x(g(t)) + x(g(t)) \int_{g(t)}^t p(s)ds + x(g^2(t)) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v)w(v)dv} du ds = 0.$$

Consequently,

$$\int_{g(t)}^t p(s)ds + w(g(t)) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v)w(v)dv} du ds = 1 - \frac{x(t)}{x(g(t))}.$$

Therefore,

$$\limsup_{t \rightarrow \infty} \left(\int_{g(t)}^t p(s)ds + w(g(t)) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v)w(v)dv} du ds \right) = 1 - \liminf_{t \rightarrow \infty} \frac{x(t)}{x(g(t))}.$$

□

The proof of the following theorem is a consequence of Lemmas 1, 2, and ([5], Lemmas 2.1.2 and 2.1.3).

Theorem 5. Assume that $k^* \leq \frac{1}{e}$ and $m, n \in \mathbb{N}$ such that

$$\limsup_{t \rightarrow \infty} \left(\int_{g(t)}^t p(s)ds + (\lambda(k^*) - \epsilon) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v)R_{m,n}(v)dv} du ds \right) > 1 - c(k^*),$$

where $\epsilon \in (0, \lambda(k^*))$. Then, every solution of Equation (1) is oscillatory.

Theorem 6. Let $\tilde{L} := \limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s)ds < 1$, $0 < k^* \leq \frac{1}{e}$,

$$\int_{g(s)}^{g(t)} p(u)du \geq \int_s^t p(u)du, \quad \text{for all } s \in [g(t), t], \quad (27)$$

and

$$A := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^{g(t)} p(s)ds. \quad (28)$$

If one of the following conditions is satisfied:

- (i) $\tilde{L} > \frac{-1 - A\lambda(k^*) + \sqrt{2 + (1 + A\lambda(k^*))^2 + 2k^*\lambda(k^*)}}{\lambda(k^*)},$
- (ii) $\tilde{L} > 1 + k^* + \frac{1}{\lambda(k^*)} + A - \sqrt{\left(1 + k^* + \frac{1}{\lambda(k^*)} + A\right)^2 - 2\left(k^* + \frac{1}{\lambda(k^*)}\right)},$

then every solution of Equation (1) is oscillatory.

Proof. Assume that Equation (1) has a nonoscillatory solution $x(t)$; as usual, we assume that $x(t)$ is an eventually positive solution. Let

$$I(t) = \int_{g(t)}^t p(s) ds + w(g(t)) \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u) e^{\int_{\tau(u)}^{g^2(t)} p(v) w(v) dv} du ds, \quad (29)$$

where $w(t) = \frac{x(g(t))}{x(t)}$. Therefore,

$$I(t) \geq \int_{g(t)}^t p(s) ds + w(g(t)) \left(\int_{g(t)}^t p(s) \int_{\tau(s)}^{g(s)} p(u) du ds + \int_{g(t)}^t p(s) \int_{g(s)}^{g(t)} p(u) du ds \right).$$

In view of [5], Lemma 2.1.2) and (28), for sufficiently small ϵ , we obtain

$$I(t) \geq \int_{g(t)}^t p(s) ds + (\lambda(k^*) - \epsilon) \left((A - \epsilon) \int_{g(t)}^t p(s) ds + \int_{g(t)}^t p(s) \int_{g(s)}^{g(t)} p(u) du ds \right).$$

By using (27), it follows that

$$I(t) \geq (1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) \int_{g(t)}^t p(s) ds + (\lambda(k^*) - \epsilon) \int_{g(t)}^t p(s) \int_s^t p(u) du ds. \quad (30)$$

However,

$$\int_{g(t)}^t p(s) \int_s^t p(u) du ds = \frac{1}{2} \left(\int_{g(t)}^t p(s) ds \right)^2.$$

Therefore, (30) implies that

$$I(t) \geq (1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) \int_{g(t)}^t p(s) ds + \frac{\lambda(k^*) - \epsilon}{2} \left(\int_{g(t)}^t p(s) ds \right)^2. \quad (31)$$

On the other hand, from [9], we have

$$\liminf_{t \rightarrow \infty} \frac{x(t)}{x(g(t))} \geq 1 - k^* - \frac{1}{\lambda(k^*)}. \quad (32)$$

Therefore, Lemma 2 and (32) imply that $I(t) < k^* + \frac{1}{\lambda(k^*)} + \epsilon$ for sufficiently large t . Thus, (31) yields

$$(1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) \int_{g(t)}^t p(s) ds + \frac{\lambda(k^*) - \epsilon}{2} \left(\int_{g(t)}^t p(s) ds \right)^2 \leq I(t) < k^* + \frac{1}{\lambda(k^*)} + \epsilon,$$

or equivalently,

$$(\lambda(k^*) - \epsilon) \Lambda^2 + 2(1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) \Lambda - 2k^* - \frac{2}{\lambda(k^*)} - 2\epsilon < 0,$$

where

$$\Lambda := \int_{g(t)}^t p(s) ds.$$

Then,

$$\Lambda < \frac{-(1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) + \sqrt{(1 + (\lambda(k^*) - \epsilon)(A - \epsilon))^2 + 2(\lambda(k^*) - \epsilon)\left(k^* + \frac{1}{\lambda(k^*)} + \epsilon\right)}}{\lambda(k^*) - \epsilon}.$$

Thus,

$$\tilde{L} \leq \frac{-(1 + (\lambda(k^*) - \epsilon)(A - \epsilon)) + \sqrt{(1 + (\lambda(k^*) - \epsilon)(A - \epsilon))^2 + 2(\lambda(k^*) - \epsilon)\left(k^* + \frac{1}{\lambda(k^*)} + \epsilon\right)}}{\lambda(k^*) - \epsilon}.$$

Now, letting $\epsilon \rightarrow 0$, we obtain

$$\tilde{L} \leq \frac{-1 - A\lambda(k^*) + \sqrt{2 + (1 + A\lambda(k^*))^2 + 2k^*\lambda(k^*)}}{\lambda(k^*)}.$$

This completes the proof of case (i).

To prove case (ii), integrating Equation (1) from $g^2(t)$ to $g(t)$, we obtain

$$x(g(t)) - x(g^2(t)) + \int_{g^2(t)}^{g(t)} p(s)x(\tau(s))ds = 0,$$

which, by using the nonincreasing nature of $x(t)$ and the assumption that $\tau(t) \leq g(t)$, implies that

$$x(g(t)) - x(g^2(t)) + x(g^2(t)) \int_{g^2(t)}^{g(t)} p(s)ds \leq 0. \quad (33)$$

In view of (27), we have

$$\int_{g^2(t)}^{g(t)} p(s)ds \geq \int_{g(t)}^t p(s)ds.$$

Substituting into (33), it follows that

$$\frac{x(g^2(t))}{x(g(t))} \geq \frac{1}{1 - \int_{g(t)}^t p(s)ds}.$$

From this and (29), we obtain

$$I(t) \geq \int_{g(t)}^t p(s)ds + \frac{1}{1 - \int_{g(t)}^t p(s)ds} \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u)du ds.$$

Again Lemma 2 and (32) imply for sufficiently small ϵ that

$$\int_{g(t)}^t p(s)ds + \frac{1}{1 - \int_{g(t)}^t p(s)ds} \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(t)} p(u)du ds \leq I(t) < k^* + \frac{1}{\lambda(k^*)} + \epsilon. \quad (34)$$

However, as in the proof of case (i), we have

$$\begin{aligned} \int_{g(t)}^t p(s) \int_{\tau(s)}^{g(s)} p(u) du ds &= \left(\int_{g(t)}^t p(s) \int_{\tau(s)}^{g(s)} p(u) du ds + \int_{g(t)}^t p(s) \int_{g(s)}^{g(t)} p(u) du ds \right) \\ &\geq \left((A - \epsilon) \int_{g(t)}^t p(s) ds + \int_{g(t)}^t p(s) \int_s^t p(u) du ds \right) \\ &= (A - \epsilon) \int_{g(t)}^t p(s) ds + \frac{1}{2} \left(\int_{g(t)}^t p(s) ds \right)^2. \end{aligned} \quad (35)$$

Combining the inequalities (34) and (35), we obtain

$$2\Lambda_1(1 - \Lambda_1) + 2(A - \epsilon)\Lambda_1 + \Lambda_1^2 - 2\alpha(\epsilon)(1 - \Lambda_1) < 0,$$

where

$$\Lambda_1 = \int_{g(t)}^t p(s) ds, \quad \alpha(\epsilon) = k^* + \frac{1}{\lambda(k^*)} + \epsilon.$$

Thus,

$$\Lambda_1^2 - 2(1 + \alpha(\epsilon) + A - \epsilon)\Lambda_1 + 2\alpha(\epsilon) > 0,$$

which implies that $\Lambda_1 < 1 + \alpha(\epsilon) + A - \epsilon - \sqrt{(1 + \alpha(\epsilon) + A - \epsilon)^2 - 2\alpha(\epsilon)}$, and hence

$$\tilde{L} = \limsup_{t \rightarrow \infty} \int_{g(t)}^t p(s) ds \leq 1 + \alpha(\epsilon) + A - \epsilon - \sqrt{(1 + \alpha(\epsilon) + A - \epsilon)^2 - 2\alpha(\epsilon)}.$$

Letting $\epsilon \rightarrow 0$, we obtain

$$\tilde{L} \leq 1 + k^* + \frac{1}{\lambda(k^*)} + A - \sqrt{\left(1 + k^* + \frac{1}{\lambda(k^*)} + A\right)^2 - 2\left(k^* + \frac{1}{\lambda(k^*)}\right)}.$$

□

Remark 1.

(i) Condition (27) is satisfied if (see [9,16])

$$p(g(t))g'(t) \geq p(t), \quad \text{eventually for all } t.$$

(ii) It is easy to show that the conclusion of Theorem 6 is valid, if $p(t) > 0$ and condition (27) is replaced by

$$\liminf_{t \rightarrow \infty} \frac{p(g(t))g'(t)}{p(t)} = 1.$$

Corollary 1. Assume that $0 < k \leq \frac{1}{e}$, $L < 1$ and $\tau(t)$ is a nondecreasing continuous function such that

$$\int_{\tau(s)}^{\tau(t)} p(u) du \geq \int_s^t p(u) du, \quad \text{for all } s \in [\tau(t), t].$$

If

$$L > \min \left\{ \frac{-1 + \sqrt{3 + 2k\lambda(k)}}{\lambda(k)}, 1 + k + \frac{1}{\lambda(k)} - \sqrt{1 + \left(k + \frac{1}{\lambda(k)}\right)^2} \right\}, \quad (36)$$

then Equation (1) is oscillatory.

Remark 2.

- 1- Condition (21), with $n = 1$ and $n = 2$, improves conditions (2), (8), (9) and (10), respectively.
- 2- Condition (23) improves condition (11).
- 3- Condition (24), with $n = 1$, improves conditions (13) with $\ell = 1$.
- 4- It is easy to see that

$$\frac{-1 + \sqrt{3 + 2k\lambda(k)}}{\lambda(k)} \leq \frac{\ln \lambda(k) - 1 + \sqrt{5 - 2\lambda(k) + 2k\lambda(k)}}{\lambda(k)},$$

for all $\lambda(k) \in [1, e]$. Therefore, condition (36) improves condition (4).

The following example illustrates the applicability and strength of our result.

Example 1. Consider the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq 2, \quad (37)$$

where (See Figure 1)

$$\tau(t) = t - 1 - \alpha \sin^2(\nu \pi(t + \alpha)) + \alpha,$$

and

$$p(t) := \begin{cases} \frac{1}{(1-\alpha)e}, & t \in [2n, 2n+1-\alpha], \\ \frac{1}{\alpha(1-\alpha)} \left(\beta - \frac{1}{e} \right) (t - 2n - 1) + \frac{\beta}{(1-\alpha)}, & t \in [2n+1-\alpha, 2n+1], \\ \frac{\beta}{(1-\alpha)}, & t \in [2n+1, 2n+2-\alpha], \\ \frac{-1}{\alpha(1-\alpha)} \left(\beta - \frac{1}{e} \right) (t - 2n - 2) + \frac{1}{(1-\alpha)e}, & t \in [2n+2-\alpha, 2n+2], \end{cases}$$

where $n \in \mathbb{N}$, $\alpha = 0.0001$, $\beta = 0.505$ and $\nu = 20,000$. Throughout our calculations, we take $g = \delta$. It is clear, from the definition of δ and τ , that

$$t - 1 \leq \tau(t) \leq \delta(t) \leq t - 1 + \alpha.$$

Notice that

$$k^* = k = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = \lim_{n \rightarrow \infty} \int_{\tau(2n+1-\alpha)}^{2n+1-\alpha} p(s) ds = \lim_{n \rightarrow \infty} \int_{2n}^{2n+1-\alpha} p(s) ds = \frac{1}{e}. \quad (38)$$

Then, $\lambda(k) = e$, and $\frac{1-k-\sqrt{1-2k-k^2}}{2} \approx 0.1365429862$.

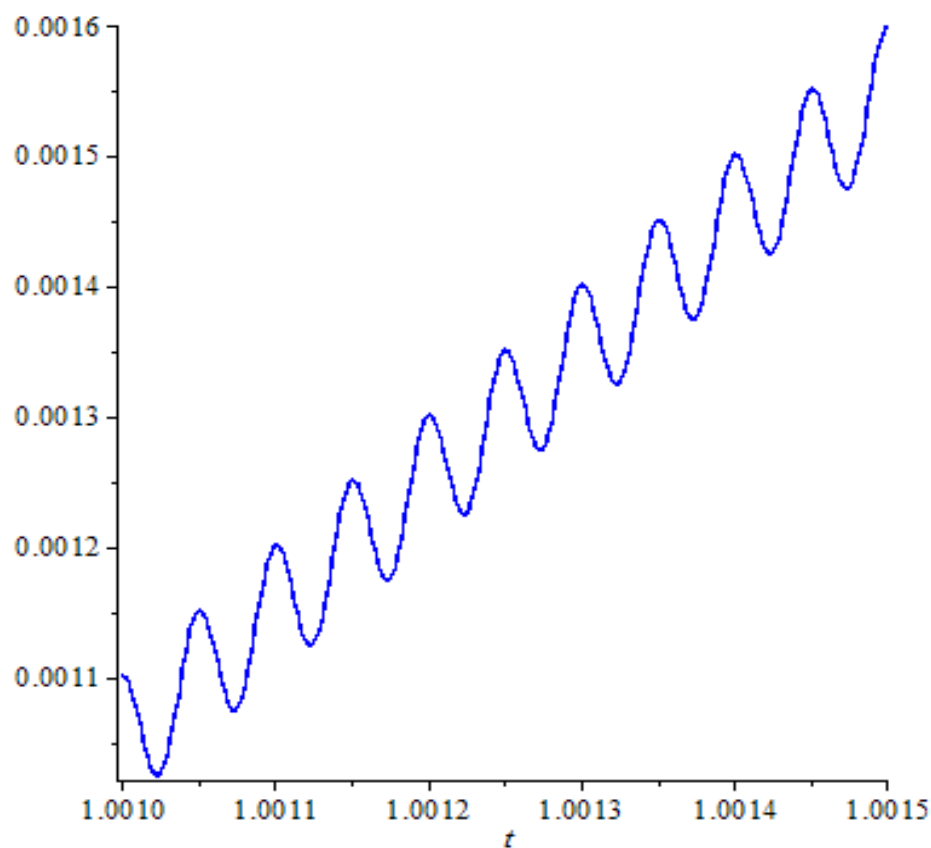


Figure 1. The graph of τ .

Since

$$p(t)R_{1,1}(t) = p(t) \left[1 + \int_{\tau(t)}^t p(s) e^{\int_{\tau(s)}^t p(u) e^{(\lambda(k)-\epsilon) \int_{\tau(u)}^u p(\eta) \left(1 + (\lambda(k)-\epsilon) \int_{\tau(\eta)}^{\delta(\eta)} p(r) dr \right) d\eta} du} ds \right],$$

for $\epsilon = 0.0001$, we have

$$p(t)R_{1,1}(t) \geq \frac{1}{(1-\alpha)e} \left[1 + \int_{t-1+\alpha}^t \frac{1}{(1-\alpha)e} e^{\int_{s-1+\alpha}^t \frac{1}{(1-\alpha)e} e^{(\lambda(k)-\epsilon) \int_{u-1+\alpha}^u \frac{1}{(1-\alpha)e} d\eta} du} ds \right] \approx 1.00006322.$$

Now, assume that

$$J(t) = \int_{\delta(t)}^t p(s) \exp \left(\int_{\tau(s)}^{\delta(t)} p(u) R_{1,1}(u) du \right) ds.$$

Then,

$$\begin{aligned}
 J(2n+2-\alpha) &= \int_{\delta(2n+2-\alpha)}^{2n+2-\alpha} p(s) \exp \left(\int_{\tau(s)}^{\delta(2n+2-\alpha)} p(u) R_{1,1}(u) du \right) ds \\
 &\geq \int_{2n+1}^{2n+2-\alpha} p(s) \exp \left(\int_{s-1+\alpha}^{2n+1-\alpha} p(u) R_{1,1}(u) du \right) ds \\
 &\geq \int_{2n+1}^{2n+2-\alpha} \frac{\beta}{(1-\alpha)} \exp \left(1.00006322 \int_{s-1+\alpha}^{2n+1-\alpha} du \right) ds \\
 &> 0.867626.
 \end{aligned}$$

Therefore,

$$\limsup_{t \rightarrow \infty} J(t) \geq \lim_{n \rightarrow \infty} J(2n+2-\alpha) \geq 0.867626 > 1 - \frac{1-k-\sqrt{1-2k-k^2}}{2} \approx 0.8634570138.$$

Consequently, Theorem (4) with $n = m = 1$ implies that Equation (37) is oscillatory. However, by using (38), condition (3) does not hold.

Let

$$J_1(t) = \int_{\delta(t)}^t p(s) \exp \left(\int_{\tau(s)}^{\delta(t)} p(u) \exp \left(\int_{\tau(u)}^u p(v) dv \right) du \right) ds.$$

Then,

$$J_1(t) \leq \int_{t-1}^t \frac{\beta}{(1-\alpha)} \exp \left(\int_{s-1}^{t-1+\alpha} \frac{\beta}{(1-\alpha)} \exp \left(\int_{u-1}^u \frac{\beta}{(1-\alpha)} dv \right) du \right) ds \approx 0.7901391991.$$

Consequently, $\limsup_{t \rightarrow \infty} J_1(t) < 0.79014$, which means that conditions (7) with $j = 3$ and (10) fail to apply.

In addition, since

$$\int_{\delta(t)}^t p(s) \exp \left(\int_{\tau(s)}^{\delta(t)} p(u) du \right) < \int_{t-1}^t \frac{\beta}{(1-\alpha)} \exp \left(\int_{s-1}^{t-1+\alpha} \frac{\beta}{(1-\alpha)} du \right),$$

it follows that

$$\limsup_{t \rightarrow \infty} \int_{\delta(t)}^t p(s) \exp \left(\int_{\tau(s)}^{\delta(t)} p(u) du \right) < 0.6571023948 < 1 - \frac{1-k-\sqrt{1-2k-k^2}}{2} \approx 0.8634570138.$$

Therefore, none of the conditions (7) with $j = 2$, (8) and (9) are satisfied.

Define

$$J_2(t) = \int_{\delta(t)}^t p(s) \int_{\tau(s)}^s p(u) \exp \left(\int_{\tau(u)}^s p(v) dv \right) du ds + c(k) \exp \left(\int_{\delta(t)}^t p(s) ds \right).$$

It follows that

$$\begin{aligned}
 J_2(t) &\leq \int_{t-1}^t \frac{\beta}{(1-\alpha)} \int_{s-1}^s \frac{\beta}{(1-\alpha)} \exp \left(\int_{u-1}^s \frac{\beta}{(1-\alpha)} dv \right) du ds + c(k) \exp \left(\int_{t-1}^t \frac{\beta}{(1-\alpha)} ds \right) \\
 &< 0.776165,
 \end{aligned}$$

so $\limsup_{t \rightarrow \infty} J_2(t) \leq 0.776165$. Thus, condition (12) with $n = 1$ fails to apply.

Now, let us define the following functions:

$$J_3(t, \epsilon) = \int_{\delta(t)}^t p(s) \exp \left((\lambda(k) - \epsilon) \int_{\tau(s)}^{\delta(t)} p(u) du \right),$$

and

$$J_4(t) = \int_{\delta(t)}^t p(s) \exp \left(\int_{\tau(s)}^{\delta(t)} p(u) F_1(u) du \right) ds,$$

where

$$F_1(t) = 1 + \int_{\delta(t)}^t p(v) \exp \left(\int_{\tau(v)}^t p(u) du \right) dv.$$

Since

$$F_1(t) \leq 1 + \int_{t-1}^t \frac{\beta}{1-\alpha} \exp \left(\int_{v-1}^t \frac{\beta}{1-\alpha} du \right) dv \approx 2.088615495,$$

and $\lambda(k) - \epsilon < e$, it follows that $J_3(t, \epsilon) < G_e(t)$ and $J_4(t) < G_{2.088615495}(t)$, where $G_\omega(t)$ is defined by

$$G_\omega(t) = \int_{\delta(t)}^t p(s) \exp \left(\omega \int_{\tau(s)}^{\delta(t)} p(u) du \right) ds, \quad \text{for } \omega > 0.$$

Next, we estimate the upper limit of $G_\omega(t)$ for $\omega = e$ and $\omega = 2.088615495$.

For $0 \leq \zeta \leq 1 - \alpha$, we have

$$\begin{aligned} G_\omega(2n + \zeta) &= \int_{\delta(2n+\zeta)}^{2n+\zeta} p(s) \exp \left(\omega \int_{\tau(s)}^{\delta(2n+\zeta)} p(u) du \right) ds \\ &\leq \int_{2n+\zeta-1}^{2n+\zeta} p(s) \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du \right) ds \\ &= \int_{2n+\zeta-1}^{2n-\alpha} p(s) \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du \right) ds \\ &\quad + \int_{2n-\alpha}^{2n} p(s) \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du \right) ds \\ &\quad + \int_{2n}^{2n+\zeta} p(s) \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du \right) ds, \end{aligned}$$

which implies that

$$\begin{aligned} G_\omega(2n + \zeta) &\leq \int_{2n+\zeta-1}^{2n-\alpha} \frac{\beta}{(1-\alpha)} \exp \left(\omega \int_{s-1}^{2n-1-\alpha} \frac{1}{(1-\alpha)e} du + \omega \int_{2n-1-\alpha}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du \right) ds \\ &\quad + \int_{2n-\alpha}^{2n} \frac{\beta}{(1-\alpha)} \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du \right) ds \\ &\quad + \int_{2n}^{2n+\zeta} \frac{1}{(1-\alpha)e} \exp \left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du \right) ds \\ &\approx \frac{1}{\omega} \left(1.372732323 e^{\omega(0.3679804513+0.1371342722\zeta)} - 0.3727323230 e^{0.0001010101010\omega(5000\zeta+1)} \right. \\ &\quad \left. - e^{0.00005050505050\omega(10000\zeta+1)} + 1.980198020 e^{0.5050505050\omega\zeta-1+0.00005050505050\omega} \right. \\ &\quad \left. - 1.980198020 e^{0.00005050505050\omega-1} \right). \end{aligned}$$

Therefore, $G_{2.088615495}(2n + \zeta) < 0.7725$ and $G_e(2n + \zeta) < 0.9162$ for all $\zeta \in [0, 1 - \alpha]$.

In addition, if $1 - \alpha \leq \zeta \leq 1$, then

$$G_{\omega}(2n + \zeta) \leq \int_{2n+\zeta-1}^{2n} p(s) \exp\left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du\right) ds \\ + \int_{2n}^{2n+\zeta} p(s) \exp\left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} p(u) du\right) ds.$$

Therefore,

$$G_{\omega}(2n + \zeta) \leq \int_{2n+\zeta-1}^{2n} \frac{\beta}{(1-\alpha)} \exp\left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du\right) ds \\ + \int_{2n}^{2n+1-\alpha} \frac{1}{(1-\alpha)e} \exp\left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du\right) ds \\ + \int_{2n+1-\alpha}^{2n+\zeta} \frac{\beta}{(1-\alpha)} \exp\left(\omega \int_{s-1}^{2n+\zeta-1+\alpha} \frac{\beta}{(1-\alpha)} du\right) ds \\ \approx \frac{1}{\omega} \left(e^{0.5051010101\omega} - e^{0.000050505050\omega(10000\zeta+1)} + 1.980198020e^{-1+0.5050505050\omega\zeta+0.000050505050\omega} \right. \\ \left. - 1.980198020e^{-1+0.5050505050\omega\zeta-0.5049494949\omega} + e^{0.0001010101010\omega(5000.0\zeta-4999)} - e^{0.000050505050\omega} \right).$$

Thus, $G_{2.088615495}(2n + \zeta) < 0.6529$ and $G_e(2n + \zeta) < 0.7899$ for all $\zeta \in [1 - \alpha, 1]$.

Using similar arguments, we obtain:

$$G_{2.088615495}(2n + \zeta + 1) < 0.7603, \quad G_e(2n + \zeta) < 0.8737 \text{ for all } \zeta \in [0, 1 - \alpha]$$

and

$$G_{2.088615495}(2n + \zeta + 1) < 0.7603, \quad G_e(2n + \zeta) < 0.8681 \text{ for all } \zeta \in [1 - \alpha, 1].$$

Then,

$$G_{2.088615495}(t) < 0.7725, \quad \text{for all } t \in [2n, 2n + 2], n \in \mathbb{N},$$

and

$$G_e(t) < 0.9162, \quad \text{for all } t \in [2n, 2n + 2], n \in \mathbb{N}.$$

Consequently,

$$\limsup_{\epsilon \rightarrow 0^+} \left(\limsup_{t \rightarrow \infty} J_3(t, \epsilon) \right) \leq \limsup_{t \rightarrow \infty} G_e(t) \leq 0.9162 < 1,$$

and

$$\limsup_{t \rightarrow \infty} J_4(t) \leq \limsup_{t \rightarrow \infty} G_{2.088615495}(t) \leq 0.7726 < 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \approx 0.8634570138.$$

Then, conditions (11) and (13) with $l = 1$ respectively fail to apply.

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References

1. Agarwal, R.P.; Berezansky, L.; Braverman, E.; Domoshnitsky, A. *Non-Oscillation Theory of Functional Differential Equations with Applications*; Springer: New York, NY, USA; Dordrecht, The Netherlands; Heidelberg, Germany; London, UK, 2012.
2. Braverman, E.; Karpuz, B. On oscillation of differential and difference equations with non-monotone delays. *Appl. Math. Comput.* **2011**, *218*, 3880–3887. [\[CrossRef\]](#)
3. El-Morshedy, H.A. On the distribution of zeros of solutions of first order delay differential equations. *Nonlinear Anal.* **2011**, *74*, 3353–3362. [\[CrossRef\]](#)
4. El-Morshedy, H.A.; Attia, E.R. New oscillation criterion for delay differential equations with non-monotone arguments. *Appl. Math. Lett.* **2016**, *54*, 54–59. [\[CrossRef\]](#)
5. Erbe, L.H.; Kong, Q.; Zhang, B.G. *Oscillation Theory for Functional Differential Equations*; Dekker: New York, NY, USA, 1995.
6. Gopalsamy, K. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1992.
7. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
8. Jaroš, J.; Stavroulakis, I.P. Oscillation tests for delay equations. *Rocky Mt. J. Math.* **1999**, *29*, 197–207. [\[CrossRef\]](#)
9. Kon, M.; Sficas, Y.G.; Stavroulakis, I.P. Oscillation criteria for delay equations. *Proc. Am. Math. Soc.* **2000**, *128*, 2989–2997. [\[CrossRef\]](#)
10. Koplatadze, R.G.; Chanturiya, T.A. On oscillatory and monotonic solutions of first order differential equations with deviating arguments. *Differential'nye Uravneniya* **1982**, *18*, 1463–1465. (In Russian)
11. Koplatadze, R.G.; Kvinikadze, G. On the oscillation of solutions of first order delay differential inequalities and equations. *Georgian Math. J.* **1994**, *1*, 675–685. [\[CrossRef\]](#)
12. Ladas, G. Sharp conditions for oscillations caused by delays. *Appl. Anal.* **1979**, *9*, 93–98. [\[CrossRef\]](#)
13. Ladas, G.; Lakshmikantham, V.; Papadakis, L.S. *Oscillations of Higher-Order Retarded Differential Equations Generated by the Retarded Arguments, in Delay and Functional Differential Equations and Their Applications*; Academic Press: New York, NY, USA, 1972.
14. Myshkis, A.D. Linear homogeneous differential equations of first order with deviating arguments. *Uspekhi Mat. Nauk* **1950**, *5*, 160–162. (In Russian)
15. Philos, C.G.; Sficas, Y.G. An oscillation criterion for first-order linear delay differential equations. *Canad. Math. Bull.* **1998**, *41*, 207–213. [\[CrossRef\]](#)
16. Sficas, Y.G.; Stavroulakis, I.P. Oscillation criteria for first-order delay equations. *Bull. London Math. Soc.* **2003**, *35*, 239–246. [\[CrossRef\]](#)
17. Stavroulakis, I.P. Oscillation criteria for delay and difference equations with non-monotone arguments. *Appl. Math. Comput.* **2014**, *226*, 661–672. [\[CrossRef\]](#)
18. Bereketoglu, H.; Karakoc, F.; Oztepe, G.S.; Stavroulakis, I.P. Oscillation of first order differential equations with several non-monotone retarded arguments. *Georgian Math. J.* **2019**. [\[CrossRef\]](#)
19. Infante, G.; Koplatadze, R.; Stavroulakis, I.P. Oscillation criteria for differential equations with several retarded arguments. *Funkcial. Ekvac.* **2015**, *58*, 347–364. [\[CrossRef\]](#)



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