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An Erdős-Ko-Rado Type Theorem via the Polynomial Method

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Abstract: A family $\mathcal{F}$ is an intersecting family if any two members have a nonempty intersection. Erdős, Ko, and Rado showed that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ holds for a k-uniform intersecting family $\mathcal{F}$ of subsets of $[n]$. The Erdős-Ko-Rado theorem for non-uniform intersecting families of subsets of $[n]$ of size at most k can be easily proved by applying the above result to each uniform subfamily of a given family. It establishes that $|\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}$ holds for non-uniform intersecting families of subsets of $[n]$ of size at most k. In this paper, we prove that the same upper bound of the Erdős-Ko-Rado Theorem for k-uniform intersecting families of subsets of $[n]$ holds also in the non-uniform family of subsets of $[n]$ of size at least k and at most $n-k$ with one more additional intersection condition. Our proof is based on the method of linearly independent polynomials.

Keywords: Erdős-Ko-Rado theorem; intersecting families; polynomial method

1. Introduction

Let $[n]$ be the set $\{1, 2, \cdots, n\}$. A family $\mathcal{F}$ of subsets of $[n]$ is *intersecting* if $F \cap F'$ is non-empty for all $F, F' \in \mathcal{F}$. A family $\mathcal{F}$ of subsets of $[n]$ is *t-intersecting* if $|F \cap F'| \geq t$ holds for any $F, F' \in \mathcal{F}$. A family $\mathcal{F}$ is *k-uniform* if it is a collection of k-subsets of $[n]$. In 1961, Erdős, Ko, and Rado [1] were interested in obtaining an upper bound on the maximum size that an intersecting k-uniform family can have and proved the following theorem which bounds the cardinality of an intersecting k-uniform family.

Theorem 1 (Erdős-Ko-Rado Theorem [1]). *If $n \geq 2k$ and $\mathcal{F}$ is an intersecting k-uniform family of subsets of $[n]$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Erdős-Ko-Rado Theorem is an important result of extremal set theory and has been an inspiration for various generalizations by many authors for over 50 years. Erdős, Ko, and Rado [1] also proved that there exists an integer $n_0(k,t)$ such that if $n \geq n_0(k,t)$, then the maximum size of a t-intersecting k-uniform family of subsets of $[n]$ is $\binom{n-t}{k-t}$. The following generalization of the Erdős-Ko-Rado Theorem was proved by Frankl [2] for $t \geq 15$, and was completed by Wilson [3] for all t. It establishes that the generalized EKR theorem is true if $n \geq (k-t+1)(t+1)$.

Theorem 2 (Generalized Erdős-Ko-Rado Theorem [2,3]). *If $n \geq (k-t+1)(t+1)$ and $\mathcal{F}$ is a t-intersecting k-uniform family of subsets of $[n]$, then we have*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

The Erdős-Ko-Rado Theorem can be restated as follows.

Theorem 3 (Erdős-Ko-Rado Theorem [1])**.** *If $\mathcal{F}$ is a family of subsets F_i of $[n]$ with $|F_i| = k$ and $|F_i| \leq n-k$ that satisfies the following two conditions, for $i \neq j$*

(a) $1 \leq |F_i \cap F_j| \leq k-1$
(b) $1 \leq |F_i \cap F_j^c| \leq k-1$

then we have

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

2. Results

The following EKR-type theorem for non-uniform intersecting families of subsets of $[n]$ of size at most k can be easily proved by applying Theorem 3 to each uniform subfamily of the given non-uniform family.

Theorem 4. *If $\mathcal{F}$ is a family of subsets F_i of $[n]$, with $|F_i| \leq k$ and $n \geq 2k$, that satisfies the following two conditions, for $i \neq j$*

(a) $1 \leq |F_i \cap F_j| \leq k-1$
(b) $1 \leq |F_i \cap F_j^c| \leq k-1$

then we have

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}.$$

In 2014, Alon, Aydinian, and Huang [4] gave the following strengthening of the bounded rank Erdős-Ko-Rado theorem by obtaining the same upper bound under a weaker condition as follows.

Theorem 5 (Alon, Aydinian, and Huang [4])**.** *Let $\mathcal{F}$ be a family of subsets of $[n]$ of size at most k, $1 \leq k \leq n-1$. Suppose that for every two subsets $A, B \in \mathcal{F}$, if $A \cap B = \emptyset$, then $|A| + |B| \leq k$. Then we have*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}.$$

Since the bound $\binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}$ is much larger than $\binom{n-1}{k-1}$, this leads to the following interesting question: when is it possible to get the same bound as in the Erdős-Ko-Rado theorem for uniform intersecting families for the non-uniform intersecting families? We answer this question in the main result of this paper, where we prove that the same upper bound of the EKR Theorem for k-uniform intersecting families of subsets of $[n]$ also holds in the non-uniform family of subsets of $[n]$ of size at least k and at most $n-k$ with one more additional intersection condition, as follows.

Theorem 6. *If $\mathcal{F}$ is a family of subsets F_i of $[n]$ with $k \leq |F_i| \leq n-k$ that satisfies the following three conditions, for $i \neq j$*

(a) $1 \leq |F_i \cap F_j| \leq k-1$
(b) $1 \leq |F_i \cap F_j^c| \leq k-1$
(c) $1 \leq |F_i^c \cap F_j^c| \leq k-1$

then we have

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Please note that if we remove the third condition in Theorem 6, we get the same bound of the Erdős-Ko-Rado theorem for k-uniform intersecting families under the same condition for subsets of $[n]$ that are of size at least k and at most $n-k$.

Erdős-Ko-Rado Theorem is a seminal result in extremal combinatorics and has been proved by various methods (see a survey in [5]). There have been many results that have generalized EKR in various ways over the decades. The aim of this paper is to give a generalization of the EKR Theorem to non-uniform families with some extra conditions. Our proof is based on the method of linearly independent multilinear polynomials.

Our paper is organized as follows. In Section 3, we will introduce our main tool, the method of linearly independent multilinear polynomials. In Section 4, we will give the proof of our main result, Theorem 6.

3. Polynomial Method

The method of linearly independent polynomials is one of the most powerful methods for counting the number of sets in various combinatorial settings. In this method, we correspond multilinear polynomials to the sets and then prove that these polynomials are linearly independent in some space. In 1975, Ray-Chaudhuri and Wilson [6] obtained the following result by using the method of linearly independent polynomials.

Theorem 7 (Ray-Chaudhuri and Wilson [6])**.** *Let $l_1, l_2, \cdots, l_s < n$ be nonnegative integers. If $\mathcal{F}$ is a k-uniform family of subsets of $[n]$ such that $|A \cap B| \in L = \{l_1, l_2, \cdots, l_s\}$ holds for every pair of distinct subsets $A, B \in \mathcal{F}$, then $|\mathcal{F}| \le \binom{n}{s}$ holds.*

In 1981, Frankl and Wilson [7] obtained the following nonuniform version of the Ray-Chaudhuri-Wilson Theorem using the polynomial method. Their proof is given underneath.

Theorem 8 (Frankl and Wilson [7])**.** *Let $l_1, l_2, \cdots, l_s < n$ be nonnegative integers. If $\mathcal{F}$ is a family of subsets of $[n]$ such that $|A \cap B| \in L = \{l_1, l_2, \cdots, l_s\}$ holds for every pair of distinct subsets $A, B \in \mathcal{F}$, then $|\mathcal{F}| \le \sum_{k=0}^{s} \binom{n}{k}$ holds.*

Proof. Let x be the n-tuple of variables $x_1, x_2, \cdots, x_n$, where x_i takes the values only 0 and 1. Then all the polynomials we will work with have the relation $x_i^2 = x_i$ in their domain. Let $F_1, F_2, \cdots, F_m$ be the distinct sets in $\mathcal{F}$, listed in non-decreasing order according to their sizes. We define the characteristic vector $v_i = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$ of F_i such that $v_{i_j} = 1$ if $j \in F_i$ and $v_{i_j} = 0$ if $j \notin F_i$. We consider the following multilinear polynomial

$$f_i(x) = \prod_{l \in L,\ l < |F_i|} (v_i \cdot x - l)$$

where $x = (x_1, x_2, \cdots, x_n)$.

Then we obtain that $f_i(v_i) \neq 0$ and $f_i(v_j) = 0$ for $j < i$. As the vectors v_i are $0-1$ vectors, we have an another multilinear polynomial $g_i(x)$ such that $f_i(x) = g_i(x)$ holds for all $x \in \{0,1\}^n$ by substituting x_k for the powers of x_k, where $k = 1, 2, \cdots, n$. Then it is easy to see that the polynomials $g_1, g_2, \cdots, g_m$ are linearly independent over $\mathbb{R}$. Since the dimension of n-variable multilinear polynomials of degree at most s is $\sum_{k=0}^{s} \binom{n}{k}$, we have

$$|\mathcal{F}| \le \sum_{k=0}^{s} \binom{n}{k}$$

finishing the proof of Theorem 8. □

In the same paper, Frankl and Wilson [7] obtained the following modular version of Theorem 7.

Theorem 9 (Frankl and Wilson [7]). *If $\mathcal{F}$ is a family of subsets of $[n]$ such that $|A \cap B| \equiv l \in L$ (mod p) holds for every pair of distinct subsets $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{n}{|L|}$ holds.*

In 1983, Deza, Frankl and Singhi [8] obtained the following modular version of Theorem 8.

Theorem 10 (Deza, Frankl and Singhi [8]). *If $\mathcal{F}$ is a family of subsets of $[n]$ such that $|A \cap B| \equiv l \in L$ (mod p) holds for every pair of distinct subsets $A, B \in \mathcal{F}$ and $|A| \not\equiv l$ (mod p) for every $A \in \mathcal{F}$, then $|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}$ holds.*

In 1991, Alon, Babai, and Suzuki [9] gave another modular version of Theorem 8 by replacing the condition of nonuniformity with the condition that the members of $\mathcal{F}$ have r different sizes as follows. Their proof was also based on the polynomial method.

Theorem 11 (Alon-Babai-Suzuki [9]). *Let $K = \{k_1, k_2, \cdots, k_r\}$ and $L = \{l_1, l_2, \cdots, l_s\}$ be two disjoint subsets of $\{0, 1, \cdots, p-1\}$, where p is a prime, and let $\mathcal{F}$ be a family of subsets of $[n]$ whose sizes modulo p are in the set K, and $|A \cap B| \pmod{p} \in L$ holds for every distinct two subsets A, B in $\mathcal{F}$, then the largest size of such a family $\mathcal{F}$ is $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$ under the conditions $r(s-r+1) \leq p-1$ and $n \geq s + \max_{1 \leq i \leq r} k_i$.*

In the same paper, Alon, Babai, and Suzuki [9] also conjectured that the statement of Theorem 11 remains true if the condition $r(s-r+1) \leq p-1$ is dropped. Recently Hwang and Kim [10] proved this conjecture of Alon, Babai and Suzuki (1991), using the method of linearly independent polynomials. This result is as follows.

Theorem 12 (Hwang and Kim [10]). *Let $K = \{k_1, k_2, \cdots, k_r\}$ and $L = \{l_1, l_2, \cdots, l_s\}$ be two disjoint subsets of $\{0, 1, \cdots, p-1\}$, where p is a prime, and let $\mathcal{F}$ be a family of subsets of $[n]$ whose sizes modulo p are in the set K, and $|A \cap B| \pmod{p} \in L$ for every distinct two subsets A, B in $\mathcal{F}$, then the largest size of such a family $\mathcal{F}$ is $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$ under the only condition that $n \geq s + \max_{1 \leq i \leq r} k_i$.*

The method of linearly independent polynomials has also been used to prove many intersection theorems about set families by Blokhuis [11], Chen and Liu [12], Furedi, Hwang, and Weichsel [13], Liu and Yang [14], Qian and Ray-Chaudhuri [15], Ramanan [16], Snevily [17,18], Wang, Wei, and Ge [19], and others.

4. Proof of the Main Result

In this section, we prove Theorem 6. As we have mentioned before, our proof is based on the polynomial method. Let x be the n-tuple of variables $x_1, x_2, \cdots, x_n$, where x_i takes the values only 0 and 1. Then all the polynomials we will work with have the relation $x_i^2 = x_i$ in their domain.

Proof of Theorem 6. The result is immediate if $|\mathcal{F}| = 1$. Suppose $|\mathcal{F}| > 1$. Let $F_1, F_2, \cdots, F_f$ be the distinct sets in $\mathcal{F}$, listed in non-decreasing order of size. We define the characteristic vector $v_i = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$ of F_i such that $v_{i_j} = 1$ if $j \in F_i$ and $v_{i_j} = 0$ if $j \notin F_i$.

We consider the following family of multilinear polynomials

$$f_i(x) = \prod_{j=1}^{k-1} (v_i \cdot x - j)$$

where $x = (x_1, x_2, \cdots, x_n)$.

Since $|F_1| \leq |F_2|$, there exists some $p \in F_2$ such that $p \notin F_1$. Let $\mathcal{G} = \{G_1, G_2, \cdots, G_g\}$ be the family of subsets of $[n]$ with the size at most $k-2$, which is listed in non-decreasing order of size, and not containing p. Next, we consider the second family of multilinear polynomials

$$g_i(x) = (x_p - 1) \prod_{j \in G_i} x_j$$

where $1 \le i \le g$. Let $\mathcal{H} = \{H_1, H_2, \cdots, H_h\}$ be the family of subsets of $[n]$ with the size at most $k-1$, which is listed in non-decreasing order of size, and containing p. Then, we consider our third and last family of multilinear polynomials

$$h_i(x) = \prod_{j=0}^{|H_i|-1}(w_i \cdot x - j) - \sum_{l:p\notin F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l^c - j)}{\prod_{j=1}^{k-1}(v_l^c \cdot v_l^c - j)} \prod_{j=1}^{k-1}(v_l^c \cdot x - j) - \sum_{l:p\in F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l - j)}{\prod_{j=1}^{k-1}(v_l \cdot v_l - j)} \prod_{j=1}^{k-1}(v_l \cdot x - j)$$

where w_i is the characteristic vector of H_i.

We claim that the functions $f_i(x)$, $g_i(x)$, and $h_i(x)$ taken together are linearly independent. Assume that

$$\sum_{i=1}^{f} \alpha_i f_i(x) + \sum_{i=1}^{g} \beta_i g_i(x) + \sum_{i=1}^{h} \gamma_i h_i(x) = 0 \tag{1}$$

We substitute the characteristic vector v_s of F_s containing p into Equation (1). Because of the $(x_p - 1)$ factor, we have

$$g_i(v_s) = 0 \quad \text{for all} \quad 1 \le i \le g.$$

Let v_l^c be the characteristic vector of F_l^c. Next, let us consider $h_i(v_s)$:

$$h_i(v_s) = \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s - j) - \sum_{l:p\notin F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l^c - j)}{\prod_{j=1}^{k-1}(v_l^c \cdot v_l^c - j)} \prod_{j=1}^{k-1}(v_l^c \cdot v_s - j) - \sum_{l:p\in F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l - j)}{\prod_{j=1}^{k-1}(v_l \cdot v_l - j)} \prod_{j=1}^{k-1}(v_l \cdot v_s - j).$$

Since $1 \le |F_l \cap F_s| \le k-1$, we have $\prod_{j=1}^{k-1}(v_l \cdot v_s - j) = 0$ except when $s = l$. Since $|F_i| \ge k$ for all i, we have

$$\begin{aligned} - \sum_{l:p\in F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l - j)}{\prod_{j=1}^{k-1}(v_l \cdot v_l - j)} \prod_{j=1}^{k-1}(v_l \cdot v_s - j) &= - \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_s - j)}{\prod_{j=1}^{k-1}(v_s \cdot v_s - j)} \prod_{j=1}^{k-1}(v_s \cdot v_s - j) \\ &= - \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s - j). \end{aligned}$$

Since $1 \le |F_l^c \cap F_s| \le k-1$ for $s \ne l$, we have $\prod_{j=1}^{k-1}(v_l^c \cdot v_s - j) = \prod_{j=1}^{k-1}(|F_l^c \cap F_s| - j) = 0$. Thus, we have

$$h_i(v_s) = \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s - j) - \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s - j) = 0 \quad \text{for all} \quad 1 \le i \le h.$$

Finally, we consider $f_i(v_s)$. Since $f_s(v_s) \ne 0$ and $1 \le |F_i \cap F_s| \le k-1$ for $i \ne s$, we get $\alpha_s = 0$ whenever $p \in F_s$.

Next, we substitute the characteristic vector v_s^c of F_s^c into Equation (1), where $p \notin F_s$. Because of the $(x_p - 1)$ factor, we have

$$g_i(v_s^c) = 0 \quad \text{for all} \quad 1 \le i \le g.$$

Next, let us consider $h_i(v_s^c)$. Since $1 \le |F_l^c \cap F_s^c| \le k-1$, we have $\prod_{j=1}^{k-1}(v_l^c \cdot v_s^c - j) = 0$ except when $s = l$. Since $n - |F_i| \ge k$, we have

$$- \sum_{l:p\notin F_l} \frac{\prod_{j=0}^{|H_i|-1}(w_i \cdot v_l^c - j)}{\prod_{j=1}^{k-1}(v_l^c \cdot v_l^c - j)} \prod_{j=1}^{k-1}(v_l^c \cdot v_s^c - j) = - \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s^c - j).$$

Since $1 \leq |F_l \cap F_s^c| \leq k-1$ for $s \neq l$, we have $\prod_{j=1}^{k-1}(v_l \cdot v_s^c - j) = \prod_{j=1}^{k-1}(|F_l \cap F_s^c| - j) = 0$. Thus, we have

$$h_i(v_s^c) = \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s^c - j) - \prod_{j=0}^{|H_i|-1}(w_i \cdot v_s^c - j) = 0 \quad \text{for all } 1 \leq i \leq h.$$

Finally we consider $f_i(v_s^c)$. Since $1 \leq |F_i \cap F_s^c| \leq k-1$, by the hypothesis $f_i(v_s^c)$ is also 0 except for $f_s(v_s^c)$. Since $f_s(v_s^c) \neq 0$, we get $\alpha_s = 0$ whenever $p \notin F_s$.

So Equation (1) is reduced to :

$$\sum_{i=1}^{g} \beta_i g_i(x) + \sum_{i=1}^{h} \gamma_i h_i(x) = 0 \tag{2}$$

Next, we substitute the characteristic vector w_s of H_s in order of increasing size into Equation (2). Now we note that $p \in H_s$. Because of the $(x_p - 1)$ factor, we have $g_i(w_s) = 0$ for all $1 \leq i \leq g$. Since the size of H_i is at most $k-1$ for all i, we have $1 \leq |F_l^c \cap H_s| \leq k-1$ for $p \in F_l^c$. Thus, the factor $\prod_{j=1}^{k-1}(v_l^c \cdot w_s - j)$ is 0. Similarly, the factor $\prod_{j=1}^{k-1}(v_l \cdot w_s - j)$ is 0 for $p \in F_l$. Thus, we have $h_i(w_s) = \prod_{j=0}^{|H_i|-1}(w_i \cdot w_s - j)$. Since $h_s(w_s) \neq 0$, and $h_i(w_s) = 0$ for $i > s$, we have $\sum_{i=1}^{h} \gamma_i h_i(w_s) = \sum_{i=1}^{s} \gamma_i h_i(w_s)$.

Recall that we substitute the vector w_s in order of increasing size. When we first plug w_1 into Equation (2), we have $\gamma_1 h_1(w_1) = 0$, and thus $\gamma_1 = 0$. Next, we plug w_2 into (2) after dropping $\gamma_1 h_1(w_1)$ term from (2). Then we have $\gamma_2 h_2(w_2) = 0$, and thus $\gamma_2 = 0$. Similarly, we have $\gamma_i = 0$ for all i.

Thus, Equation (1) becomes

$$\sum_{i} \beta_i g_i(x) = 0. \tag{3}$$

Next, we substitute the characteristic vector y_s of G_s in order of increasing size into Equation (3). Thus, we have

$$g_i(y_s) = (y_{s_p} - 1) \prod_{j \in G_i} y_{s_j} = - \prod_{j \in G_i} y_{s_j} \quad \text{for all } 1 \leq i \leq g.$$

Recall that we substitute the vector y_s in order of increasing size. Please note that $g_i(0)$ is the empty product, which is taken to be 1. When we first plug y_1 into Equation (3), we have $g_1(y_1) \neq 0$ and $g_i(y_1) = 0$ for all $i > 1$, and thus $\beta_1 = 0$. Next, we plug y_2 into (3) after dropping $\beta_1 g_1(x)$ term from (3). Then we have $g_2(y_2) \neq 0$ and $g_i(y_2) = 0$ for all $i > 2$, and thus $\beta_2 = 0$. Similarly, we have $\beta_i = 0$ for all i.

This concludes that all the polynomials $f_i(x)$, $g_i(x)$, and $h_i(x)$ are linearly independent. We found $|\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}|$ linearly independent polynomials. All these polynomials are of degree less than or equal to $k-1$. The space of these multilinear polynomials has dimension $\sum_{i=0}^{k-1}\binom{n}{i}$. We have

$$|\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}| \leq \sum_{i=0}^{k-1}\binom{n}{i}.$$

Since $|\mathcal{G}| = \sum_{i=0}^{k-2}\binom{n-1}{i}$ and $|\mathcal{H}| = \sum_{i=0}^{k-2}\binom{n-1}{i}$, we have $|\mathcal{F}| + 2\sum_{i=0}^{k-2}\binom{n-1}{i} \leq \sum_{i=0}^{k-1}\binom{n}{i}$. This gives us

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

finishing the proof of Theorem 6. □

5. Conclusions

We have answered the following question: when is it possible to get the same bound of the Erdős-Ko-Rado theorem for uniform intersecting families in the non-uniform intersecting families? Since the EKR-type bound for the non-uniform family of subsets of $[n]$, which is $\binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}$, is much larger than $\binom{n-1}{k-1}$, this question is interesting and deserves further study.

Please note that if we can delete the condition (c) in Theorem 6, we can get the same bound of the Erdős-Ko-Rado theorem for k-uniform intersecting families under the same condition for non-uniform intersecting families of size at least k and at most $n - k$. Another intriguing question motivated by our result is the problem of getting the same bound of Theorem 6 without the condition (c) or finding a better bound for the non-uniform intersecting families than the previous results by the others.

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References

1. Erdős, P.; Ko, C.; Rado, R. Intersection theorem for systems of finite sets. *Q. J. Math. Oxf. Ser.* **1961**, *12*, 313–320. [CrossRef]
2. Frankl, P. The Erdős-Ko-Rado theorem is true for $n = ckt$. In *Combinatorics, Proceedings of the Fifth Hungarian Colloquium, Keszthely*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1978; Volume 1, pp. 365–375.
3. Wilson, R. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica* **1984**, *4*, 247–257. [CrossRef]
4. Alon, N.; Aydinian, H.; Huang, H. Maximizing the number of nonnegative subsets. *SIAM J. Discret. Math.* **2014**, *28*, 811–816. [CrossRef]
5. Deza, M.; Frankl, P. Erdős-Ko-Rado theorem–22 years later. *SIAM J. Algebr. Discret. Methods* **1983**, *4*, 419–431. [CrossRef]
6. Ray-Chaudhuri, D.; Wilson, R. On t-designs. *Osaka J. Math.* **1975**, *12*, 737–744.
7. Frankl, P.; Wilson, R. Intersection theorems with geometric consequences. *Combinatorica* **1981**, *1*, 357–368. [CrossRef]
8. Deza, M.; Frankl, P.; Singhi, N. On functions of strength t. *Combinatorica* **1983**, *3*, 331–339. [CrossRef]
9. Alon, N.; Babai, L.; Suzuki, H. Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems. *J. Comb. Theory Ser. A* **1991**, *58*, 165–180. [CrossRef]
10. Hwang, K.; Kim, Y. A proof of Alon-Babai-Suzuki's Conjecture and Multilinear Polynomials. *Eur. J. Comb.* **2015**, *43*, 289–294. [CrossRef]
11. Blokhuis, A. Solution of an extremal problem for sets using resultants of polynomials. *Combinatorica* **1990**, *10*, 393–396. [CrossRef]
12. Chen, W.Y.C.; Liu, J. Set systems with L-intersections modulo a prime number. *J. Comb. Theory Ser. A* **2009**, *116*, 120–131. [CrossRef]
13. Füredi, Z.; Hwang, K.; Weichsel, P. A proof and generalization of the Erős-Ko-Rado theorem using the method of linearly independent polynomials. In *Topics in Discrete Mathematics*; Algorithms Combin. 26; Springer: Berlin, Germnay, 2006; pp. 215–224.
14. Liu, J.; Yang, W. Set systems with restricted k-wise L-intersections modulo a prime number. *Eur. J. Comb.* **2014**, *36*, 707–719. [CrossRef]
15. Qian, J.; Ray-Chaudhuri, D. On mod p Alon-Babai-Suzuki inequality. *J. Algebr. Comb.* **2000**, *12*, 85–93. [CrossRef]
16. Ramanan, G. Proof of a conjecture of Frankl and Füredi. *J. Comb. Theory Ser. A* **1997**, *79*, 53–67. [CrossRef]
17. Snevily, H. On generalizations of the de Bruijn-Erdős theorem. *J. Comb. Theory Ser. A* **1994**, *68*, 232–238. [CrossRef]

18. Snevily, H. A sharp bound for the number of sets that pairwise intersect at k positive values. *Combinatorica* **2003**, *23*, 527–532. [CrossRef]
19. Wang, X.; Wei, H.; Ge, G. A strengthened inequality of Alon-Babai-Suzuki's conjecture on set systems with restricted intersections modulo p. *Discret. Math.* **2018**, *341*, 109–118. [CrossRef]