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Grey Fuzzy Multiple Attribute Group Decision-Making Methods Based on Interval Grey Triangular Fuzzy Numbers Partitioned Bonferroni Mean

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Abstract: Considering the characteristics such as fuzziness and greyness in real decision-making, the interval grey triangular fuzzy number is easy to express fuzzy and grey information simultaneously. And the partition Bonferroni mean (PBM) operator has the ability to calculate the interrelationship among the attributes. In this study, we combine the PBM operator into the interval grey triangular fuzzy numbers to increase the applicable scope of PBM operators. First of all, we introduced the definition, properties, expectation, and distance of the interval grey triangular fuzzy numbers, and then we proposed the interval grey triangular fuzzy numbers partitioned Bonferroni mean (IGTFPBM) and the interval grey triangular fuzzy numbers weighted partitioned Bonferroni mean (IGTFWPBM), the adjusting of parameters in the operator can bring symmetry effect to the evaluation results. After that, a novel method based on IGTFWPBM is developed for solving the grey fuzzy multiple attribute group decision-making (GFMAGDM) problems. Finally, we give an example to expound the practicability and superiority of this method.

Keywords: GFMAGDM; interval grey triangular fuzzy number; partitioned Bonferroni mean

1. Introduction

Multiple attribute group decision-making (MAGDM) is the process of comparing and choosing the best alternative by analyzing the evaluations of multiple attributes aggregated from decision-makers. And MAGDM problems have a wide application in real life, such as effect evaluation of environmental protection policies formulated by the government [1], the evaluation of international cooperation plans for energy development [2], the assessment of the comprehensive strength of different schools [3], the comparison of various schemes by enterprises in business negotiations [4], and the selection of enterprises' purchasing schemes [5]. Most of the MAGDM problems exist fuzziness and uncertainty, which is because people's ideology is subjective and complex. Besides, decision-makers may fall into the predicament of the lack of information when evaluating alternatives, which shows the characteristics of grey decision making. In practical decision making, fuzziness and greyness are often existing at the same time, which is called the grey fuzzy multiple attribute group decision-making (GFMAGDM) problem.

To solve GFMGADM problems completely and improve the accuracy of decision results, more and more research has been carried out and expanded from the expression forms of grey information and fuzzy information [6], the integrated calculation method of grey information and fuzzy information [7], the construction of decision system based on grey fuzzy information [8], and the application problems based on grey fuzzy information [9,10]. Bu and Zhang [11] interpreted the grey degree as a level of uncertainty of information, converted the grey numbers into the three-parameter interval numbers, and gave an approach of interval numbers to sort all alternatives. However, they did not give a definite formula when comparing any two alternatives. Based on the literature [11], Luo and Liu [12] used the optimization theory and the maximum entropy criterion to give the method about GFMGADM problems under the conditions that the attributes' weights are known or unknown. Jin and Lou [13,14] considered the fuzziness and greyness in the decision-making process, calculated the distinctions between positive and negative ideal solutions of all alternatives and used satisfaction as weights to adjust the attributes' ranking order. The above researches mainly introduced a method about the grey part of grey fuzzy numbers, which can pragmatically handle the problem of lacking information. However, the fuzzy part shown as exact numbers is difficult to represent the complexity of objective things and fuzziness of people's thinking. Zhu et al. [15] tried to express the fuzzy part with interval numbers and combine with the grey part to express grey fuzzy numbers. Jin and Liu [16] proposed the interval grey linguistic variables, the fuzzy part was represented by interval grey linguistic variables and introduced an approach to undertake GFMADM problems. Interval grey trapezoid fuzzy linguistic variables proposed by Yin et al. [17] can use more information to improve the accuracy of evaluation. Wang and Wang [18] proposed grey linguistic 2-tuple terms to express the priority option of a decision-maker in a grey fuzzy environment. The above expressions used to solve GFMADM problems to reflect fuzzy information and grey information are relatively simple, especially in the fuzzy part, the fuzzy information is expressed by interval numbers and linguistic variables. Although the calculation process has the advantage of simplicity and rapidity, the calculation results are often rough and difficult to be applied to accurate evaluation problems in the military field and medical field.

In recent years, there are extensive researches on decision methods and operators in a grey fuzzy environment. Jin and Liu [16] constructed the TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method based on interval grey linguistic variables to solve the decision-making problem in a grey fuzzy environment. Wang et al. [19] extended the traditional SIR (Superiority and Inferiority Ranking) model to interval grey linguistic variables and proposed the SIR Choquet method considering the independence of multi-attribute variables. Liu and Zhang [20] proposed the interval grey linguistic variables weighted geometric aggregation (IGLWGA) operator, the interval grey linguistic variables ordered weighted geometric aggregation (IGLOWGA) operator, and the interval grey linguistic variable hybrid weighted geometric aggregation (IGLHWGA) operator, and developed a new method referred to IGLHWGA operators. Liu [21] proposed the interval grey linguistic variables weighted aggregation (IGLWA) operator and the interval grey linguistic variables weighted aggregation (IGLHWA) operator and applied them to a method of sorting the alternatives. Liu et al. [22] proposed the interval grey uncertain linguistic generalized ordered weighted averaging (IGULGOWA) operator and the interval grey uncertain linguistic generalized hybrid averaging (IGULGHA) operator to solve GFMADM problems according to interval grey uncertain linguistic information. The above methods and operators do not consider the interrelationship among attribute variables. However, in the process of evaluation, there may be some correlation among the attribute variables. Back in 1950, Bonferroni [23] proposed the Bonferroni mean (BM), which were to be judged alternatives by establishing conjunction between each attribute variable. Yager [24–26] deepened the understanding of the Bonferroni mean in 2009, firstly it was proposed that BM could be used to calculate the interrelationship among any pair of attributes and suggested replace the simple weighted operator with other more complex operators. At present, BM has been widely used in different fuzzy information and different environments due to its ability to determine the optimal alternative according to interrelationship [27–32]. Tian et al. [33] proposed the grey linguistic weighted Bonferroni mean operator to solve GFMADM problems which exit interrelationship between each attribute. However, BM assumes that there exists an interrelationship between any two attributes, and it is difficult to match the actual decision process. Focusing on this problem, Dutta and Guha [29] proposed the partitioned Bonferroni mean (PBM) which is based on the premise that all attributes are divided into some classes based on the interrelationship of each attribute variable in the actual decision, any attribute variable in each class is required to be only related to attribute variables in the same class and be not related to attribute variables in other classes. For example, when choosing the mobile phone, we may evaluate mobile phones from the following four aspects: appearance design (A_1), the inner performance (A_2), brand influence (A_3), and after-sales service (A_4). The four attributes can be divided into two classes: $P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4\}$. It is not difficult to find that A_1 is unrelated to A_2, and there is no interrelationship between P_1 and P_2. After that, more and more scholars have perfected PBM operators. The intuitionistic uncertain linguistic partitioned Bonferroni mean (IULPBM) operator [34] and the intuitionistic fuzzy interaction partitioned Bonferroni mean (IFWIPBM) operator [35] are proposed to calculate the internal correlation of attribute variables in an intuitionistic fuzzy environment. Yin et al. [36] developed the trapezoidal fuzzy two-dimensional linguistic partitioned Bonferroni mean (TF2DLPBM) operator for solving MAGDM problems. Liu and Liu [37] proposed the PBM operator for two-dimensional uncertain linguistic variables (2DULVs) and developed a new method.

In the actual decision-making process, different decision-makers have different cognition of the alternatives, and the same decision-maker has different cognition degrees of different attributes of the same alternative. Therefore, the comparability of the evaluation results of different decision-makers is weak, and the reliability of the comprehensive results is insufficient. At present, the existing fuzzy variables are mostly used for the decision-makers subjective evaluation information of the evaluation program, and there are few pieces of research on the objective cognition degree in the evaluation process. In this paper, we proposed the interval grey triangular fuzzy numbers which fuzzy part in the form of triangular fuzzy numbers and the grey part expressed by interval numbers. The fuzzy part of the interval grey triangular fuzzy numbers is used to express the subjective evaluation of decision-makers on the attribute variables of alternatives, and the grey part is used to indicate the objective cognition degree of the decision-maker to the attribute variables of the alternatives. The increase of the grey part is a supplement to the reliability of the original fuzzy evaluation. When the value of the grey part is smaller, it means that the decision-maker knows more about the alternatives and his evaluation result is more convincing, in other words, the reliability of the fuzzy part is higher. Compared with the existing form of expression of uncertain information, the interval grey triangular fuzzy numbers can reflect decision-makers' subjective evaluation and objective cognition of evaluation objects, the decision-making information is more comprehensive. At the same time, the interval grey triangular fuzzy numbers can transform with interval grey linguistic variables, interval grey interval number and other grey fuzzy forms, which has a wide scope of application.

Until now, in the grey fuzzy environment, there is no method considering the internal structure and interrelationship among the attributes. And then, we extend the PBM operator to the interval grey triangular fuzzy numbers, partitioned all attributes into several classes and ensured that any attribute is only interrelated to other attributes in the same class, after all these, we proposed the interval grey triangular fuzzy numbers partitioned Bonferroni mean (IGTFPBM) and the interval grey triangular fuzzy numbers weighted partitioned Bonferroni mean (IGTFWPBM). Based on the IGTFWPBM operators, we introduced an approach for solving GFMAGDM problems. The method proposed in this paper can consider the interrelationship among attributes, classify and integrate the calculation of attributes, which is more consistent with the decision logic of practical problems. The parameters p and q in the operator can reflect the decision maker's attitude towards the hierarchical relationship of attributes, and the decision-maker can obtain different decision ordering by adjusting the parameter size. In addition, symmetric adjustment of parameters p and q will produce a symmetry effect on the expectation of the alternative.

The remainder of this paper is organized as follows. We introduce some conception of the triangular fuzzy number, grey math, and PBM briefly in Section 2. Section 3 gives the definition, calculation rules, distances, and comparative methods of the interval grey triangular fuzzy numbers. We proposed the IGTFPBM operators and IGTFWPBM operators by considering the interrelationship in the fuzzy and grey information in Section 4. In Section 5, we developed an approach for undertaking the GFMAGDM problems based on IGTFWPBM. In Section 6, we gave an instance to explain the new method. Finally, we discussed the conclusion in Section 7.

2. Preliminaries

In this section, we recalled the form of the triangular fuzzy numbers and grey math briefly, and we reviewed the concepts of BM and the PBM, which are foundations of the decision-making methods we proposed.

2.1. The Triangular Fuzzy Number

Definition 1. *[38,39] A triangular fuzzy number $\widetilde{f}$ is defined as $\widetilde{f} = (f^L, f^M, f^R)$, which satisfies the condition $0 < f^L \leq f^M \leq f^R$, and the calculation rules of membership function $\mu(x) : R \rightarrow [0,1]$* are shown as follows:

$$\mu(x) = \begin{cases} \frac{x-f^L}{f^M-f^L} & x \in (f^L, f^M\ ) \\ 1 & x = f^M \\ \frac{x-f^R}{f^M-f^R} & x \in (f^M, f^R) \\ 0 & other \end{cases} \tag{1}$$

when $f^L = f^M = f^R$, the triangular fuzzy number can be transformed into a crisp number.

2.2. Grey Fuzzy Math

Definition 2. *[16,40] Let $\underset{\otimes}{\widetilde{A}}$ be grey fuzzy set in space X. $\widetilde{A}$ is the fuzzy subset in the space X = {x} and $\underset{\otimes}{A}$ is the grey subset in the space X = {x}. The membership degree $\mu_{\widetilde{A}}(x)$ of $x \in X$ is the fuzzy number in the interval [0, 1], and the grey degree $v_{\widetilde{A}}(x)$ of $x \in X$ is the grey number in the interval [0, 1].*

$$\underset{\otimes}{\widetilde{A}} = \left\{(x, \mu_{\widetilde{A}}(x), v_{\widetilde{A}}(x))\middle| x \in X\right\} \tag{2}$$

And the simplified form is:

$$\underset{\otimes}{\widetilde{A}} = \left(\widetilde{A}, \underset{\otimes}{A}\right) \tag{3}$$

where $\widetilde{A} = \left\{(x, \mu_{\widetilde{A}}(x)\middle| x \in X\right\}$ is called the fuzzy part of $\underset{\otimes}{\widetilde{A}}$, and $\underset{\otimes}{A} = \left\{(x, v_{\widetilde{A}}(x))\middle| x \in X\right\}$ is called the grey part of $\underset{\otimes}{\widetilde{A}}$.

2.3. Partitioned Bonferroni Mean

Definition 3. *[23] For any $p, q \geq 0$, the BM operator as follows:*

$$\mathrm{BM}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{4}$$

where $a_i (i = 1, 2, \ldots, n)$ is the set of a non-negative real number, p and q are the parameters of decision-makers. BM operators can calculate the interrelationship between any attributes. However, the operator is limited by the assumption that there is an interrelationship among each attribute, so some independent attributes cannot be included in the calculation. Under the circumstances, Dutta and Guha [31] proposed PBM.

Definition 4. *[29] For any $p, q \geq 0$, the PBM operator as follows*:

$$\mathrm{PBM}^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{e} \left(\sum_{h=1}^{e} \left(\frac{1}{|P_h|} \sum_{i \in P_h} a_i^p \left(\frac{1}{|P_h| - 1} \sum_{\substack{j \in P_h \\ j \neq i}} a_j^q \right) \right)^{\frac{1}{p+q}} \right) \tag{5}$$

where $a_i (i = 1, 2, \ldots, n)$ is the set of a non-negative real number, e is the number of classes and P_h is number of attributes in the same class.

3. Interval Grey Triangular Fuzzy Numbers

In this section, we built the interval grey triangular fuzzy numbers. After that, we gave the definition, operation rules, distance, and comparing methods of the interval grey triangular fuzzy numbers.

3.1. The Definition of Interval Grey Triangular Fuzzy Numbers

Definition 6. *Let $\underset{\otimes}{\widetilde{A}} = \left(\widetilde{A}, \underset{\otimes}{A} \right)$ be any grey fuzzy number, if the fuzzy part $\widetilde{A}$ is a triangular fuzzy number (f_A^L, f_A^M, f_A^U), and its grey part $\underset{\otimes}{A}$ is closed interval $[g_A^L, g_A^U]$ in the interval [0, 1],* then *$\underset{\otimes}{\widetilde{A}}$ is called an interval grey triangular fuzzy number.*

$$\underset{\otimes}{\widetilde{A}} = \left((f_A^L, f_A^M, f_A^U), [g_A^L, g_A^U] \right) \tag{6}$$

It is worth emphasizing that the fuzzy part (triangular fuzzy number) and grey part (the interval grey number) are mutually independent. In the traditional sense, both fuzzy information and grey information can express uncertain information. Fuzzy information represents the subjective evaluation of the decision-maker on the alternatives, while grey information represents the incomplete cognition of decision-makers to the objective existence of certain characteristics of alternatives [41]. The fuzzy part of the interval grey triangular fuzzy numbers expresses the subjective evaluation of the decision-maker on the attributes of the alternatives, and the grey part indicates the objective cognition degree of the decision-maker to the attributes of alternatives, which is to evaluate the alternatives from two different dimensions. Therefore, the fuzzy part (triangular fuzzy number) and grey part (the interval grey number) are mutually independent, and the form used to express grey information and fuzzy information is also consistent with some research results [16,20–22].

3.2. The Operation Rules of the Interval Grey Triangular Fuzzy Numbers

Let $\underset{\otimes}{\widetilde{A}} = \left((f_A^L, f_A^M, f_A^U), [g_A^L, g_A^U]\right)$ and $\underset{\otimes}{\widetilde{B}} = \left((f_B^L, f_B^M, f_B^U), [g_B^L, g_B^U]\right)$ be any two-interval grey triangular fuzzy numbers. The operation rules of interval grey triangular fuzzy numbers are defined as bellows:

$$\underset{\otimes}{\widetilde{A}} + \underset{\otimes}{\widetilde{B}} = \left((f_A^L + f_B^L, f_A^M + f_B^M, f_A^U + f_B^U), [\max(g_A^L, g_B^L), \max(g_A^U, g_B^U)]\right) \tag{7}$$

$$\underset{\otimes}{\widetilde{A}} - \underset{\otimes}{\widetilde{B}} = \left((f_A^L - f_B^L, f_A^M - f_B^M, f_A^U - f_B^U), [\max(g_A^L, g_B^L), \max(g_A^U, g_B^U)]\right) \tag{8}$$

$$\begin{aligned}\underset{\otimes}{\widetilde{A}} \times \underset{\otimes}{\widetilde{B}} = \big(&(\min(f_A^L \times f_B^L, f_A^L \times f_B^U, f_A^U \times f_B^L, f_A^U \times f_B^U), f_A^M \times f_B^M, \\ &\max(f_A^L \times f_B^L, f_A^L \times f_B^U, f_A^U \times f_B^L, f_A^U \times f_B^U), [\max(g_A^L, g_B^L), \max(g_A^U, g_B^U)]\big)\end{aligned} \tag{9}$$

$$\begin{aligned}\underset{\otimes}{\widetilde{A}} / \underset{\otimes}{\widetilde{B}} = \big(&(\min(f_A^L / f_B^L, f_A^L / f_B^U, f_A^U / f_B^L, f_A^U / f_B^U), f_A^M \times f_B^M, \\ &\max(f_A^L / f_B^L, f_A^L / f_B^U, f_A^U / f_B^L, f_A^U / f_B^U), [\max(g_A^L, g_B^L), \max(g_A^U, g_B^U)]\big) \quad f_B^L \neq 0,\ f_B^M \neq 0,\ f_B^U \neq 0\end{aligned} \tag{10}$$

$$k\underset{\otimes}{\widetilde{A}} = \left((kf_A^L, kf_A^M, kf_A^U), [g_A^L, g_A^U]\right) \quad (k \geq 0) \tag{11}$$

$$\left(\underset{\otimes}{\widetilde{A}}\right)^r = \left((f_A^L)^r, (f_A^M)^r, (f_A^U)^r, [g_A^L, g_A^U]\right) \quad (r > 0) \tag{12}$$

Supposed that $\underset{\otimes}{\widetilde{A}} = \left((f_A^L, f_A^M, f_A^U), [g_A^L, g_A^U]\right)$, $\underset{\otimes}{\widetilde{B}} = \left((f_B^L, f_B^M, f_B^U), [g_B^L, g_B^U]\right)$ and $\underset{\otimes}{\widetilde{C}} = \left((f_C^L, f_C^M, f_C^U), [g_C^L, g_C^U]\right)$ be the three interval grey triangular fuzzy numbers. The interval grey triangular fuzzy numbers satisfied the following properties.

$$\underset{\otimes}{\widetilde{A}} + \underset{\otimes}{\widetilde{B}} = \underset{\otimes}{\widetilde{B}} + \underset{\otimes}{\widetilde{A}} \tag{13}$$

$$\underset{\otimes}{\widetilde{A}} \times \underset{\otimes}{\widetilde{B}} = \underset{\otimes}{\widetilde{B}} \times \underset{\otimes}{\widetilde{A}} \tag{14}$$

$$\underset{\otimes}{\widetilde{A}} + \underset{\otimes}{\widetilde{B}} + \underset{\otimes}{\widetilde{C}} = \underset{\otimes}{\widetilde{A}} + \left(\underset{\otimes}{\widetilde{B}} + \underset{\otimes}{\widetilde{C}}\right) \tag{15}$$

$$\underset{\otimes}{\widetilde{A}} \times \underset{\otimes}{\widetilde{B}} \times \underset{\otimes}{\widetilde{C}} = \underset{\otimes}{\widetilde{A}} \times \left(\underset{\otimes}{\widetilde{B}} \times \underset{\otimes}{\widetilde{C}}\right) \tag{16}$$

$$\underset{\otimes}{\widetilde{A}} \times \left(\underset{\otimes}{\widetilde{B}} + \underset{\otimes}{\widetilde{C}}\right) = \underset{\otimes}{\widetilde{A}} \times \underset{\otimes}{\widetilde{B}} + \underset{\otimes}{\widetilde{A}} \times \underset{\otimes}{\widetilde{C}} \tag{17}$$

$$(\lambda_1 + \lambda_2)\underset{\otimes}{\widetilde{A}} = \lambda_1\underset{\otimes}{\widetilde{A}} + \lambda_2\underset{\otimes}{\widetilde{A}} \tag{18}$$

3.3. The Distance between the Two-Interval Grey Triangular Fuzzy Numbers

Definition 7. *Let $\underset{\otimes_1}{\widetilde{A}}$ and $\underset{\otimes_2}{\widetilde{A}}$ be any two-interval grey triangular fuzzy numbers. Y is the set of the interval grey triangular fuzzy numbers, d be the mapping, $d: Y \times Y \to R$. If $d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}\right)$, and satisfied the following formula:*

(1) $d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}\right) \geq 0$, $d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_1}{\widetilde{A}}\right) = 0$

(2) $d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}\right) = d\left(\underset{\otimes_2}{\widetilde{A}}, \underset{\otimes_1}{\widetilde{A}}\right)$;

(3) $d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}\right) + d\left(\underset{\otimes_2}{\widetilde{A}}, \underset{\otimes_3}{\widetilde{A}}\right) \geq d\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_3}{\widetilde{A}}\right)$

Then $d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}\right)$ *is called the distance between the interval grey triangular fuzzy numbers.*

Definition 8. *Let* $\underset{\otimes 1}{\widetilde{A}} = \left((f_1^L, f_1^M, f_1^U), [g_1^L, g_1^U]\right)$ *and* $\underset{\otimes 2}{\widetilde{A}} = \left((f_2^L, f_2^M, f_2^U), [g_2^L, g_2^U]\right)$ be any two-interval grey triangular fuzzy numbers variables, then the distance between $\underset{\otimes 1}{\widetilde{A}}$ *and* $\underset{\otimes 2}{\widetilde{A}}$ *is defined as follows*:

$$\begin{aligned} d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}\right) = \ & \tfrac{1}{6}\Big(\left|(1-g_1^L)f_1^L - (1-g_2^L)f_2^L\right| \\ & +\left|(1-g_1^U)f_1^L - (1-g_2^U)f_2^L\right| \\ & +\left|(1-g_1^L)f_1^M - (1-g_2^L)f_2^M\right| \\ & +\left|(1-g_1^U)f_1^M - (1-g_2^U)f_2^M\right| \\ & +\left|(1-g_1^L)f_1^U - (1-g_2^L)f_2^U\right| \\ & +\left|(1-g_1^U)f_1^U - (1-g_2^U)f_2^U\right|\Big) \end{aligned} \tag{19}$$

The distance measure between the interval grey triangular fuzzy numbers satisfies the following properties:

(1) Let $\underset{\otimes 1}{\widetilde{A}}$ and $\underset{\otimes 2}{\widetilde{A}}$ be the two-interval grey triangular fuzzy numbers. When $d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}\right)$ tend to 0, $\underset{\otimes 1}{\widetilde{A}}$ close to $\underset{\otimes 2}{\widetilde{A}}$, when $d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}\right) = 0$, $\underset{\otimes 1}{\widetilde{A}} = \underset{\otimes 2}{\widetilde{A}}$.

(2) Let $\underset{\otimes 1}{\widetilde{A}}$, $\underset{\otimes 2}{\widetilde{A}}$ and $\underset{\otimes 3}{\widetilde{A}}$ be the three interval grey triangular fuzzy numbers. The necessary and sufficient condition for $\underset{\otimes 1}{\widetilde{A}}$ to be closer to $\underset{\otimes 3}{\widetilde{A}}$ than $\underset{\otimes 2}{\widetilde{A}}$ is that $d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 3}{\widetilde{A}}\right) < d\left(\underset{\otimes 2}{\widetilde{A}}, \underset{\otimes 3}{\widetilde{A}}\right)$. If $d\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 3}{\widetilde{A}}\right) = d\left(\underset{\otimes 2}{\widetilde{A}}, \underset{\otimes 3}{\widetilde{A}}\right)$, the distance between $\underset{\otimes 1}{\widetilde{A}}$ and $\underset{\otimes 3}{\widetilde{A}}$ is equal to the distance between $\underset{\otimes 2}{\widetilde{A}}$ and $\underset{\otimes 3}{\widetilde{A}}$.

3.4. The Comparing Method of Interval Grey Triangular Fuzzy Numbers

Definition 9. *Let* $\underset{\otimes}{\widetilde{A}} = \left((f_A^L, f_A^M, f_A^U), [g_A^L, g_A^U]\right)$ *be an interval grey triangular fuzzy numbers variable, then the expectation of interval grey triangular fuzzy numbers is defined as below:*

$$I\left(\underset{\otimes}{\widetilde{A}}\right) = \frac{1}{2(r+1)}\left(rf^L + (1+r)f^M + f^U\right) \times \frac{(1-g_A^U) + r(1-g_A^L)}{r+1} \tag{20}$$

Let $\underset{\otimes 1}{\widetilde{A}} = \left((f_1^L, f_1^M, f_1^U), [g_1^L, g_1^U]\right)$ and $\underset{\otimes 2}{\widetilde{A}} = \left((f_2^L, f_2^M, f_2^U), [g_2^L, g_2^U]\right)$ be any two-interval grey triangular fuzzy numbers variables, We can decide by comparing the expected size of $\underset{\otimes 1}{\widetilde{A}}$ and $\underset{\otimes 2}{\widetilde{A}}$, if $I\left(\underset{\otimes 1}{\widetilde{A}}\right) \geq I\left(\underset{\otimes 2}{\widetilde{A}}\right)$, then $\underset{\otimes 1}{\widetilde{A}} \geq \underset{\otimes 2}{\widetilde{A}}$, or vice versa.

The comparison method of a grey part was proposed by Jin and Liu [16], the comparison method of fuzzy part reference interval number [42], and the solution process of the comparison method of triangular fuzzy number is as follows:

Let $\widetilde{f} = (f^L, f^M, f^R)$ be the triangular fuzzy number, then the $\alpha-$ cut set of $\widetilde{f}$ is defined as $\widetilde{f^\alpha} = \{x \big| \mu(x) \geq \alpha\}, \alpha \in [0,1]$, $\widetilde{f^\alpha}$ is closed interval $\left[\widetilde{f_L^\alpha}, \widetilde{f_U^\alpha}\right]$, then $\widetilde{f_L^\alpha} = (f^M - f^L)\alpha + f^L$, and $\widetilde{f_U^\alpha} = (f^M - f^U)\alpha + f^U$.

Let $\widetilde{f} = (f^L, f^M, f^R)$ be the triangular fuzzy number, then:

$$\phi_\rho(f^L, f^M, f^R) = \int_0^1 \int_0^1 \frac{d\rho}{dy}\left(\widetilde{f_U^\alpha} - y\left(\widetilde{f_U^\alpha} - \widetilde{f_L^\alpha}\right)\right) dy d\alpha \tag{21}$$

Then ϕ is called the fuzzy OWA (F-OWA) operator.

Specially, when $\rho(y) = y^r$ (r > 0), then:

$$\begin{aligned} \phi_\rho(f^L, f^M, f^R) &= \tfrac{1}{r+1}\Big(\int_0^1 \widetilde{f}_U^\alpha d\alpha + r\int_0^1 \widetilde{f}_L^\alpha d\alpha\Big) \\ &= \tfrac{1}{r+1}\Big(\int_0^1 ((f^M - f^U)\alpha + f^U)d\alpha + r\int_0^1 ((f^M - f^L)\alpha + f^L)d\alpha\Big) \\ &= \tfrac{1}{2(r+1)}\Big(rf^L + (1+r)f^M + f^U\Big) \end{aligned} \tag{22}$$

Aggregate the fuzzy part of $\underset{\otimes}{\widetilde{A}}$ by F-OWA operator, transform the triangular fuzzy number (f_A^L, f_A^M, f_A^U) into real numbers that calculating the value of $\phi_\rho(f^L, f^M, f^R)$.

4. The Interval Grey Triangular Fuzzy Numbers Partitioned Bonferroni Mean Operators and Interval Grey Triangular Fuzzy Numbers Weighted Partitioned Bonferroni Mean Operators

In this section, we applied the PBM into the fuzzy and grey information and proposed the IGTFPBM and IGTFWPBM operators.

4.1. The Interval Grey Triangular Fuzzy Numbers Partitioned Bonferroni Mean Operators

Definition 10. *Let* $\underset{\otimes k}{\widetilde{A}} = \big((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\big)$ *($k = 1, 2, \cdots, n$) be a group of interval grey triangular fuzzy numbers. For any $p, q \geq 0$, the IGTFPBM:* $\Omega^n \to \Omega$, if

$$\text{IGTFPBM}^{p.q}\Big(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\Big) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h \\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}} \tag{23}$$

where Ω is the set of all interval grey triangular fuzzy numbers variables and $|P_h|$ denotes the cardinality of P_h.

Theorem 1. *Let* $\underset{\otimes k}{\widetilde{A}} = \big((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\big)$ *($k = 1, 2, \cdots, n$) be a collection of the interval grey triangular fuzzy number, then the result aggregated by IGTFPBM operators is still an interval grey triangular fuzzy numbers variable, and also*

$$\begin{aligned} &\text{IGTFPBM}^{p.q}\Big(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\Big) = \\ &\left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^L\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h \\ j\neq i}}\left(f_j^L\right)^q\right)\right)^{\frac{1}{p+q}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^M\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h \\ j\neq i}}\left(f_j^M\right)^q\right)\right)^{\frac{1}{p+q}},\right.\right. \\ &\left.\left.\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^U\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h \\ j\neq i}}\left(f_j^U\right)^q\right)\right)^{\frac{1}{p+q}}\right], \left[\max_{\substack{i,j \\ j\neq i}}(g_i^L, g_j^L), \max_{\substack{i,j \\ j\neq i}}(g_i^U, g_j^U)\right]\right) \end{aligned} \tag{24}$$

Proof. Because of the operational rules of the interval grey triangular fuzzy numbers variables, we have:

$$\underset{\otimes i}{\widetilde{A}}^p = \Big((f_i^L)^p, (f_i^M)^p, (f_i^U)^p, [g_i^L, g_i^U]\Big), \underset{\otimes j}{\widetilde{A}}^q = \Big((f_j^L)^q, (f_j^M)^q, (f_j^U)^q, [g_j^L, g_j^U]\Big)$$

and:

$$\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)^q=\left(\left[\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}(f_j^L)^q,\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}(f_j^M)^q,\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}(f_j^U)^q\right],\left[\max_j(g_j^L),\max_j(g_j^U)\right]\right)$$

Then:

$$\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p\times\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)=$$

$$\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^L\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^L\right)^q\right),\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^M\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^M\right)^q\right)\right.$$

$$\left.\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^U\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^U\right)^q\right)\right),\left[\max_{\substack{i,j\\ j\neq i}}(g_i^L,g_j^L),\max_{\substack{i,j\\ j\neq i}}(g_i^U,g_j^U)\right]\right)$$

and:

$$\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p\times\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}}=$$

$$\left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^L\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^L\right)^q\right)\right)^{\frac{1}{p+q}},\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^M\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^M\right)^q\right)\right)^{\frac{1}{p+q}},\right.\right.$$

$$\left.\left.\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^U\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(f_j^U\right)^q\right)\right)^{\frac{1}{p+q}}\right],\left[\max_{\substack{i,j\\ j\neq i}}(g_i^L,g_j^L),\max_{\substack{i,j\\ j\neq i}}(g_i^U,g_j^U)\right]\right)$$

which completes the proof. □

Theorem 2. *(Idempotency): Let* $p, q \geq 0$, $\underset{\otimes_k}{\widetilde{A}} = \left((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\right)$, $\underset{\otimes}{\widetilde{A}} = \left((f^L, f^M, f^U), [g^L, g^U]\right)$*, and all* $\underset{\otimes_k}{\widetilde{A}}$ $(k = 1, 2, \cdots, n)$ *is equal, i.e.,* $\underset{\otimes_k}{\widetilde{A}} = \underset{\otimes}{\widetilde{A}}$*,then,*

$$\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}, \cdots, \underset{\otimes_n}{\widetilde{A}}\right) = \underset{\otimes}{\widetilde{A}} \tag{25}$$

Proof. Since $\underset{\otimes_k}{\widetilde{A}} = \underset{\otimes}{\widetilde{A}}$, then:

$$\begin{aligned}
&\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}, \cdots, \underset{\otimes_n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes}{\widetilde{A}}\right)^p \times \left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}} \left(\underset{\otimes}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}} \\
&= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}} \left(f^L\right)^{p+q}\right)^{\frac{1}{p+q}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}} \left(f^M\right)^{p+q}\right)^{\frac{1}{p+q}},\right.\right. \\
&\left.\left.\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}} \left(f^U\right)^{p+q}\right)^{\frac{1}{p+q}}\right], [g^L, g^U]\right) \\
&= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\left(f^L\right)^{p+q}\right)^{\frac{1}{p+q}}, \frac{1}{e}\sum_{h=1}^{e}\left(\left(f^M\right)^{p+q}\right)^{\frac{1}{p+q}}, \frac{1}{e}\sum_{h=1}^{e}\left(\left(f^U\right)^{p+q}\right)^{\frac{1}{p+q}}\right], [g^L, g^U]\right) \\
&= \left((f^L, f^M, f^U), [g^L, g^U]\right) = \underset{\otimes}{\widetilde{A}}
\end{aligned}$$

which completes the proof. □

Theorem 3. *(Commutativity) Let* $\left(\underset{\otimes_1}{\widetilde{A}'}, \underset{\otimes_2}{\widetilde{A}'}, \cdots, \underset{\otimes_n}{\widetilde{A}'}\right)$ *be any permutation of* $\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}, \cdots, \underset{\otimes_n}{\widetilde{A}}\right)$*, then:*

$$\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes_1}{\widetilde{A}}', \underset{\otimes_2}{\widetilde{A}}', \cdots, \underset{\otimes_n}{\widetilde{A}}'\right) = \mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}, \cdots, \underset{\otimes_n}{\widetilde{A}}\right) \tag{26}$$

Proof. Since $\left(\underset{\otimes_1}{\widetilde{A}'}, \underset{\otimes_2}{\widetilde{A}'}, \cdots, \underset{\otimes_n}{\widetilde{A}'}\right)$ is any permutation of $\left(\underset{\otimes_1}{\widetilde{A}}, \underset{\otimes_2}{\widetilde{A}}, \cdots, \underset{\otimes_n}{\widetilde{A}}\right)$, then:

$$\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes_1}{\widetilde{A}'}, \underset{\otimes_2}{\widetilde{A}'}, \cdots, \underset{\otimes_n}{\widetilde{A}'}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes_i}{\widetilde{A}'}\right)^p \times \left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}} \left(\underset{\otimes_j}{\widetilde{A}'}\right)^q\right)\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}} \left(f_i'^L\right)^p\left(f_j'^L\right)^q\right)^{\frac{1}{p+q}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}} \left(f_i'^M\right)^p\left(f_j'^M\right)^q\right)^{\frac{1}{p+q}},\right.\right.$$

$$\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i'^U\right)^p\left(f_j'^U\right)^q\right)^{\frac{1}{p+q}}\Bigg],\left[\max_{\substack{i,j\\ j\neq i}}\left(g_i'^L,g_j'^L\right),\ \max_{\substack{i,j\\ j\neq i}}\left(g_i'^U,g_j'^U\right)\right]\Bigg)$$

$$=\left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^L\right)^p\left(f_j^L\right)^q\right)^{\frac{1}{p+q}},\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^M\right)^p\left(f_j^M\right)^q\right)^{\frac{1}{p+q}},\right.\right.$$

$$\left.\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^U\right)^p\left(f_j^U\right)^q\right)^{\frac{1}{p+q}}\right],\left[\max_{\substack{i,j\\ j\neq i}}\left(g_i^L,g_j^L\right),\ \max_{\substack{i,j\\ j\neq i}}\left(g_i^U,g_j^U\right)\right]\Bigg)$$

$$=\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p\times\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}}=\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}},\underset{\otimes 2}{\widetilde{A}},\cdots,\underset{\otimes n}{\widetilde{A}}\right)$$

which completes the proof. □

Theorem 4. (Monotonicity) If $\underset{\otimes k}{\widetilde{A}}\leq\underset{\otimes k}{\widetilde{A}}'$ $(k=1,2,\cdots,n)$, then:

$$\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}},\underset{\otimes 2}{\widetilde{A}},\cdots,\underset{\otimes n}{\widetilde{A}}\right)\leq\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}}',\underset{\otimes 2}{\widetilde{A}}',\cdots,\underset{\otimes n}{\widetilde{A}}'\right)\tag{27}$$

Proof.

$$\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}},\underset{\otimes 2}{\widetilde{A}},\cdots,\underset{\otimes n}{\widetilde{A}}\right)=\frac{1}{e}\left(\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ i\neq j}}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p\times\left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}}\right)$$

$$\leq\frac{1}{e}\left(\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ i\neq j}}\left(\left(\underset{\otimes i}{\widetilde{A}}'\right)^p\times\left(\underset{\otimes j}{\widetilde{A}}'\right)^q\right)\right)^{\frac{1}{p+q}}\right)=\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}}',\underset{\otimes 2}{\widetilde{A}}',\cdots,\underset{\otimes n}{\widetilde{A}}'\right)$$

□

Theorem 5. *(Symmetry) Let $p, q \geq 0$, $\underset{\otimes k}{\widetilde{A}} = \left((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\right)$ $(k = 1, 2, \cdots, n)$ be a collection of the interval grey triangular fuzzy number, then*

$$\mathrm{IGTFPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \mathrm{IGTFPBM}^{q,p}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) \tag{28}$$

Proof.

$$\mathrm{IGTFPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\left(\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)} \sum_{\substack{i, j \in P_h \\ i \neq j}} \left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}}\right)$$

$$= \frac{1}{e}\left(\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)} \sum_{\substack{i, j \in P_h \\ i \neq j}} \left(\left(\underset{\otimes i}{\widetilde{A}}\right)^q \times \left(\underset{\otimes j}{\widetilde{A}}\right)^p\right)\right)^{\frac{1}{p+q}}\right) = \mathrm{IGTFPBM}^{q,p}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right)$$

when $e = 1$ and $P_1 = n$, then, the IGTFPBM operators reduce to interval grey triangular fuzzy numbers Bonferroni mean (IGTFBM).

$$\begin{aligned} &\mathrm{IGTFPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \left(\frac{1}{|P_1|} \sum_{i \in P_1}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\frac{1}{|P_1|-1} \sum_{\substack{j \in P_1 \\ j \neq i}} \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}} \\ &= \left(\frac{1}{n} \sum_{i=1}^{n}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\frac{1}{n-1} \sum_{\substack{j = 1 \\ j \neq i}}^{n} \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \sum_{i=1}^{n}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \sum_{\substack{j = 1 \\ j \neq i}}^{n} \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}} \\ &= \mathrm{IGTFBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) \end{aligned} \tag{29}$$

We gave some special cases of the IGTFPBM operators as follows:
Case 1. If $q = 0$, then, Equations (23) and (24) are transformed as follows:

$$\begin{aligned} &\mathrm{IGTFPBM}^{p,0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e} \sum_{h=1}^{e}\left(\frac{1}{|P_h|} \sum_{i \in P_h} \left(\underset{\otimes i}{\widetilde{A}}\right)^p\right)^{\frac{1}{p}} \\ &= \left(\left[\frac{1}{e} \sum_{h=1}^{e}\left(\frac{1}{|P_h|} \sum_{i \in P_h} \left(f_i^L\right)^p\right)^{\frac{1}{p}}, \frac{1}{e} \sum_{h=1}^{e}\left(\frac{1}{|P_h|} \sum_{i \in P_h} \left(f_i^M\right)^p\right)^{\frac{1}{p}}, \frac{1}{e} \sum_{h=1}^{e}\left(\frac{1}{|P_h|} \sum_{i \in P_h} \left(f_i^U\right)^p\right)^{\frac{1}{p}}\right], \right. \\ &\left. \left[\max_i(g_i^L), \max_i(g_i^U)\right] \right) \end{aligned} \tag{30}$$

Case 2. If $p = 1$ and $q = 0$, then, Equations (23) and (24) are transformed as follows:

$$\begin{aligned}
&\mathrm{IGTFPBM}^{1,0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\underset{\otimes i}{\widetilde{A}}\right)\right) \\
&= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^L\right)\right), \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^M\right)\right), \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^U\right)\right)\right],\right. \\
&\left.\left[\max_i(g_i^L), \max_i(g_i^U)\right]\right)
\end{aligned} \tag{31}$$

Case 3. If $p = 2$ and $q = 0$, then, Equations (23) and (24) are transformed as follows:

$$\begin{aligned}
&\mathrm{IGTFPBM}^{2,0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\underset{\otimes i}{\widetilde{A}}\right)^2\right)^{\frac{1}{2}} \\
&= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^L\right)^2\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^M\right)^2\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(f_i^U\right)^2\right)^{\frac{1}{2}}\right],\right. \\
&\left.\left[\max_i(g_i^L), \max_i(g_i^U)\right]\right)
\end{aligned} \tag{32}$$

Case 4. If $p = 1$ and $q = 1$, then, Equations (23) and (24) are transformed as follows:

$$\begin{aligned}
&\mathrm{IGTFPBM}^{1,1}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)\times\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\underset{\otimes j}{\widetilde{A}}\right)\right)\right)\right)^{\frac{1}{2}} \\
&= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^L f_j^L\right)\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^M f_j^M\right)\right)^{\frac{1}{2}},\right.\right. \\
&\left.\left.\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(f_i^U f_j^U\right)\right)^{\frac{1}{2}}\right], \left[\max_{\substack{i,j\\ j\neq i}}\left(g_i^L, g_j^L\right), \max_{\substack{i,j\\ j\neq i}}\left(g_i^U, g_j^U\right)\right]\right)
\end{aligned} \tag{33}$$

□

4.2. The Interval Grey Triangular Fuzzy Numbers Weighted Partitioned Bonferroni Mean Operator

IGTFPBM can capture the interrelationship, that operator assumes that the attributes are equally important, however, this is not in line with fact. Therefore, we establish the IGTFWPBM considering the different attributes have different importance and then, we gave the related properties of new operators.

Definition 11. *Let* $\underset{\otimes k}{\widetilde{A}} = \left((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\right)$ *($k = 1, 2, \cdots, n$) be a group of the interval grey triangular fuzzy numbers variables. For any $p, q \geq 0$, the IGTFWPBM: $\Omega^n \rightarrow \Omega$, if*

$$\mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\left(\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i \underset{\otimes i}{\widetilde{A}}\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}}\left(\omega_j \underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}} \tag{34}$$

where Ω is the set of all interval grey triangular fuzzy numbers, and $\omega_i (i = 1, 2, \cdots, n)$ represents the importance of each attribute variable and satisfies the conditions: $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$. Then WIGTFPBM called WIGTFPBM aggregation operator.

Theorem 5. *Let* $\underset{\otimes k}{\widetilde{A}} = \left((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U]\right)$ *($k = 1, 2, \cdots, n$) be a collection of the interval grey triangular fuzzy numbers variables, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\underset{\otimes k}{\widetilde{A}}$ ($k = 1, 2, \cdots, n$), $\omega_i \geq 0$, $\sum_{i=1}^{n} \omega_i = 1$, then, the result is still an interval grey triangular fuzzy numbers variable, and also:*

$$\begin{aligned}
&\mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \left(\left[\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^L\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}}\left(\omega_j f_j^L\right)^q\right)\right)\right)^{\frac{1}{p+q}},\right.\right. \\
&\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^M\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}}\left(\omega_j f_j^M\right)^q\right)\right)\right)^{\frac{1}{p+q}}, \\
&\left.\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^U\right)^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}}\left(\omega_j f_j^U\right)^q\right)\right)\right)^{\frac{1}{p+q}}\right], \left.\left[\max_{\substack{i,j \\ j\neq i}}\left(g_i^L, g_j^L\right), \max_{\substack{i,j \\ j\neq i}}\left(g_i^U, g_j^U\right)\right]\right)
\end{aligned} \tag{35}$$

The proof of this theorem is similar to that of Theorem 1, so we left it out here.

Theorem 6. *(Commutativity) Let* $\left(\underset{\otimes 1}{\widetilde{A}'}, \underset{\otimes 2}{\widetilde{A}'}, \cdots, \underset{\otimes n}{\widetilde{A}'}\right)$ *be any permutation of* $\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right)$*, then*

$$\mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}'}, \underset{\otimes 2}{\widetilde{A}'}, \cdots, \underset{\otimes n}{\widetilde{A}'}\right) = \mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) \tag{36}$$

The proof was omitted here, please refer to the proof of Theorem 3.

Theorem 7. (Monotonicity) If $\underset{\otimes k}{\widetilde{A}} \leq \underset{\otimes k}{\widetilde{A}}'$ ($k = 1, 2, \cdots, n$), then:

$$\mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) \leq \mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes 1}{\widetilde{A}'}, \underset{\otimes 2}{\widetilde{A}'}, \cdots, \underset{\otimes n}{\widetilde{A}'}\right) \tag{37}$$

The proof is omitted here, please refer to the proof of Theorem 4.

Theorem 8. *(Symmetry) Let* $p, q \geq 0$, $\underset{\otimes k}{\widetilde{A}} = \left((f_k^L, f_k^M, f_k^U), [g_k^L, g_k^U] \right)$ $(k = 1, 2, \cdots, n)$ *be a collection of the interval grey triangular fuzzy number, then:*

$$\mathrm{IGTFWPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \mathrm{IGTFWPBM}^{q.p}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) \tag{38}$$

The proof is omitted here, please refer to the proof of Theorem 5.

In particular, the IGTFPBM operator is one special case of IGTFWPBM operator, the conversion relationship between IGTFPBM operator and IGTFWPBM operator is shown as follows:

$$\mathrm{IGTFWPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i \in P_h}\left(\left(\frac{1}{n}\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\frac{1}{|P_h|-1}\sum_{\substack{j \in P_h \\ j \neq i}}\left(\frac{1}{n}\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)\right)^{\frac{1}{p+q}}$$

$$= \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}}\left(\left(\frac{1}{n}\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\frac{1}{n}\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}}$$

$$= \frac{1}{e}\sum_{h=1}^{e}\left(\left(\frac{1}{n}\right)^{p+q} \times \frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}}$$

$$= \frac{1}{ne}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i, j \in P_h \\ j \neq i}}\left(\left(\underset{\otimes i}{\widetilde{A}}\right)^p \times \left(\underset{\otimes j}{\widetilde{A}}\right)^q\right)\right)^{\frac{1}{p+q}} = \frac{1}{n}\mathrm{IGTFPBM}^{p.q}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right)$$

And then, we discussed some special cases of IGTFWPBM as follows:
Case 1. If $q = 0$, then, Equation (34) and Equation (35) are transformed as follows:

$$\mathrm{IGTFPBM}^{p.0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i \in P_h}\left(\omega_i \underset{\otimes i}{\widetilde{A}}\right)^p\right)^{\frac{1}{p}}$$
$$= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i \in P_h}\left(\omega_i f_i^L\right)^p\right)^{\frac{1}{p}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i \in P_h}\left(\omega_i f_i^M\right)^p\right)^{\frac{1}{p}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i \in P_h}\left(\omega_i f_i^U\right)^p\right)^{\frac{1}{p}}\right], \left[\max_i(g_i^L), \max_i(g_i^U)\right]\right)$$

Case 2. If $p = 1$ and $q = 0$, then, Equations (34) and (35) are transformed as follows:

$$\mathrm{IGTFPBM}^{1.0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i \underset{\otimes i}{\widetilde{A}}\right)\right)$$
$$= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^L\right)\right), \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^M\right)\right), \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^U\right)\right)\right], \left[\max_i(g_i^L), \max_i(g_i^U)\right]\right)$$

Case 3. If $p = 2$ and $q = 0$, then, Equations (34) and (35) are transformed as follows:

$$\mathrm{IGTFPBM}^{2.0}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i \underset{\otimes i}{\widetilde{A}}\right)^2\right)^{\frac{1}{2}}$$
$$= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^L\right)^2\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^M\right)^2\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\omega_i f_i^U\right)^2\right)^{\frac{1}{2}}\right], \left[\max_i(g_i^L), \max_i(g_i^U)\right]\right)$$

Case 4. If $p = 1$ and $q = 1$, then, Equations (34) and (35) are transformed as follows:

$$\mathrm{IGTFPBM}^{1.1}\left(\underset{\otimes 1}{\widetilde{A}}, \underset{\otimes 2}{\widetilde{A}}, \cdots, \underset{\otimes n}{\widetilde{A}}\right) = \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{i\in P_h}\left(\left(\omega_i \underset{\otimes i}{\widetilde{A}}\right)\times\left(\frac{1}{|P_h|-1}\sum_{\substack{j\in P_h\\ j\neq i}}\left(\omega_j \underset{\otimes j}{\widetilde{A}}\right)\right)\right)\right)^{\frac{1}{2}}$$
$$= \left(\left[\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(\omega_i f_i^L \omega_j f_j^L\right)\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(\omega_i f_i^M \omega_j f_j^M\right)\right)^{\frac{1}{2}}, \frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|(|P_h|-1)}\sum_{\substack{i,j\in P_h\\ j\neq i}}\left(\omega_i f_i^U \omega_j f_j^U\right)\right)^{\frac{1}{2}}\right], \left[\max_{\substack{i,j\\ j\neq i}}(g_i^L, g_j^L), \max_{\substack{i,j\\ j\neq i}}(g_i^U, g_j^U)\right]\right)$$

5. A GFMAGDM Method Based on IGTFWPBM

In this section, we introduced a method to solve the GFMAGDM problems in the context of grey and fuzzy information. In real decision-making, the group of decision-makers $\{D_1, D_2, \ldots D_P\}$ needs to evaluate the optimal decision from the set of m alternatives $X = \{X_1, X_2, \ldots X_m\}$ based on the n attributes $G = \{G_1, G_2, \ldots G_n\}$. Because some attributes may be interrelated with other attributes, we divided n attributes into e classes $P = \{P_1, P_2, \ldots P_e\}$, and ensured that each attribute is only interrelated with other attributes in the same class. The weights of the attributes can be expressed by ω_j,$\omega_j \geq 0$, $\sum_{j=1}^{n}\omega_j = 1$. The weights of the decision-makers can be expressed by γ_k,$\gamma_k \geq 0$ and $\sum_{k=1}^{n}\gamma_k = 1$. Suppose that $\underset{\otimes}{\widetilde{R}}^k = \left[\underset{\otimes ij}{\widetilde{A}}^k\right]_{m\times n}$ is the decision matrix where $\underset{\otimes ij}{\widetilde{A}}^k = \left((f_{ijk}^L, f_{ijk}^M, f_{ijk}^U), [g_{ijk}^L, g_{ijk}^U]\right)(k = 1, 2, \cdots, p)$ represents the structure of the interval grey triangular fuzzy numbers, which expresses that the decision maker D_k evaluate the G_j attribute of the X_i attribute. We collected the above information about each alternative and ranked the order of alternatives. And the specific steps are as follows:

Step 1. Normalizing the interval grey triangular fuzzy numbers.

In general, attributes have directionality, and we changed some attributes' direction to ensure that benefit attributes and cost attributes have the same direction. Meanwhile, we should normalize the attributes because of the differences in evaluation standards from different decision-makers.

Suppose $\underset{\otimes}{\widetilde{V}}^k = \left(\underset{\otimes ij}{\widetilde{B}}^k\right)_{m\times n}$ is the normalized matrix of decision matrix $\underset{\otimes}{\widetilde{R}}^k = \left[\underset{\otimes ij}{\widetilde{A}}^k\right]_{m\times n}$ where $\underset{\otimes ij}{\widetilde{B}}^k = \left((f_{ijk}^{L}{}', f_{ijk}^{M}{}', f_{ijk}^{U}{}'), [g_{ijk}^L, g_{ijk}^U]\right)$, and the normalization method is as follows:

(1) For benefit attributes:

$$(f_{ijk}^{L}{}', f_{ijk}^{M}{}', f_{ijk}^{U}{}') = \left(\frac{f_{ijk}^L}{f_k^U}, \frac{f_{ijk}^M}{f_k^U}, \frac{f_{ijk}^U}{f_k^U}\right) \tag{39}$$

where $f_k^U = \max\limits_{i,j}(f_{ijk}^U)$.

(2) For cost attributes:

$$(f_{ijk}^{L}{}', f_{ijk}^{M}{}', f_{ijk}^{U}{}') = \left(\frac{f_k^L}{f_{ijk}^L}, \frac{f_k^L}{f_{ijk}^M}, \frac{f_k^L}{f_{ijk}^U}\right) \tag{40}$$

where $f_k^L = \min\limits_{i,j}(f_{ijk}^L)$.

Step 2. Aggregating the information from decision-makers.

Suppose $\underset{\otimes}{\widetilde{U}} = \left(\underset{\otimes ij}{\widetilde{C}}\right)_{m\times n}$ is the group decision matrix by aggregating from normalized matrix $\underset{\otimes}{\widetilde{V}}^k = \left(\underset{\otimes ij}{\widetilde{B}}^k\right)_{m\times n}$ by weighting method. Where $\underset{\otimes ij}{\widetilde{C}} = \left((f_{ij}^{L}{}'', f_{ij}^{M}{}'', f_{ij}^{U}{}''), [g_{ij}^L, g_{ij}^U]\right)$, as follows:

$$(f_{ij}^{L}{}'', f_{ij}^{M}{}'', f_{ij}^{U}{}'') = \left(\sum_{k=1}^{p}\gamma_k f_{ijk}^{L}{}', \sum_{k=1}^{p}\gamma_k f_{ijk}^{M}{}', \sum_{k=1}^{p}\gamma_k f_{ijk}^{U}{}'\right) \tag{41}$$

Step 3. Calculating the comprehensive evaluation value.

Calculating each alternative's evaluation value by IGTFWPBM operators. Where $\underset{\otimes i}{\widetilde{C}} = \left((f_i^L, f_i^M, f_i^U), [g_i^L, g_i^U]\right)$ as follows:

$$
\begin{aligned}
&(f_i^L, f_i^M, f_i^U) = \mathrm{IGTFWPBM}^{p,q}\left(\underset{\otimes i1}{\widetilde{C}}, \underset{\otimes i2}{\widetilde{C}}, \cdots, \underset{\otimes in}{\widetilde{C}}\right) \\
&= \left(\left[\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{j_1 \in P_h}(\omega_{j_1} f_{ij_1}^{L}{}'')^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j_2 \in P_h \\ j_2 \neq j_1}}(\omega_{j_2} f_{ij_2}^{L}{}'')^q\right)\right)\right)^{\frac{1}{p+q}},\right.\right. \\
&\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{j_1 \in P_h}(\omega_{j_1} f_{ij_1}^{M}{}'')^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j_2 \in P_h \\ j_2 \neq j_1}}(\omega_{j_2} f_{ij_2}^{M}{}'')^q\right)\right)\right)^{\frac{1}{p+q}}, \\
&\left.\left(\frac{1}{e}\sum_{h=1}^{e}\left(\frac{1}{|P_h|}\sum_{j_1 \in P_h}(\omega_{j_1} f_{ij_1}^{U}{}'')^p\left(\frac{1}{|P_h|-1}\sum_{\substack{j_2 \in P_h \\ j_2 \neq j_1}}(\omega_{j_2} f_{ij_2}^{U}{}'')^q\right)\right)\right)^{\frac{1}{p+q}}\right], \left.\left[\max_{\substack{j_1, j_2 \\ j_2 \neq j_1}}(g_{ij_1}^L, g_{ij_2}^L), \max_{\substack{j_1, j_2 \\ j_2 \neq j_1}}(g_{ij_1}^U, g_{ij_2}^U)\right]\right)
\end{aligned}
\tag{42}
$$

Step 4. Calculating the expectation for the alternative's IGTFWPBM operator.

$$
I\left(\underset{\otimes i}{\widetilde{C}}\right) = \frac{1}{2(r+1)}\left(rf_i^L + (1+r)f_i^M + f_i^U\right) \times \frac{(1-g_i^U) + r(1-g_i^L)}{r+1} \tag{43}
$$

Step 5. Ranking the alternatives.

Ranking the alternatives based on the expectation value $I\left(\underset{\otimes i}{\widetilde{C}}\right)$ and selecting the best alternative.

6. Illustrative Examples

In this section, we gave an example to explain the application of IGTFWPBM operators in solving GFMGADM problems in Section 6.1. In Section 6.2, the variation of parameters on the evaluation results and the symmetry effect with the expectations are discussed. Section 6.3 and 6.4 analyze the effectiveness and advantages of IGTFWPBM operators by adding examples.

6.1. Application of IGTFWPBM Operators

Example 1. *A financial institution needs to assess the credit ratings of four small and micro enterprises and make loans on the best enterprise. Generally speaking, financial institutions provide credit ratings based on the investigation from the financial information and the no-financial information of small and micro-enterprises. The financial information is acquired by collecting financial data, on-spot verification, visiting the related business and other means. For the moment, the four small and micro enterprises have the similar financial information and little to rank the order of them. Thus, the financial institution set a team of three experts with the weight vector* $\gamma = (0.4, 0.35, 0.25)^T$ *to conduct the field survey on the no-financial information of each enterprise and grade the following aspects.*

G_1: The leadership qualities
G_2: The employees' qualities
G_3: The contract performance
G_4: The industry situation

G_5: The product competitiveness

In the process of actual research, it is difficult to get comprehensive information on each small and micro enterprise, therefore, the assessment process has greyness. Because the peoples' thinking is subjective and complicated, the grading process exhibits fuzziness. Considering the interrelationship among the above five aspects, we part them into two classes. The first class is composed of G_1,G_2, and G_3 which shows the enterprise strength, and the second class is composed of G_4 and G_5 which expresses the external environment of the enterprise. The attribute set: $P_1 = \{G_1, G_2, G_3\}$ and $P_2 = \{G_4, G_5\}$ with the weight vector $\omega = (0.2, 0.25, 0.1, 0.25, 0.2)^T$.

The decision process for the four companies is as follows:

The three decision matrixes as Tables 1–3.

Table 1. Decision matrix $\underset{\otimes}{\widetilde{R}}^1$.

	A_1	A_2	A_3	A_4	A_5
X_1	$([4,5,6],[0.2,0.4])$	$([1,2,3],[0.2,0.3])$	$([4,5,6],[0.4,0.4])$	$([2,3,4],[0.5,0.5])$	$([4,5,6],[0.2,0.4])$
X_2	$([3,4,5],[0.1,0.2])$	$([4,5,6],[0.5,0.5])$	$([2,3,4],[0.4,0.5])$	$([3,4,5],[0.4,0.4])$	$([4,5,6],[0.3,0.4])$
X_3	$([2,3,4],[0.2,0.3])$	$([3,4,5],[0.3,0.3])$	$([3,4,5],[0.2,0.3])$	$([4,5,6],[0.2,0.3])$	$([2,3,4],[0.2,0.4])$
X_4	$([4,6,6],[0.3,0.4])$	$([1,2,3],[0.2,0.3])$	$([2,3,4],[0.2,0.4])$	$([2,3,4],[0.5,0.6])$	$([2,3,4],[0.3,0.5])$

Table 2. Decision matrix $\underset{\otimes}{\widetilde{R}}^2$.

	A_1	A_2	A_3	A_4	A_5
X_1	$([3,4,5],[0.2,0.3])$	$([2,3,4],[0.4,0.5])$	$([2,3,4],[0.1,0.3])$	$([4,5,6],[0.2,0.2])$	$([2,3,4],[0.3,0.4])$
X_2	$([4,5,6],[0.3,0.4])$	$([2,3,4],[0.2,0.3])$	$([3,4,5],[0.4,0.5])$	$([2,3,4],[0.2,0.4])$	$([1,2,3],[0.2,0.4])$
X_3	$([3,4,5],[0.2,0.3])$	$([3,4,5],[0.3,0.3])$	$([1,2,3],[0.2,0.4])$	$([2,3,4],[0.4,0.4])$	$([1,2,3],[0.2,0.2])$
X_4	$([4,5,6],[0.4,0.5])$	$([3,4,5],[0.2,0.4])$	$([1,2,3],[0.3,0.4])$	$([1,2,3],[0.3,0.4])$	$([3,4,5],[0.2,0.3])$

Table 3. Decision matrix $\underset{\otimes}{\widetilde{R}}^3$.

	A_1	A_2	A_3	A_4	A_5
X_1	$([4,5,6],[0.4,0.5])$	$([2,3,4],[0.2,0.3])$	$([3,4,5],[0.3,0.3])$	$([3,4,5],[0.2,0.4])$	$([3,4,5],[0.2,0.3])$
X_2	$([3,4,5],[0.1,0.2])$	$([4,5,6],[0.1,0.2])$	$([1,2,3],[0.3,0.4])$	$([2,3,4],[0.3,0.3])$	$([4,5,6],[0.2,0.4])$
X_3	$([3,4,5],[0.1,0.2])$	$([4,5,6],[0.2,0.3])$	$([0,0,1],[0.3,0.4])$	$([3,4,5],[0.2,0.3])$	$([4,5,6],[0.3,0.4])$
X_4	$([2,3,4],[0.3,0.4])$	$([2,3,4],[0.4,0.5])$	$([3,4,5],[0.1,0.3])$	$([4,5,6],[0.2,0.3])$	$([3,4,5],[0.2,0.4])$

Calculating the normalized matrixes based on Equations (40) and (41) as Tables 4–6.

Table 4. Normalized decision matrix $\underset{\otimes}{\widetilde{V}}^1$.

	A_1	A_2	A_3	A_4	A_5
X_1	$\begin{pmatrix}[0.667,0.833,1.0],\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.167,0.333,0.5],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.667,0.833,1.0],\\ [0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.4,0.5]\end{pmatrix}$
X_2	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.1,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.667,0.833,1.0],\\ [0.5,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.4,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.4,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.667,0.833,1.0],\\ [0.3,0.4]\end{pmatrix}$
X_3	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.3,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.667,0.833,1.0],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.2,0.4]\end{pmatrix}$
X_4	$\begin{pmatrix}[0.833,1.00,1.00],\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.167,0.333,0.5],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.667,0.833],\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.6,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.333,0.5,0.667],\\ [0.3,0.5]\end{pmatrix}$

Table 5. Normalized decision matrix $\underset{\otimes}{\widetilde{V}}^{2}$.

	A_1	A_2	A_3	A_4	A_5
X_1	([0.5, 0.667, 0.833], [0.2, 0.3])	([0.333, 0.5, 0.667], [0.4, 0.5])	([0.333, 0.5, 0.667], [0.1, 0.3])	([0.833, 1.00, 1.00], [0.2, 0.2])	([0.333, 0.5, 0.667], [0.3, 0.4])
X_2	([0.667, 0.833, 1.0], [0.3, 0.4])	([0.333, 0.5, 0.667], [0.2, 0.3])	([0.5, 0.667, 0.833], [0.4, 0.5])	([0.333, 0.5, 0.667], [0.2, 0.4])	([0.167, 0.333, 0.5], [0.2, 0.4])
X_3	([0.5, 0.667, 0.833], [0.2, 0.3])	([0.5, 0.667, 0.833], [0.3, 0.3])	([0.167, 0.333, 0.5], [0.2, 0.4])	([0.333, 0.5, 0.667], [0.4, 0.4])	([0.167, 0.333, 0.5], [0.2, 0.2])
X_4	([0.667, 0.833, 1.0], [0.4, 0.5])	([0.5, 0.667, 0.833], [0.2, 0.4])	([0.167, 0.333, 0.5], [0.3, 0.4])	([0.5, 0.667, 0.833], [0.3, 0.4])	([0.5, 0.667, 0.833], [0.2, 0.3])

Table 6. Normalized decision matrix $\underset{\otimes}{\widetilde{V}}^{3}$.

	A_1	A_2	A_3	A_4	A_5
X_1	([0.667, 0.833, 1.0], [0.4, 0.5])	([0.333, 0.5, 0.667], [0.2, 0.3])	([0.5, 0.667, 0.833], [0.3, 0.3])	([0.5, 0.667, 0.833], [0.2, 0.4])	([0.5, 0.667, 0.833], [0.2, 0.3])
X_2	([0.5, 0.667, 0.833], [0.1, 0.2])	([0.667, 0.833, 1.0], [0.1, 0.2])	([0.167, 0.333, 0.5], [0.3, 0.4])	([0.333, 0.5, 0.667], [0.3, 0.3])	([0.667, 0.833, 1.0], [0.2, 0.4])
X_3	([0.5, 0.667, 0.833], [0.1, 0.2])	([0.667, 0.833, 1.0], [0.2, 0.3])	([0.00, 0.00, 0.167], [0.3, 0.4])	([0.5, 0.667, 0.833], [0.2, 0.3])	([0.667, 0.833, 1.0], [0.3, 0.4])
X_4	([0.333, 0.5, 0.667], [0.3, 0.4])	([0.333, 0.5, 0.667], [0.4, 0.5])	([0.5, 0.667, 0.833], [0.1, 0.3])	([0.667, 0.833, 1.0], [0.2, 0.3])	([0.5, 0.667, 0.833], [0.2, 0.4])

We get an aggregated matrix $\underset{\otimes}{\widetilde{U}}$ by aggregating the assessment information which is shown in Table 7:

Table 7. Aggregated matrix $\underset{\otimes}{\widetilde{U}}$.

	A_1	A_2	..	A_4	A_5
X_1	([0.617, 0.783, 0.950], [0.4, 0.5])	([0.267, 0.433, 0.600], [0.4, 0.5])	([0.517, 0.683, 0.850], [0.4, 0.4])	([0.533, 0.700, 0.817], [0.5, 0.5])	([0.450, 0.617, 0.783], [0.4, 0.5])
X_2	([0.550, 0.717, 0.883], [0.3, 0.4])	([0.567, 0.733, 0.900], [0.5, 0.5])	([0.333, 0.500, 0.667], [0.4, 0.5])	([0.399, 0.567, 0.733], [0.4, 0.4])	([0.383, 0.550, 0.717], [0.3, 0.4])
X_3	([0.433, 0.600, 0.767], [0.2, 0.3])	([0.550, 0.717, 0.883], [0.3, 0.3])	([0.250, 0.367, 0.533], [0.3, 0.4])	([0.517, 0.683, 0.850], [0.4, 0.4])	([0.517, 0.683, 0.850], [0.3, 0.4])
X_4	([0.633, 0.800, 0.900], [0.4, 0.5])	([0.550, 0.717, 0.883], [0.3, 0.3])	([0.333, 0.500, 0.667], [0.3, 0.4])	([0.483, 0.650, 0.817], [0.6, 0.6])	([0.433, 0.600, 0.767], [0.3, 0.5])

Utilizing the IGTFWPBM operators to aggregate the comprehensive information of all alternatives. Suppose $p = 1$ and $q = 1$, the computed result is as follows:

$$\underset{\otimes}{\widetilde{C}} = \begin{bmatrix} [0.0935, 0.1275, 0.1586], [0.50, 0.50] \\ [0.0885, 0.1220, 0.1555], [0.50, 0.50] \\ [0.0960, 0.1284, 0.1620], [0.40, 0.40] \\ [0.0887, 0.1226, 0.1543], [0.60, 0.60] \end{bmatrix}$$

And then, calculate the expectation of all alternatives $\underset{\otimes i}{\widetilde{c}}(i = 1, 2, \ldots, 4)$ based on Equation (43), for example, the calculative process of expectation $I\left(\underset{\otimes 1}{\widetilde{C}}\right)$ is shown as follows:

$$I\left(\underset{\otimes 1}{\widetilde{C}}\right) = \frac{0.0935 + 2 \times 0.1275 + 0.1586}{4} \times \frac{(1 - 0.5) + 2 \times (1 - 0.5)}{3} = 0.0634$$

And the other alternatives' expectations are as below:

$$I\left(\underset{\otimes 2}{\widetilde{C}}\right) = 0.0609, \quad I\left(\underset{\otimes 3}{\widetilde{C}}\right) = 0.0772, \quad I\left(\underset{\otimes 4}{\widetilde{C}}\right) = 0.0488$$

Scince $I\left(\underset{\otimes 3}{\widetilde{C}}\right) > I\left(\underset{\otimes 1}{\widetilde{C}}\right) > I\left(\underset{\otimes 2}{\widetilde{C}}\right) > I\left(\underset{\otimes 4}{\widetilde{C}}\right)$, the final ranking of all alternatives is that $X_3 > X_1 > X_2 > X_4$. The best alternative is X_3.

6.2. Discussion

In the above calculation process, parameters p and q represent the subjective preferences of decision-makers. Decision-makers can rise p and q to show their positive preferences. Similarly, decision-makers can reduce p and q to express negative preferences. To explain the effect of changing parameters on the sorting results of alternatives, we utilized the different parameters in the IGTFWPBM operators and the ranking results calculated by different p and q are shown in Table 8.

Table 8. The expectation obtained with interval grey triangular fuzzy numbers weighted partitioned Bonferroni mean (IGTFWPBM) and rankings of the alternatives.

p,q	$E(\underset{\otimes i}{\tilde{z}})$	ranking
$p=1, q=0$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0647, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0628, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0798, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0501$	$X_3 > X_1 > X_2 > X_4$
$p=0, q=1$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0647, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0628, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0798, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0501$	$X_3 > X_1 > X_2 > X_4$
$p=1, q=2$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0643, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0629, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0800, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0499$	$X_3 > X_1 > X_2 > X_4$
$p=2, q=1$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0643, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0629, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0800, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0499$	$X_3 > X_1 > X_2 > X_4$
$p=2, q=2$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0641, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0634, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0804, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0501$	$X_3 > X_1 > X_2 > X_4$
$p=1, q=5$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0684, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0679, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0868, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0536$	$X_3 > X_1 > X_2 > X_4$
$p=5, q=5$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0659, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0675, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0853, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0526$	$X_3 > X_2 > X_1 > X_4$
$p=1, q=10$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0731, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0723, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0929, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0570$	$X_3 > X_1 > X_2 > X_4$
$p=10, q=10$	$E\left(\underset{\otimes 1}{\tilde{z}}\right)=0.0671, E\left(\underset{\otimes 2}{\tilde{z}}\right)=0.0696, E\left(\underset{\otimes 3}{\tilde{z}}\right)=0.0875, E\left(\underset{\otimes 4}{\tilde{z}}\right)=0.0539$	$X_3 > X_2 > X_1 > X_4$

We can adjust p and q from the IGTFWPBM operators to change the preference of decision-makers. From Table 8, the best alternative is always X_3 and the worst alternative is always X_4. The ranking of alternatives X_1 and X_2 may be different with the change of the parameters.

For $p = 5$ or $q = 5$, if we take different values of $q(p)$ from 0 to 20, the ranking sort is changed which shown in Figures 1 and 2.

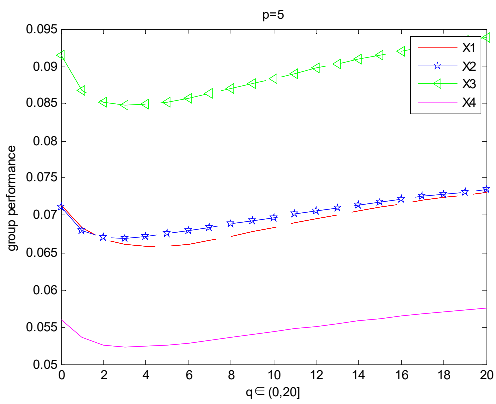

Figure 1. The expectation of all alternatives when $p = 5$ and $q \in [0, 20]$.

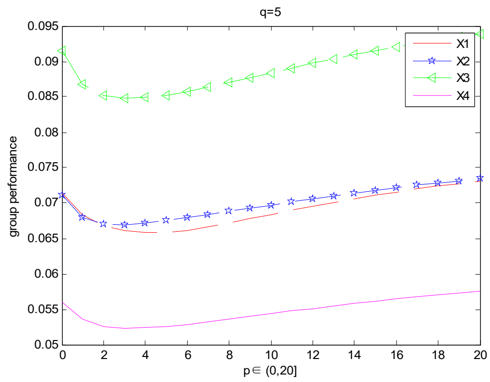

Figure 2. The expectation of all alternatives when $q = 5$ and $p \in [0, 20]$.

Observing Figures 1 and 2. It is not difficult to see that when $p = 5$ or $q = 5$, changing another parameter from 0 to 20, the expectation curves of all attributes first drop and then rise. We can find the alternative X_3 is the best one and alternative X_4 is the worst one. When $p \leq 1.92$ ($q \leq 1.92$) the alternative X_1 is better than alternative X_2. And when $p > 1.92(q > 1.92)$, the alternative X_2 is the better one between X_1 and X_2.

If we select different parameters, and change of expectation of each alternative's IGTFWPBM operators are drawn in Figures 3–6.

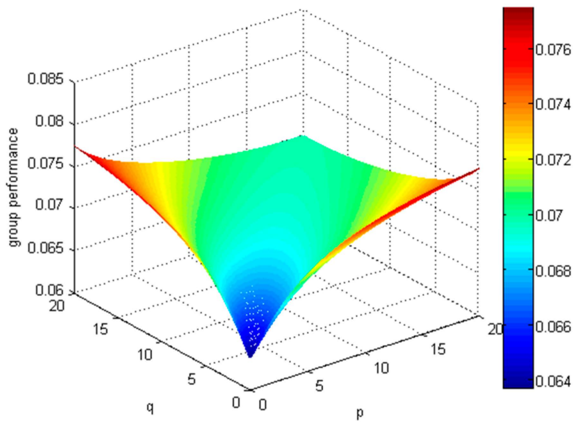

Figure 3. Change of expectation of X_1 when $p, q \in [0, 20]$.

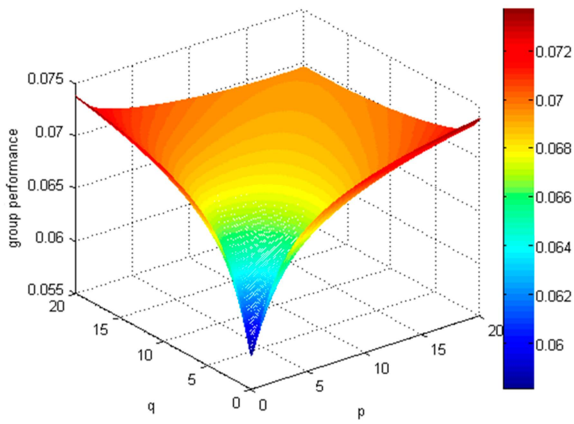

Figure 4. Change of expectation of X_2 when $p, q \in [0, 20]$.

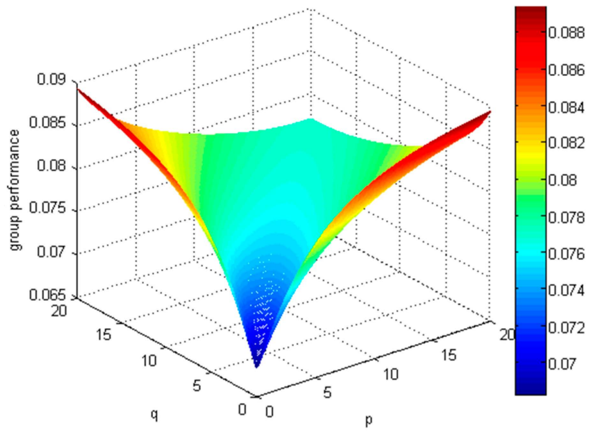

Figure 5. Change of expectation of X_3 when $p, q \in [0, 20]$.

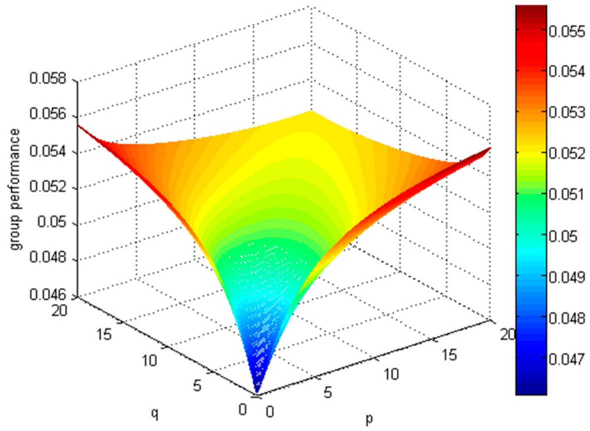

Figure 6. Change of expectation of X_4 when $p, q \in [0, 20]$.

By comparing Figures 3–6, we find that the overall expectation level of X_3 is the highest and the overall expectation level of X_4. The overall expectation level of X_1 and X_2 is roughly the same, but the overall fluctuation of X_1 is smaller than X_2. In general, the parameters are larger, the decision-makers' attitude is more positive, and the expectation of an alternative's IGTFWPBM operator is greater.

According to Figures 3–6, the change of expectation of four selected alternatives is symmetrical. For example, when $p = 1$ and $q = 0$, the expectation of four selected alternatives are $E\Big(\underset{\otimes 1}{\widetilde{z}}\Big) = 0.0647, E\Big(\underset{\otimes 2}{\widetilde{z}}\Big) = 0.0628, E\Big(\underset{\otimes 3}{\widetilde{z}}\Big) = 0.0798, E\Big(\underset{\otimes 4}{\widetilde{z}}\Big) = 0.0501$, which is consistent with the evaluation results when $p = 0$ and $q = 1$. Therefore, in order to distinguish the differences in the hierarchical relationships among the attributes, we can adjust the unilateral parameter (p or q).

6.3. Verification of the Effectiveness

In order to further illustrate the method in more cases of applicability, new Examples 2 and 3 are used to interpret how the IGTFWPBM operators solve GFMGADM problems in the case of no relationship among attribute variables and the case of the existing relationship among attribute variables. In this section, we explain the calculation process in detail and verify the effectiveness of the decision method based on IGTFWPBM operators by comparing it with other achievements.

Example 2. *[16]: To evaluate the technological innovation ability of the four enterprises $\{A_1, A_2, A_3, A_4\}$ and there are four evaluating attributes which are the ability of innovative resources investment (C_1), the ability of innovation management (C_2), the ability of innovation tendency (C_3), and the ability of research and development (C_4). There are three experts to evaluate four enterprises. Suppose that $\gamma = (0.4, 0.32, 0.28)^T$ is the expert weight vector, and $\omega = (0.3, 0.2, 0.2, 0.3)^T$ is attribute weight vector, and the attribute values given by the* *experts take the form of interval grey triangular fuzzy numbers shown in Tables 9–11. It is worth emphasizing that in literature [16], the fuzzy part is represented by linguistic variables, we transformed linguistic variables into triangular fuzzy numbers through the transformation method [42].*

Table 9. The attribute values of each attribute to four enterprises given by expert e_1.

	C_1	C_2	C_3	C_4
A_1	$([0.667, 0.833, 1.000], [0.2, 0.3])$	$([0.167, 0.333, 0.500], [0.4, 0.4])$	$([0.667, 0.833, 1.000], [0.5, 0.5])$	$([0.333, 0.500, 0.667], [0.2, 0.4])$
A_2	$([0.500, 0.667, 0.833], [0.4, 0.4])$	$([0.667, 0.833, 1.000], [0.4, 0.5])$	$([0.333, 0.500, 0.667], [0.1, 0.2])$	$([0.500, 0.667, 0.833], [0.5, 0.5])$
A_3	$([0.333, 0.500, 0.667], [0.2, 0.3])$	$([0.500, 0.667, 0.833], [0.3, 0.3])$	$([0.500, 0.667, 0.833], [0.3, 0.3])$	$([0.667, 0.833, 1.000], [0.2, 0.3])$
A_4	$([0.833, 1.000, 1.000], [0.5, 0.6])$	$([0.000, 0.167, 0.333], [0.2, 0.2])$	$([0.333, 0.500, 0.667], [0.2, 0.4])$	$([0.333, 0.500, 0.667], [0.3, 0.4])$

Table 10. The attribute values of each attribute to four enterprises given by expert e_2.

	C_1	C_2	C_3	C_4
A_1	$([0.500, 0.667, 0.833], [0.1, 0.3])$	$([0.333, 0.500, 0.667], [0.2, 0.3])$	$([0.333, 0.500, 0.667], [0.2, 0.2])$	$([0.833, 1.000, 1.000], [0.4, 0.5])$
A_2	$([0.667, 0.833, 1.000], [0.4, 0.5])$	$([0.333, 0.500, 0.667], [0.3, 0.4])$	$([0.500, 0.667, 0.833], [0.2, 0.4])$	$([0.333, 0.500, 0.667], [0.2, 0.3])$
A_3	$([0.500, 0.667, 0.833], [0.2, 0.4])$	$([0.500, 0.667, 0.833], [0.2, 0.3])$	$([0.000, 0.167, 0.333], [0.4, 0.4])$	$([0.333, 0.500, 0.667], [0.3, 0.3])$
A_4	$([0.667, 0.833, 1.000], [0.3, 0.4])$	$([0.500, 0.667, 0.833], [0.4, 0.5])$	$([0.000, 0.167, 0.333], [0.3, 0.4])$	$([0.500, 0.667, 0.833], [0.2, 0.4])$

Table 11. The attribute values of each attribute to four enterprises given by expert e_3.

	C_1	C_2	C_3	C_4
A_1	$([0.667, 0.833, 1.000], [0.2, 0.4])$	$([0.333, 0.500, 0.667], [0.3, 0.3])$	$([0.500, 0.667, 0.833], [0.4, 0.5])$	$([0.500, 0.667, 0.833], [0.2, 0.3])$
A_2	$([0.500, 0.667, 0.833], [0.3, 0.3])$	$([0.667, 0.833, 1.000], [0.3, 0.4])$	$([0.000, 0.167, 0.333], [0.1, 0.2])$	$([0.333, 0.500, 0.667], [0.1, 0.2])$
A_3	$([0.500, 0.667, 0.833], [0.2, 0.3])$	$([0.667, 0.833, 1.000], [0.3, 0.4])$	$([0.000, 0.000, 0.000], [0.1, 0.2])$	$([0.500, 0.667, 0.833], [0.2, 0.3])$
A_4	$([0.333, 0.500, 0.667], [0.2, 0.3])$	$([0.333, 0.500, 0.667], [0.1, 0.3])$	$([0.500, 0.667, 0.833], [0.3, 0.4])$	$([0.667, 0.833, 1.000], [0.4, 0.5])$

We get an aggregated matrix by aggregating the assessment information which is shown in Table 12:

Table 12. The aggregated information of each attribute to four enterprises.

	C_1	C_2	C_3	C_4
A_1	$([0.614, 0.780, 0.947], [0.2, 0.4])$	$([0.267, 0.433, 0.600], [0.4, 0.4])$	$([0.513, 0.680, 0.847], [0.5, 0.5])$	$([0.540, 0.707, 0.820], [0.4, 0.5])$
A_2	$([0.553, 0.720, 0.886], [0.4, 0.5])$	$([0.560, 0.726, 0.893], [0.4, 0.5])$	$([0.340, 0.507, 0.673], [0.2, 0.4])$	$([0.400, 0.567, 0.733], [0.5, 0.5])$
A_3	$([0.433, 0.600, 0.767], [0.2, 0.4])$	$([0.547, 0.714, 0.880], [0.3, 0.4])$	$([0.253, 0.373, 0.540], [0.4, 0.4])$	$([0.513, 0.680, 0.847], [0.3, 0.3])$
A_4	$([0.640, 0.807, 0.907], [0.5, 0.6])$	$([0.253, 0.420, 0.587], [0.4, 0.5])$	$([0.327, 0.493, 0.660], [0.3, 0.4])$	$([0.480, 0.647, 0.813], [0.4, 0.5])$

Because in this case, the attribute variables are independent from each other, we divided the attribute variables into four groups, each group has only one attribute variable. Utilizing the IGTFWPBM operators to aggregate the comprehensive information of all alternatives. Suppose $p = 1$, $q = 1$, the computed result as follows:

$$\underset{\otimes}{\widetilde{A}} = \begin{bmatrix} [0.1255, 0.1672, 0.2048], [0.50, 0.50] \\ [0.1165, 0.1582, 0.1998], [0.50, 0.50] \\ [0.1110, 0.1504, 0.1920], [0.40, 0.40] \\ [0.1130, 0.1547, 0.1913], [0.50, 0.60] \end{bmatrix}$$

And then, the expectation of all enterprises $\underset{\otimes i}{\widetilde{A}}(i = 1, 2, \ldots, 4)$ are shown as follows:

$$I\left(\underset{\otimes 1}{\widetilde{A}}\right) = 0.0831, I\left(\underset{\otimes 2}{\widetilde{A}}\right) = 0.0791, I\left(\underset{\otimes 3}{\widetilde{A}}\right) = 0.0906, I\left(\underset{\otimes 4}{\widetilde{A}}\right) = 0.0665$$

Therefore, the ranking order of the four enterprises is that $A_3 \succ A_1 \succ A_2 \succ A_4$. Enterprise 3 is more innovative, while enterprise 4 is less innovative. In order to demonstrate the effectiveness of this method in solving the decision-making problem where there is no interrelationship among attribute variables, we selected the existing decision method with the grey fuzzy environment [16,19–22], and the comparison results are shown in Table 13.

Table 13. Comparison of the proposed operators with other aggregation operators/methods in cases where there is no relationship among attributes.

Methods	Aggregation Operator/Method	Whether Captures the Relationship among Attributes	Ranking
Jin and Liu [16]	Topsis	No	$X_3 > X_1 > X_2 > X_4$
Liu [20]	IGLOWA	No	$X_3 > X_1 > X_2 > X_4$
Liu [21]	IGLHWA	No	$X_3 > X_1 > X_2 > X_4$
Liu et al. [22]	IGULGHA	No	$X_3 > X_2 > X_1 > X_4$
Wang et al. [19]	SIR Choquet	Yes	$X_3 > X_1 > X_2 > X_4$
Proposed method	IGTFWPBM	Yes	$X_3 > X_1 > X_2 > X_4$

In methods [16,19–22], the fuzzy parts are represented by linguistic variables or uncertain linguistic variables. Compared with linguistic variables and interval numbers, the fuzzy part expressed by triangular fuzzy numbers can measure the fuzziness of natural linguistic variables, which is the main distinction of a new method from existing methods. For ensuring the consistency of expert evaluation information, we used a method proposed by Liu [43] to transform the linguistic variables into the triangular fuzzy numbers. As shown in Table 13, it is not difficult to find that when there is no interrelationship among attributes, the method proposed in this paper is consistent with the decision results of the other five methods. It indicates that the IGTFWPBM operator is effective in solving the decision problem where there is no interrelationship among attribute variables. At this time, IGTFWPBM is simplified to an IGTFWBM operator, and the calculation process is relatively simple.

Example 3. *[33] ABC Nonferrous Metals Co. Ltd. is a largely state-owned company whose main business is the deep processing of nonferrous metals. In order to expand its main business, the overseas investment department decided to select a pool of alternatives from several foreign countries based on preliminary surveys. Five countries* $\{a_1, a_2, a_3, a_4, a_5\}$ *were evaluated through surveys and screening. Many factors affect the investment environment and four factors are chosen based on the experience of the department personnel, including* c_1*: resources (such as the suitability of the minerals and their exploration);* c_2*: politics and policy (such as corruption and political risks);* c_3*: economy (such as development vitality and the stability); and* c_4*: infrastructure (such as railway and highway facilities). The weight vector of the factors is* $\omega = (0.25, 0.22, 0.35, 0.18)$*. The evaluation information is shown in Table 14. There is a close correlation among the four evaluation factors. The energy reserve can influence the direction of policy formulation, and economic development depends on the resource reserve and related policies and regulations. The infrastructure depends on economic development, and the completeness of infrastructure is more conducive to the rapid development of the economy. Therefore, we classify the four attribute variables into one class for discussion.*

Table 14. The attribute values of each attribute to five countries.

	c_1	c_2	c_3	c_4
a_1	$([0.500, 0.667, 0.833], 0.6, [0.6, 0.6])$	$([0.333, 0.500, 0.667], 0.7, [0.4, 0.4])$	$([0.500, 0.667, 0.833], 0.8, [0.5, 0.5])$	$([0.500, 0.667, 0.833], 0.8, [0.3, 0.3])$
a_2	$([0.333, 0.500, 0.667], 0.7, [0.5, 0.5])$	$([0.500, 0.667, 0.833], 0.6, [0.4, 0.4])$	$([0.000, 0.167, 0.333], 0.6, [0.2, 0.2])$	$([0.333, 0.500, 0.667], 0.7, [0.4, 0.4])$
a_3	$([0.500, 0.667, 0.833], 0.5, [0.1, 0.1])$	$([0.667, 0.833, 1.000], 0.6, [0.5, 0.5])$	$([0.333, 0.500, 0.667], 0.7, [0.6, 0.6])$	$([0.333, 0.500, 0.667], 0.5, [0.5, 0.5])$
a_4	$([0.333, 0.500, 0.667], 0.4, [0.5, 0.5])$	$([0.333, 0.500, 0.667], 0.5, [0.3, 0.3])$	$([0.500, 0.667, 0.833], 0.6, [0.8, 0.8])$	$([0.500, 0.667, 0.833], 0.9, [0.3, 0.3])$
a_5	$([0.333, 0.500, 0.667], 0.6, [0.4, 0.4])$	$([0.500, 0.667, 0.833], 0.8, [0.3, 0.3])$	$([0.333, 0.500, 0.667], 0.7, [0.5, 0.5])$	$([0.333, 0.500, 0.667], 0.6, [0.5, 0.5])$

Transform the assessment values into interval grey triangular fuzzy numbers, and the grey linguistic numbers $\left(h_{\theta_{i,j}}, \mu_{i,j}, \nu_{i,j}\right)$ [33] is replaced by $\left(\left(f_a^L, f_a^M, f_a^U\right), \mu_{i,j}, \nu_{i,j}\right)$, where the lower and upper bounds are equal. As $\mu_{i,j}$ independent of $\left(f_a^L, f_a^M, f_a^U\right)$ and $\nu_{i,j}$, so in the process of computing $\mu_{i,j}$ and interval grey triangular fuzzy numbers can be calculated independently. And the calculation method of $\mu_{i,j}$ can be referred to the literature [16]. Utilizing the IGTFWPBM operators to aggregate the comprehensive information of all alternatives. Suppose p = 1 and q = 1, the computed result as follows:

$$\underset{\otimes}{\widetilde{a}} = \begin{bmatrix} [0.2406, 0.3344, 0.4276], 0.2288, [0.60, 0.60] \\ [0.1409, 0.2633, 0.3608], 0.1875, [0.50, 0.50] \\ [0.2339, 0.3283, 0.4220], 0.1654, [0.60, 0.60] \\ [0.2292, 0.3229, 0.4161], 0.1704, [0.50, 0.50] \\ [0.2142, 0.3080, 0.4015], 0.1944, [0.50, 0.50] \end{bmatrix}$$

And then, we calculate the expectation of all alternatives $\{a_1, a_2, a_3, a_4, a_5\}$, for example, the calculative process of expectation $I\left(\underset{\otimes 1}{\widetilde{a}}\right)$ is shown as follows:

$$I\left(\underset{\otimes 1}{\widetilde{a}}\right) = \frac{0.2406 + 2 \times 0.3344 + 0.4276}{4} \times \frac{(1-0.6) + 2 \times (1-0.6)}{3} \times 0.2288 = 0.0306$$

And the other alternatives' expectations are as shown below:

$$I\left(\underset{\otimes 2}{\widetilde{a}}\right) = 0.0241, I\left(\widetilde{a}_3\right) = 0.0217, I\left(\underset{\otimes 4}{\widetilde{a}}\right) = 0.0110 I\left(\underset{\otimes 5}{\widetilde{a}}\right) = 0.0299$$

Therefore, the ranking of all countries is that $a_1 > a_5 > a_2 > a_3 > a_4$. By comparison, we find the ranking calculated by IGTFWPBM operator is consistent with methods [16,19,33], which illustrates the effectiveness of the GFMAGDM method based on the IGTFWPBM operator. To better illustrate the superiority of the method based on the IGTFWPBM operator, we compared the existing methods in a grey fuzzy environment in Section 6.4.

6.4. Comparative Analysis

To elaborate on the validity and superiority of a new method based on the IGTFWPBM operators, we selected three examples and compared them with various methods of decision-making in the existing grey fuzzy environment [16,19–22,33]. It is not difficult to find that IGTFWPBM operators are valid in cases where there is no relation among attributes (Example 2) and a case where a simple relationship among attributes (Example 3).

To demonstrate the superiority of the proposed method, we took advantage of the decision-making problem that how to evaluate the economic development of the five countries (Example 3) to compare it with five existing grey fuzzy methods [16,19–21,33]. The ranking orders calculated by existing methods are represented as Table 15.

Table 15. Comparison of the proposed operators with other aggregation operators/methods where there exits relationship among attributes.

Methods	Aggregation Operator/Method	Whether Captures the Relationship among Attributes	Ranking
Jin et al. [16]	Topsis	No	$X_1 > X_5 > X_2 > X_3 > X_4$
Liu [20]	IGLOWA	No	$X_1 > X_5 > X_3 > X_2 > X_4$
Liu [21]	IGLHWA	No	$X_5 > X_1 > X_3 > X_2 > X_4$
Wang et al. [19]	SIR Choquet	Yes	$X_1 > X_5 > X_2 > X_3 > X_4$
Tian et al. [33]	GLWBM	Yes	$X_1 > X_5 > X_2 > X_3 > X_4$
Proposed method	IGTFWPBM	Yes	$X_1 > X_5 > X_2 > X_3 > X_4$

It is worth emphasizing that to ensure the comparability of calculation methods, linguistic variables are transformed into triangular fuzzy numbers through the transformation method [43]. As shown in Table 15, the method proposed in this paper is consistent with the method that takes into account the relationship among attributes, and slightly different from the other three methods that do not capture the relationship among the attributes. This is because none of the aggregated operators consider the relationship among the attributes in methods [16,20,21], which does not conform to the decision logic in this example. The method proposed in this paper and methods [19,33] can overcome this defect and improve the accuracy of decision making.

To further illustrate the superiority of IGTFWPBM operators, two methods [19,30] that take account of attribute relationships are compared with the method in this paper according to Example 1. Furthermore, new attributes are added based on the original example, which further complicates the decision-making process. The comparison results are shown in Table 16.

Table 16. Comparison of the proposed operators with other aggregation operators/methods where there exits interrelationship among attributes.

Aggregation Operator/Method	The Number of Attributes			
	5	6	7	8
IGTFWPBM	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_1 > X_2 > X_4$
GLWBM	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_2 > X_1 > X_4$	$X_3 > X_2 > X_1 > X_4$	$X_3 > X_2 > X_4 > X_1$
SIR Choquet	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_1 > X_2 > X_4$	$X_3 > X_2 > X_1 > X_4$	$X_3 > X_1 > X_2 > X_4$

As shown in Table 16, when the number of attributes is 5 in the original case, the sorting results of the three methods are consistent. However, with the increasing number of attributes, the order of the SIR Choquet method [19] and GLWBM operator [33] changes. The order of the GLWBM operator fluctuates greatly. When the number of GLWBM operators increases to six attributes, the calculated order becomes $X_3 > X_2 > X_1 > X_4$; when the attribute variable reaches eight, the permutation order becomes $X_3 > X_2 > X_4 > X_1$. The order of the SIR Choquet method changes when the number of attributes is seven, and when the number of attributes is eight, the order changes back to the original order. However, the order of IGTFWPBM operator has not changed, because IGTFWPBM operator can fully consider the interrelationship among attributes variables through classifying attributes for calculation, which greatly improves the accuracy and stability of the decision-making process. At the same time, in the calculation of IGTFWPBM operator, the decision-maker can adjust the parameters p and q to show his or her attitude towards the internal grouping relationship among attributes, which is more consistent with the actual decision-making process.

To sum up, we compare the IGTFWPBM operator with the six methods. Under the condition that the attributes are independent of each other, the calculated results of the IGTFWPBM operator are consistent with those of other methods applied in the grey fuzzy environment, indicating that the IGTFWPBM operator is effective. When there are internal relations among attributes, the method proposed in this paper considering the correlation among attributes is more accurate than the traditional grey fuzzy operator and method. When faced with more attribute variables and more complex relationships, The IGTFWPBM operator's ability to calculate internal relationships and the setting of regulating parameters can help improve the accuracy and stability of the decision-making process, which are the notable advantages of a new method.

7. Conclusions

In this study, considering the fuzziness and greyness in real decision-making, we developed a new method for solving GFMAGDM problems. There are three main contributions to this study. In the first phase, we proposed the interval grey triangular fuzzy numbers which the grey part expressed by interval numbers and the fuzzy part in the form of triangular fuzzy numbers, after that, we gave the definitions, properties, distance, and expectations of the interval grey triangular fuzzy numbers. In the second phase, we proposed the IGTFPBM operators, combined the interval grey triangular fuzzy numbers with PBM, and gave the weighted form IGTFWPBM. The proposed operators can build the partition structure and calculate the interrelationship among the attributes. In the third phase, we developed a novel method based on IGTFWPBM. We applied the method to the example that an investment institution made decision-making to choose the best one from four small and micro-enterprises. And then, we showed the ranking results under the different parameters. The adjustment of parameters will have a symmetrical effect on the expectation of the alternative, so the importance of the variable groups can be distinguished by the unilateral adjustment of parameters. Finally, we compared the proposed method with five existing methods to illustrate the effectiveness and superiority of the new method.

In future work, we will improve the decision-making accuracy of grey fuzzy variables and apply it to high-precision medical and military fields. In terms of theoretical research, we hope to choose a more complex form to express fuzzy parts, such as Gaussian fuzzy number [44,45], intuitionistic fuzzy sets [46,47], and rough sets [48]. At the same time, BPM operators can be simplified to improve computational efficiency. On the application side, we will apply the IGTFPBM operators into more complex real-world decision cases [49,50] and further explore the application of the IGTFPBM operators in a big data environment [51].

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