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A Boundary Value Problem for Noninsulated Magnetic Regime in a Vacuum Diode

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Abstract: In this paper, we study the stationary boundary value problem derived from the magnetic (non) insulated regime on a plane diode. Our main goal is to prove the existence of non-negative solutions for that nonlinear singular system of second-order ordinary differential equations. To attain such a goal, we reduce the boundary value problem to a singular system of coupled nonlinear Fredholm integral equations, then we analyze its solvability through the existence of fixed points for the related operators. This system of integral equations is studied by means of Leray-Schauder's topological degree theory.

Keywords: relativistic Vlasov-Maxwell system; magnetic insulation; singular boundary value problem; Leray-Schauder degree theory

MSC: 35Q83; 34A12; 34B15; 45B05; 45D05; 47H11

1. Motivation

High energy devices such as vacuum diodes are designed to work with extremely high applied voltages. The saturation of the current due to the self-consistent electric and magnetic field is a nonlinear phenomenon under electron transport. Langmuir and Compton [1] started the investigation of this phenomenon and established explicit formulae for the saturation current in the plane and symmetric diode cases. They assumed that the current saturates at a maximal value determined by the condition that the electric field vanishes at the emission cathode. This condition is referred to as the Child-Langmuir condition and the diode is said to operate under a space charge limited or a Child-Langmuir regime.

Here two basic regimes are possible: the first one when electrons reach the anode—non-insulated diode—and the second one is when due to the extremely high electric and magnetic field applied, the electrons rotate back to the cathode, the so-called "insulated diode". For the latter case, there is an electronic layer outside of which electromagnetic field is equal to zero (see Langmuir and Compton [1]).

The regime of "noninsulated diode" is described by the following nonlinear two-point coupled second order boundary value problem:

$$\begin{aligned} \frac{d^2\varphi(x)}{dx^2} &= j_x \frac{1+\varphi(x)}{\sqrt{(1+\varphi(x))^2-1-a(x)^2}}, \qquad \varphi(0)=0, \quad \varphi(1)=\varphi_L, \\ \frac{d^2a(x)}{dx^2} &= j_x \frac{a(x)}{\sqrt{(1+\varphi(x))^2-1-a(x)^2}}, \qquad a(0)=0, \quad a(1)=a_L. \end{aligned} \tag{I}$$

where $j_x > 0$ is a constant (not depending on x), φ is the potential of electric field, a the potential of magnetic field and $x \in [0, 1]$. To see the setting of the problem and the complete derivation of system (I), see e.g., [2].

The existence of solutions for system (I) was studied by Abdallah et al. [2] by a shooting method with $\beta = a'(0)$ and j_x as shooting parameters. The strategy was: given the values of β and j_x, solve (I) with the Cauchy conditions $\varphi(0) = 0$, $a(0) = 0$, $\varphi'(0) = 0$, $a'(0) = \beta$, and then adjust the values in order to fulfill the conditions $\varphi(1) = \varphi_L$ and $a(1) = a_L$.

In this paper, we analyze the existence of (non-negative) solutions for the coupled second-order boundary value problem (I) by transforming this boundary value problem (BVP) into a coupled system of singular nonlinear Fredholm integral equation, then we investigate conditions to assure the non-negativity of the image functions resulting from the evaluating of these integral equations. Finally, the existence of such solutions is guarantee by using the classical Leray-Schauder topological degree theory.

We recall that some analytical and numerical aspects of Child-Langmuir's regime were recently given by Abdallah et al. [2] and Dulov and Sinitsyn [3]. In addition, we would like to point out that a number of interesting results in the theory and applications of functional group methods under the conditions of model symmetry and bifurcation of the desired solutions are presented in the works of Sidorov and Sinitsyn [4,5] and in the monograph [6]. The efficient iterative method for branching solutions construction based on explicit and implicit parametrizations is proposed by Sidorov [7]. The readers may refer to seminal manuscript [8], where the generalized initial value problem, known as the Showalter-Sidorov problem, was introduced.

2. Non-Negative Solutions for Boundary Value Problem (I)

Now, we are interested in the existence of non-negative solutions for the boundary value problem derived from the noninsulated regime for a vacuum plane diode. After the substitutions $\varphi + 1 =: u$, $a =: v$, boundary value problem (I) reads as:

$$\begin{aligned} u''(x) &= j_x \frac{u(x)}{\sqrt{u^2(x) - 1 - v^2(x)}}, \quad u(0) = 1,\ u(1) = \varphi_L + 1 = p, \\ v''(x) &= j_x \frac{v(x)}{\sqrt{u^2(x) - 1 - v^2(x)}}, \quad v(0) = 0,\ v(1) = a_L = q. \end{aligned} \tag{1}$$

By singular we mean that $u^2(x) = 1 + v^2(x)$ is allowed. First, we reduce this system of BVP's to a coupled system of integral equations: By integrating by parts and using the boundary conditions for the second order BVP

$$u''(x) = j_x \frac{u(x)}{\sqrt{u^2(x) - 1 - v^2(x)}}, \quad u(0) = 1,\ u(1) = p,$$

we have:

$$u'(x) = u'(0) + j_x \int_0^x \frac{u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds$$

hence, integrating one more time, we obtain:

$$u(x) = 1 + u'(0)x + j_x \int_0^x \frac{(x-s)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds.$$

Using the boundary condition $u(1) = p$, we get

$$u'(0) = p - 1 - j_x \int_0^1 \frac{(1-s)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds.$$

Then, we reduce the second order boundary value problem to the following integral equation

$$\begin{aligned} u(x) =& 1 + (p-1)x - j_x \cdot x \int_0^1 \frac{(1-s)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds + j_x \int_0^x \frac{(x-s)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds \\ =& 1 + (p-1)x + j_x \int_0^1 \frac{x(s-1)u(s) + \chi_{(0,x)}(s)(x-s)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds \\ =& 1 + (p-1)x + j_x \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds \end{aligned}$$

where, $\chi_{(0,x)}(s)$ is the characteristic function of the interval $(0, x)$ and

$$G(s,x) = \begin{cases} s(x-1), & 0 \le s \le x \le 1 \\ x(s-1), & 0 \le x \le s \le 1. \end{cases}$$

In the same fashion, we can reduce the second order BVP

$$v''(x) = j_x \frac{v(x)}{\sqrt{u^2(x) - 1 - v^2(x)}}, \quad v(0) = 0,\ v(1) = q.$$

to the integral equation

$$\begin{aligned} v(x) =& qx - j_x \cdot x \int_0^1 \frac{(1-s)v(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds + j_x \int_0^x \frac{(x-s)v(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds \\ =& qx + j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds. \end{aligned}$$

In this way, the existence of solutions for system (1) is equivalent to investigate the existence of solutions for the following coupled system of nonlinear singular Fredholm integral equations:

$$\begin{aligned} u(x) &= 1 + (p-1)x + j_x \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds \\ v(x) &= qx + j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds. \end{aligned} \tag{2}$$

The equations are posed in the Banach space $X = (C^1[0,1], \mathbb{R})$ endowed with the norm $\|f\|_\infty = \max\{|f(x)| \ : \ x \in [0,1]\}$. By $X \times X$ we denote the product space, which is a Banach space under the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$, we define the operator $F(u,v) : X \times X \longrightarrow X \times X$ by the formula

$$F(u,v) = (F_1(u,v), F_2(u,v))$$

where, for each $x \in [0,1]$,

$$
\begin{aligned}
F_1(u,v)(x) =& 1 + (p-1)x + j_x \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
F_2(u,v)(x) =& qx + j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds.
\end{aligned}
$$

Notice that for $u, v \in X$, the mappings $F_1(u,v)$ and $F_2(u,v)$ are well-defined if $\sqrt{u^2(s)-1-v^2(s)} \in \mathbb{R}$, so we require that

$$u^2(s) \geq 1 + v^2(s), \quad \text{for all } s \in [0,1]. \tag{3}$$

In particular, $|u(s)| \geq 1$ for all $s \in [0,1]$, but $|u| \not\equiv 1$. Moreover, we have the following inequality.

Lemma 1. *Under condition* (3)*, the inequality*

$$|uu' - vv'| \geq 1 \tag{4}$$

holds for all $s \in [0,1]$.

Proof. We should prove that $uu' - vv' \geq 1$, or $uu' - vv' \leq -1$. If we assume that $uu' - vv' < 1$, then by integrating the inequality above we have $u^2(s)/2 < s + v^2(s)/2 + K$ for any constant K, but from (3) we have

$$\frac{1}{2} + \frac{v^2(s)}{2} < s + \frac{v^2(s)}{2} + K,$$

i.e., $1/2 < s + K$ for any K, which is false. Take, for instance, $K = -1/2$. Similarly, if $uu' - vv' > -1$, by integrating we get $u^2(s)/2 > -s + v^2(s)/2 + C$ for any $C \in \mathbb{R}$ and all $s \in [0,1]$. However, this is not necessarily true for constants $C < 1/2$. $\square$

In the operator theory scheme, the existence of a solution for system (2) is equivalent to the existence of a fixed point for the operator $F(u,v)$ on $X \times X$. Since we want to find solutions for system (1) representing physical meaning, we are interested in the existence of non-negative solutions for this system. Thus, we will assume that the boundary conditions satisfy that $p \geq 1$ and $q \geq 0$.

Let P be the cone of all non-negative functions on X. i.e.,

$$P = \{f \in C^1([0,1], \mathbb{R}) \ : \ f(x) \geq 0, \text{ for all } x \in [0,1]\}.$$

Our fist step is to find conditions such that $F(u,v) : X \times X \longrightarrow P \times P$. That is, conditions under which, for each $x \in [0,1]$, the next inequalities are satisfied.

$$F_1(u,v)(x) = 1 + (p-1)x + j_x \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \geq 0 \tag{5}$$

$$F_2(u,v)(x) = qx + j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \geq 0. \tag{6}$$

Equivalently, inequalities (5) and (6) are satisfied, for all $x \in [0,1]$, if the following inequalities hold (respectively):

$$-j_x \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s)-1-v^2(s)}}ds \leq 1+(p-1)x \tag{7}$$

$$-j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}}ds \leq qx. \tag{8}$$

Note that $0 \leq -G(s,x) \leq 1$, for all $s,x \in [0,1]$. Thus, for any $f \in X$,

$$\begin{aligned}
-j_x \int_0^1 \frac{G(s,x)f(s)}{\sqrt{u^2(s)-1-v^2(s)}}ds &\leq j_x \int_0^1 \frac{f(s)}{\sqrt{u^2(s)-1-v^2(s)}}ds \\
&\leq \left| j_x \int_0^1 \frac{u(s)u'(s)-v(s)v'(s)}{\sqrt{u^2(s)-1-v^2(s)}} \frac{f(s)}{u(s)u'(s)-v(s)v'(s)}ds \right| \\
&\leq j_x \int_0^1 \frac{\left| \left(\sqrt{u^2(s)-1-v^2(s)} \right)' \right| |f(s)|}{|u(s)u'(s)-v(s)v'(s)|}ds.
\end{aligned}$$

From Hölder's inequality, we have

$$\begin{aligned}
\int_0^1 \frac{\left| \left(\sqrt{u^2(s)-1-v^2(s)} \right)' \right| |f(s)|}{|u(s)u'(s)-v(s)v'(s)|}ds &\leq \int_0^1 \left| \left(\sqrt{u^2(s)-1-v^2(s)} \right)' \right| ds \\
&\quad \times \left\| \frac{|f|}{|uu'-vv'|} \right\|_\infty \\
&\leq \sqrt{p^2-1-q^2} \left\| \frac{|f|}{|uu'-vv'|} \right\|_\infty.
\end{aligned}$$

On the other hand, inequality (4) implies

$$\left\| \frac{|f|}{|uu'-vv'|} \right\|_\infty \leq \|f\|_\infty.$$

Hence, we obtain the following estimate:

$$-j_x \int_0^1 \frac{G(s,x)f(s)}{\sqrt{u^2(s)-1-v^2(s)}}ds \leq j_x \sqrt{p^2-1-q^2} \|f\|_\infty, \quad \text{for all } x \in [0,1]. \tag{9}$$

Therefore, inequality (7), and so inequality (5), holds if

$$j_x \sqrt{p^2-1-q^2} \|u\|_\infty \leq 1.$$

That is, inequality (5) holds for any $u \in X$ satisfying

$$\|u\|_\infty \leq \frac{1}{j_x \sqrt{p^2-1-q^2}}.$$

Now, we find conditions to guarantee that inequality (8) (consequently, inequality (6)) is satisfied.

$$\begin{aligned}
-j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds =& j_x \int_0^1 \frac{x(1-s)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
&+ j_x \int_0^x \frac{(s-x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
\leq& j_x \left| \int_0^1 \frac{x(1-s)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \right| \\
&+ j_x \left| \int_0^x \frac{(s-x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \right|.
\end{aligned}$$

Note that

$$\max_{s\in[0,1]} \{x(1-s)\} = x \quad \text{and} \quad s-x \leq 0.$$

Then, $s-x \leq x$ for all $x \in [0,1]$. These facts and estimate (9) give

$$\begin{aligned}
-j_x \int_0^1 \frac{G(s,x)v(s)}{\sqrt{u^2(s)-1-v^2(s)}} ds \leq& j_x \int_0^1 \frac{x|v(s)|}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
&+ j_x \int_0^x \frac{x|v(s)|}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
\leq& j_x \int_0^1 \frac{x|v(s)|}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
&+ j_x \int_0^1 \frac{x|v(s)|}{\sqrt{u^2(s)-1-v^2(s)}} ds \\
\leq& 2xj_x\sqrt{p^2-1-q^2}\|v\|_\infty.
\end{aligned}$$

In this way, inequality (8) holds, for all $x \in [0,1]$, if

$$2xj_x\sqrt{p^2-1-q^2}\|v\|_\infty \leq qx,$$

i.e., $F_2(u,v)(x) \geq 0$ for all $v \in X$ satisfying,

$$\|v\|_\infty \leq \frac{q}{2j_x\sqrt{p^2-1-q^2}}.$$

Let us denote by $\overline{\Omega}_1 := \overline{B}\left(\frac{1}{j_x\sqrt{p^2-1-q^2}}\right)$ the closed ball in X centered in 0 with radii $1/j_x\sqrt{p^2-1-q^2}$, and $\overline{\Omega}_2 := \overline{B}\left(\frac{q}{2j_x\sqrt{p^2-1-q^2}}\right)$ stand for the closed ball in X centered in 0 with radii $q/2j_x\sqrt{p^2-1-q^2}$.

We just proved that:

Proposition 1. *The operator $F(u,v)$ applies $\overline{\Omega}_1 \times \overline{\Omega}_2$ into $P \times P$.*

Remark 1. *Notice that from the triangle inequality and estimate (9), we get that in* $\overline{\Omega}_1 \times \overline{\Omega}_2 \subset X \times X$,

$$\begin{aligned}\|F_1(u,v)\|_\infty \leq& p+1,\\ \|F_2(u,v)\|_\infty \leq& 2q.\end{aligned}$$

Hence, the operator $F(u,v)$ *satisfies*

$$\|F(u,v)\| \leq \max\{p+1, 2q\}.$$

To show the existence of at least a fixed point for the operator $F(u,v)$ on $\Omega_1 \times \Omega_2$ (which implies the existence of non-negative solutions for system (1)) we will use the well-known Leray-Schauder topological degree theory (see e.g., [9]). More precisely, we are going to compute $d(\varphi, \Omega_1 \times \Omega_2, (0,0))$, where φ is the compact perturbation of the identity given by $\varphi(u,v) = I_{X\times X}(u,v) - F(u,v)$. Here $I_{X\times X}$ denotes the identity map on $X \times X$.

First, we should prove that $F(u,v)$ is, in fact, completely continuous.

Proposition 2. *The operator* $F(u,v) : \overline{\Omega}_1 \times \overline{\Omega}_2 \longrightarrow P \times P$ *is continuous and compact.*

Proof. Let $(u,v) \in \overline{\Omega}_1 \times \overline{\Omega}_2$. First, we are going to prove the continuity of the operator $F(u,v)$. To do that, we are going to prove that the mappings

$$\begin{aligned}F_1(u,v) :& \overline{\Omega}_1 \times \overline{\Omega}_2 \longrightarrow P,\\ F_2(u,v) :& \overline{\Omega}_1 \times \overline{\Omega}_2 \longrightarrow P\end{aligned}$$

are continuous. Let $(u,v) \in \overline{\Omega}_1 \times \overline{\Omega}_2$ such that $(u_n, v_n) \rightrightarrows (u,v)$. We prove the continuity of $F_1(u,v)$. To prove the continuity of $F_2(u,v)$ is similar. Since,

$$\frac{G(s,x)u_n(s)}{\sqrt{u_n^2(s) - 1 - v_n^2(s)}} \rightrightarrows \frac{G(s,x)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}}$$

then, we conclude

$$\begin{aligned}\lim_{n\to\infty} F_1(u_n,v_n)(x) = \lim_{n\to\infty} j_x \int_0^1 \frac{G(s,x)u_n(s)}{\sqrt{u_n^2(s) - 1 - v_n^2(s)}} ds \\ =& j_x \int_0^1 \lim_{n\to\infty} \frac{G(s,x)u_n(s)}{\sqrt{u_n^2(s) - 1 - v_n^2(s)}} ds \\ =& \int_0^1 \frac{G(s,x)u(s)}{\sqrt{u^2(s) - 1 - v^2(s)}} ds = F_1(u,v)(x),\end{aligned}$$

uniformly on $\overline{\Omega}_1 \times \overline{\Omega}_2$. Hence, $F_1(u,v)$ is continuous.

To prove the equicontinuity of $F(u,v)$ $\left(\overline{\Omega}_1 \times \overline{\Omega}_2\right)$, let $z_n = (u_n, v_n)$ be a sequence in $\overline{\Omega}_1 \times \overline{\Omega}_2$ and $x_1 \leq x_2$. Then,

$$\begin{aligned}&|F_1(u_n,v_n)(x_1) - F_1(u_n,v_n)(x_2)| \\ &\qquad \leq (p-1)|x_1 - x_2| + j_x \int_0^1 \frac{|G(s,x_1) - G(s,x_2)| u_n(s)}{\sqrt{u_n^2(s) - 1 - v_n^2(s)}} ds,\end{aligned}$$

where

$$|G(s,x_1)-G(s,x_2)| = \begin{cases} |s(x_1-1)-s(x_2-1)|, & 0 \le s \le x_1 \le x_2 \le 1 \\ |x_1(s-1)-x_2(s-1)|, & 0 \le x_1 \le x_2 \le s \le 1 \\ |s(x_1-1)-x_2(s-1)|, & 0 \le x_1 \le s \le x_2 \le 1 \end{cases}$$
$$= \begin{cases} |s(x_1-x_2)|, & 0 \le s \le x_1 \le x_2 \le 1 \\ |(s-1)(x_1-x_2)|, & 0 \le x_1 \le x_2 \le s \le 1 \\ |s(x_1-x_2)-s+x_2|, & 0 \le x_1 \le s \le x_2 \le 1. \end{cases}$$

In all these cases, $|G(s,x_1)-G(s,x_2)| \to 0$, as $|x_1-x_2| \to 0$. Therefore,

$$|F_1(u_n,v_n)(x_1) - F_1(u_n,v_n)(x_2)| \to 0, \quad \text{as } |x_1-x_2| \to 0.$$

Similarly, we have

$$\begin{aligned} &|F_2(u_n,v_n)(x_1) - F_2(u_n,v_n)(x_2)| \\ &\qquad \le q|x_1-x_2| + j_x \int_0^1 \frac{|G(s,x_1)-G(s,x_2)|v_n(s)}{\sqrt{u_n^2(s)-1-v_n^2(s)}} ds \\ &\qquad \to 0, \quad \text{as } |x_1-x_2| \to 0. \end{aligned}$$

Thus, from the Arzelá-Ascoli theorem, the operator $F(u,v)$ is compact. □

The existence of at least one non-negative solution for the coupled singular system (1) is given in the next theorem.

Theorem 1. *BVP* (1) *has at least one non-negative solution* (u,v), *provide that*

$$j_x < \min\left\{ \frac{1}{(p+1)\sqrt{p^2-1-q^2}}, \frac{1}{4\sqrt{p^2-1-q^2}} \right\}. \tag{10}$$

Proof. Let be the compact perturbation of the identity $\varphi(u,v) = I_{X\times X}(u,v) - F(u,v)$. We are going to exhibit an admissible homotopy between φ and $I_{X\times X}$. Let the map $H((u,v),t) \in C((\Omega_1 \times \Omega_2) \times [0,1], X \times X)$ given by

$$H((u,v),t) = t\varphi(u,v) + (1-t)I_{X\times X}(u,v).$$

Notice that

$$\begin{aligned} H((u,v),0) =& I_{X\times X}(u,v) = (u,v), \\ H((u,v),1) =& \varphi(u,v). \end{aligned}$$

We are going to show that H never assumes the value $(0,0)$ on the boundary $\partial\Omega_1 \times \partial\Omega_2$ of the set $\Omega_1 \times \Omega_2$. In fact, for $t=0$

$$H((u,v),0) = (u,v) \neq (0,0) \quad \text{over } \partial\Omega_1 \times \partial\Omega_2.$$

For $t=1$,

$$H((u,v),1) = \varphi(u,v) = (u - F_1(u,v), v - F_2(u,v)).$$

If $H((u,v),1) = (0,0)$ we have

$$\begin{aligned} u - F_1(u,v) =& 0, \\ v - F_2(u,v) =& 0. \end{aligned}$$

Inequalities above imply

$$\begin{aligned} \|F_1(u,v)\|_\infty =& \|u\|_\infty, \\ \|F_2(u,v)\|_\infty =& \|v\|_\infty. \end{aligned}$$

From Remark 1, we conclude

$$\begin{aligned} p+1 \geq& \|F_1(u,v)\|_\infty = \frac{1}{j_x\sqrt{p^2-1-q^2}}, \\ 2q \geq& \|F_2(u,v)\|_\infty = \frac{q}{2j_x\sqrt{p^2-1-q^2}}. \end{aligned}$$

However, this contradicts (10), therefore $H((u,v),1) \neq (0,0)$. Finally, let $t_0 \in (0,1)$. If $H((u,v),t_0) = (0,0)$, then

$$\begin{aligned} t_0\varphi(u,v) + (1-t_0)I_{X\times X}(u,v) =& (0,0), \\ t_0(u,v) - t_0F(u,v) + (u,v) - t_0(u,v) =& (0,0), \\ (u,v) - t_0F(u,v) =& (0,0). \end{aligned}$$

We have the following scalar equalities

$$\begin{aligned} t_0F_1(u,v) =& u, \\ t_0F_2(u,v) =& v \end{aligned}$$

which are equivalent to:

$$\begin{aligned} t_0\|F_1(u,v)\|_\infty =& \|u\|_\infty, \\ t_0\|F_2(u,v)\|_\infty =& \|v\|_\infty. \end{aligned}$$

Again, Remark 1 gives us

$$t_0(p+1) \geq \frac{1}{j_x\sqrt{p^2-1-q^2}}$$

implying that

$$j_x \geq \frac{1}{t_0(p+1)\sqrt{p^2-1-q^2}} \geq \frac{1}{(p+1)\sqrt{p^2-1-q^2}}$$

and

$$2t_0q \geq \frac{q}{2j_x\sqrt{p^2-1-q^2}}$$

which implies

$$j_x \geq \frac{1}{4t_0\sqrt{p^2-1-q^2}} \geq \frac{1}{4\sqrt{p^2-1-q^2}}.$$

The inequalities above contradict condition (10). So, H never assumes the value $(0,0)$ on $\partial\Omega_1 \times \partial\Omega_2$. Hence, H is an admissible homotopy between φ and $I_{X\times X}$. Therefore,

$$d(\varphi, \Omega_1 \times \Omega_2, (0,0)) = d(I_{X\times X}, \Omega_1 \times \Omega_2, (0,0)) = 1.$$

This means that the equation $(u, v) - F(u, v) = (0, 0)$ has at least one solution on $\Omega_1 \times \Omega_2$. $\square$

Remark 2. *Note that condition (10) can be rewritten as*

$$j_x < \begin{cases} \frac{1}{4\sqrt{p^2-1-q^2}}, & \textit{if } 1 \leq p < 3, \\ \frac{1}{(p+1)\sqrt{p^2-1-q^2}}, & \textit{if } p \geq 3. \end{cases}$$

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