

Article

Existence Results for a Nonlocal Coupled System of Differential Equations Involving Mixed Right and Left Fractional Derivatives and Integrals

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Abstract: In this paper, we study the existence and uniqueness of solutions for a new kind of nonlocal four-point fractional integro-differential system involving both left Caputo and right Riemann–Liouville fractional derivatives, and Riemann–Liouville type mixed integrals. The Banach and Schaefer fixed point theorems are used to obtain the desired results. An example illustrating the existence and uniqueness result is presented.

Keywords: fractional differential equations; fractional derivative; systems; existence; fixed point theorems

1. Introduction

Fractional-order boundary value problems involving different kinds of fractional derivatives and boundary conditions have been investigated by many researchers in recent years. The literature on the topic is now much enriched and contains a wide variety of results, for instance, see the texts [1–3] and articles [4–10]. On the other hand, there has also been shown a great interest in the fractional differential systems in view of the occurrence of such systems in the mathematical models of physical and engineering problems. In [11], the authors carried out dynamical analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system. A delay fractional order model was proposed for the co-infection of malaria and HIV/AIDS in [12]. Chaos synchronization in fractional differential systems was explained in the article [13]. For details on diffusion and reactions in fractals and disordered Systems, we refer the reader to the text [14]. Using the Riemann–Liouville fractional operator, the unsteady axial Couette flow of fractional second grade fluid and fractional Maxwell fluid between two infinitely long concentric circular cylinders was studied in [15]. In a survey [16], the authors collected the results on the fractional analogue of Bhalekar–Gejji system, Lorenz system, Liu system, Chen system and Rössler system as a characteristic representative of fractional order autonomous dynamical system. For more applications of fractional calculus on bioengineering, anomalous diffusion of contamination, earth system dynamics, open channel flow, transient flow, physical models, fluid mechanics, viscoelastic fluids, etc., we refer the reader to the articles [17–26]. In view of extensive occurrence of couples fractional differential systems in a variety of mathematical models, many authors turned to the theoretical development of such systems, for example, see [27–31].

However, there are fewer works on boundary value problems containing both left and right fractional derivatives. Such problems constitute a special class of Euler–Lagrange equations,

and facilitate the study of variational principles [32]. In [33], the authors applied a probabilistic approach to study equations involving both left-sided and right-sided generalized operators of Caputo type, and showed a relationship between these equations and two-sided exit problems for certain Levy processes. In [34], the author related the study of fully mixed and multidimensional extensions of the Caputo and Riemann–Liouville derivatives with Markov processes. The left-sided and right-sided fractional derivatives were used to formulate the fractional diffusion–advection equation to study anomalous superdiffusive transport phenomena in [35]. For further details, we refer the reader to the articles [36–39]. In a more recent work [40], the authors investigated the existence of solutions for a new kind of integro-differential equation involving right-Caputo and left-Riemann–Liouville fractional derivatives of different orders and right-left Riemann–Liouville fractional integrals equipped with nonlocal boundary conditions.

In this paper, motivated by aforementioned work on mixed fractional differential equations, we introduce and study a new coupled system of nonlinear fractional differential equations, involving left Caputo and right Riemann–Liouville fractional derivatives of different orders and a pair of nonlinearities with one of them in terms of mixed fractional integrals in each equation of the system, equipped with four-point nonlocal coupled boundary conditions given by

$$\begin{cases} {}^C D_{1-}^{\alpha_1} {}^{RL} D_{0+}^{\beta_1} x(t) + \lambda_1 I_{1-}^{p_1} I_{0+}^{q_1} h_1(t, x(t), y(t)) = f_1(t, x(t), y(t)), & t \in [0, 1], \\ {}^C D_{1-}^{\alpha_2} {}^{RL} D_{0+}^{\beta_2} y(t) + \lambda_2 I_{1-}^{p_2} I_{0+}^{q_2} h_2(t, x(t), y(t)) = f_2(t, x(t), y(t)), & t \in [0, 1], \\ x(0) = x(\xi) = 0, & x(1) = \delta y(\mu), \\ y(0) = y(\eta) = 0, & y(1) = \rho x(\omega), \end{cases} \quad 0 < \xi < \mu < \eta < \omega < 1, \quad (1)$$

where ${}^C D_{1-}^{\alpha_1}, {}^C D_{1-}^{\alpha_2}$ denote the right Caputo fractional derivatives of orders $1 < \alpha_1, \alpha_2 < 2$ and ${}^{RL} D_{0+}^{\beta_1}, {}^{RL} D_{0+}^{\beta_2}$ denote the left Riemann–Liouville fractional derivatives of orders $0 < \beta_1, \beta_2 < 1$, $I_{1-}^{p_1}, I_{1-}^{p_2}$ and $I_{0+}^{q_1}, I_{0+}^{q_2}$ denote the right and left Riemann–Liouville fractional integrals of orders $p_1, p_2, q_1, q_2 > 0$ respectively, $f_1, f_2, h_1, h_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions and $\delta, \rho, \lambda_1, \lambda_2 \in \mathbb{R}$.

The rest of the paper is organized as follows. In Section 2 we outline the basic concepts from fractional calculus. In Section 3 we first prove an auxiliary lemma for the linear variant of the problem (1). Then we derive the existence and uniqueness result for the problem (1) by applying Banach's fixed point theorem, while the existence result is established via Schaefer's fixed point theorem. An example illustrating the uniqueness result is also presented.

2. Preliminaries

In this section we recall some related definitions of fractional calculus needed in our study.

Definition 1 ([41]). The left and right Riemann–Liouville fractional integrals of order $\beta > 0$ for $g \in L_1[a, b]$, existing almost everywhere on $[a, b]$, are respectively defined by

$$I_{a+}^{\beta} g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds \quad \text{and} \quad I_{b-}^{\beta} g(t) = \int_t^b \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} g(s) ds.$$

Also, according to the classical theorem of Vallee-Poussin and the Young convolution theorem, $I_{a+}^{\beta} g, I_{b-}^{\beta} g \in L_1[a, b], \beta > 0$.

Lemma 1. For $g \in L_p[a, b], 1 \leq p < \infty$ and $q_1, q_2 > 0$, the following relations hold almost everywhere on $[a, b]$:

$$I_{b-}^{q_1} I_{b-}^{q_2} g(x) = I_{b-}^{q_1+q_2} g(x), \quad I_{a+}^{q_1} I_{a+}^{q_2} g(x) = I_{a+}^{q_1+q_2} g(x).$$

Of course, if $g \in C[a, b]$ or $q_1 + q_2 > 1$, then the above relations hold for each $x \in [a, b]$.

Definition 2 ([1]). For $g \in AC^n[a, b]$, the left Riemann–Liouville and the right Caputo fractional derivatives of order $\beta \in (n-1, n]$, $n \in \mathbb{N}$, existing almost everywhere on $[a, b]$, are respectively defined by

$${}^{RL}D_{a+}^{\beta}g(t) = \frac{d^n}{dt^n} \int_a^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)}g(s)ds \quad \text{and} \quad {}^CD_{b-}^{\beta}g(t) = (-1)^n \int_t^b \frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)}g^{(n)}(s)ds.$$

3. Existence and Uniqueness Results

The following lemma, dealing with a linear variant of the problem (1), plays an important role in the forthcoming analysis.

Lemma 2. Let $F_1, F_2, H_1, H_2 \in C([0, 1], \mathbb{R})$. Then the integral solution of the linear coupled system

$$\begin{cases} {}^CD_{1-}^{\alpha_1} {}^{RL}D_{0+}^{\beta_1}x(t) + \lambda_1 I_{1-}^{p_1} I_{0+}^{q_1} H_1(t) = F_1(t), & t \in [0, 1], \\ {}^CD_{1-}^{\alpha_2} {}^{RL}D_{0+}^{\beta_2}y(t) + \lambda_2 I_{1-}^{p_2} I_{0+}^{q_2} H_2(t) = F_2(t), & t \in [0, 1], \end{cases} \quad (2)$$

subject to the boundary conditions of the problem (1), is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds + u_1(t) \kappa_1 \int_0^{\xi} \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds \\ & + u_2(t) \left[-\kappa_2 \int_0^{\eta} \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds - \kappa_3 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds \right. \\ & + \kappa_3 \delta \int_0^{\mu} \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds - \kappa_4 \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds \\ & \left. + \kappa_4 \rho \int_0^{\omega} \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds \right] + u_3(t) \kappa_5 \int_0^{\xi} \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds, \end{aligned} \quad (3)$$

$$\begin{aligned} y(t) = & \int_0^t \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds + v_1(t) \gamma_1 \int_0^{\eta} \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds \\ & + v_2(t) \left[-\gamma_2 \int_0^{\xi} \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds - \gamma_3 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds \right. \\ & + \gamma_3 \delta \int_0^{\mu} \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds - \gamma_4 \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds \\ & \left. + \gamma_4 \rho \int_0^{\omega} \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} J_1(s)ds \right] + v_3(t) \gamma_5 \int_0^{\eta} \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} J_2(s)ds, \end{aligned} \quad (4)$$

where

$$\begin{aligned} J_1(s) &= I_{1-}^{\alpha_1} F_1(s) - \lambda_1 I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} H_1(s), \quad J_2(s) = I_{1-}^{\alpha_2} F_2(s) - \lambda_2 I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} H_2(s), \\ \begin{cases} u_1(t) = \frac{-t^{\beta_1}}{\Lambda\Gamma(\beta_1+1)}, & u_2(t) = \frac{t^{\beta_1}[\epsilon_2(\beta_1+1) - \epsilon_1 t]}{\Lambda\Gamma(\beta_1+2)}, & u_3(t) = \frac{t^{\beta_1+1}}{\Lambda\Gamma(\beta_1+2)}, \\ v_1(t) = \frac{t^{\beta_2}}{\Lambda\Gamma(\beta_2+1)}, & v_2(t) = \frac{t^{\beta_2}[\epsilon_3 t - \epsilon_4(\beta_2+1)]}{\Lambda\Gamma(\beta_2+1)}, & v_3(t) = \frac{-t^{\beta_2+1}}{\Lambda\Gamma(\beta_2+2)}, \end{cases} \end{aligned} \quad (5)$$

$$\begin{cases} \kappa_1 = [\epsilon_3(\epsilon_6\epsilon_{12} - \epsilon_8\epsilon_{10}) + \epsilon_4(\epsilon_7\epsilon_{10} - \epsilon_6\epsilon_{11})], & \kappa_2 = (\epsilon_7\epsilon_{12} - \epsilon_8\epsilon_{11}), & \kappa_3 = (\epsilon_4\epsilon_{11} - \epsilon_3\epsilon_{12}), \\ \kappa_4 = (\epsilon_3\epsilon_8 - \epsilon_4\epsilon_7), & \kappa_5 = [\epsilon_3(\epsilon_5\epsilon_{12} - \epsilon_8\epsilon_9) + \epsilon_4(\epsilon_7\epsilon_9 - \epsilon_5\epsilon_{11})], \end{cases} \quad (6)$$

$$\begin{cases} \gamma_1 = [\epsilon_1(\epsilon_8\epsilon_{10} - \epsilon_6\epsilon_{12}) + \epsilon_2(\epsilon_5\epsilon_{12} - \epsilon_8\epsilon_9)], & \gamma_2 = (\epsilon_5\epsilon_{10} - \epsilon_6\epsilon_9), & \gamma_3 = (\epsilon_2\epsilon_9 - \epsilon_1\epsilon_{10}), \\ \gamma_4 = (\epsilon_1\epsilon_6 - \epsilon_2\epsilon_5), & \gamma_5 = [\epsilon_1(\epsilon_7\epsilon_{10} - \epsilon_6\epsilon_{11}) + \epsilon_2(\epsilon_5\epsilon_{11} - \epsilon_7\epsilon_9)], \end{cases} \quad (7)$$

$$\left\{ \begin{array}{l} \epsilon_1 = \frac{\xi^{\beta_1}}{\Gamma(\beta_1 + 1)}, \quad \epsilon_2 = \frac{\xi^{\beta_1+1}}{\Gamma(\beta_1 + 2)}, \quad \epsilon_3 = \frac{\eta^{\beta_2}}{\Gamma(\beta_2 + 1)}, \quad \epsilon_4 = \frac{\eta^{\beta_2+1}}{\Gamma(\beta_2 + 2)}, \\ \epsilon_5 = \frac{1}{\Gamma(\beta_1 + 1)}, \quad \epsilon_6 = \frac{1}{\Gamma(\beta_1 + 2)}, \quad \epsilon_7 = \frac{-\delta\mu^{\beta_2}}{\Gamma(\beta_2 + 1)}, \quad \epsilon_8 = \frac{-\delta\mu^{\beta_2+1}}{\Gamma(\beta_2 + 2)}, \\ \epsilon_9 = \frac{-\rho\omega^{\beta_1}}{\Gamma(\beta_1 + 1)}, \quad \epsilon_{10} = \frac{-\rho\omega^{\beta_1+1}}{\Gamma(\beta_1 + 2)}, \quad \epsilon_{11} = \frac{1}{\Gamma(\beta_2 + 1)}, \quad \epsilon_{12} = \frac{1}{\Gamma(\beta_2 + 2)}, \end{array} \right. \quad (8)$$

and it is assumed that

$$\Lambda = \epsilon_1 [\epsilon_3(\epsilon_6\epsilon_{12} - \epsilon_8\epsilon_{10}) + \epsilon_4(\epsilon_7\epsilon_{10} - \epsilon_6\epsilon_{11})] + \epsilon_2 [\epsilon_3(\epsilon_8\epsilon_9 - \epsilon_5\epsilon_{12}) + \epsilon_4(\epsilon_5\epsilon_{11} - \epsilon_7\epsilon_9)] \neq 0. \quad (9)$$

Proof. Applying the right fractional integral $I_{1-}^{\alpha_1}$ to the first equation in the system (2), followed by operator $I_{0+}^{\beta_1}$ to the resulting equation, we get

$$x(t) = I_{0+}^{\beta_1} I_{1-}^{\alpha_1} F_1(t) - \lambda_1 I_{0+}^{\beta_1} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} H_1(t) + c_0 \frac{t^{\beta_1}}{\Gamma(\beta_1 + 1)} + c_1 \frac{t^{\beta_1+1}}{\Gamma(\beta_1 + 2)} + c_2 t^{\beta_1-1}. \quad (10)$$

Similarly, applying the right fractional integrals $I_{1-}^{\alpha_2}$ and $I_{0+}^{\beta_2}$ successively to the second equation in the system (2), we obtain

$$y(t) = I_{0+}^{\beta_2} I_{1-}^{\alpha_2} F_2(t) - \lambda_2 I_{0+}^{\beta_2} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} H_2(t) + c_3 \frac{t^{\beta_2}}{\Gamma(\beta_2 + 1)} + c_4 \frac{t^{\beta_2+1}}{\Gamma(\beta_2 + 2)} + c_5 t^{\beta_2-1}, \quad (11)$$

where $c_i, i = 0, 1, 2, 3, 4, 5$ are unknown arbitrary constants. Using the boundary conditions $x(0) = 0$ and $y(0) = 0$ in Equations (10) and (11), we get $c_2 = 0$ and $c_5 = 0$. Then, using the remaining boundary conditions of the problem (1) in the resulting equations, we get a system of equations in c_0, c_1, c_3 and c_4 given by

$$\begin{aligned} c_0 \frac{\xi^{\beta_1}}{\Gamma(\beta_1 + 1)} + c_1 \frac{\xi^{\beta_1+1}}{\Gamma(\beta_1 + 2)} &= -A_1, \\ c_3 \frac{\eta^{\beta_2}}{\Gamma(\beta_2 + 1)} + c_4 \frac{\eta^{\beta_2+1}}{\Gamma(\beta_2 + 2)} &= -A_2, \\ c_0 \frac{1}{\Gamma(\beta_1 + 1)} + c_1 \frac{1}{\Gamma(\beta_1 + 2)} - c_3 \frac{\delta\mu^{\beta_2}}{\Gamma(\beta_2 + 1)} - c_4 \frac{\delta\mu^{\beta_2+1}}{\Gamma(\beta_2 + 2)} &= (\delta A_4 - A_3), \\ -c_0 \frac{\rho\omega^{\beta_1}}{\Gamma(\beta_1 + 1)} - c_1 \frac{\rho\omega^{\beta_1+1}}{\Gamma(\beta_1 + 2)} + c_3 \frac{1}{\Gamma(\beta_2 + 1)} + c_4 \frac{1}{\Gamma(\beta_2 + 2)} &= (\rho A_6 - A_5), \end{aligned} \quad (12)$$

where

$$A_1 = I_{0+}^{\beta_1} J_1(\xi), \quad A_2 = I_{0+}^{\beta_2} J_2(\eta), \quad A_3 = I_{0+}^{\beta_1} J_1(1), \quad A_4 = I_{0+}^{\beta_2} J_2(\mu), \quad A_5 = I_{0+}^{\beta_2} J_2(1), \quad A_6 = I_{0+}^{\beta_1} J_1(\omega).$$

Using the notations (6)–(8), we solve the system (12) for c_0, c_1, c_3, c_4 by Matlab to find that

$$\begin{aligned} c_0 &= \frac{1}{\Lambda} \{ -A_1\kappa_1 - A_2\epsilon_2\kappa_2 + (\delta A_4 - A_3)\epsilon_2\kappa_3 + (\rho A_6 - A_5)\epsilon_2\kappa_4 \}, \\ c_1 &= \frac{-1}{\Lambda} \{ -A_1\kappa_5 - A_2\epsilon_1\kappa_2 + (\delta A_4 - A_3)\epsilon_1\kappa_3 + (\rho A_6 - A_5)\epsilon_1\kappa_4 \}, \\ c_3 &= \frac{-1}{\Lambda} \{ -A_1\epsilon_4\gamma_2 - A_2\gamma_1 + (\delta A_4 - A_3)\epsilon_4\gamma_3 + (\rho A_6 - A_5)\epsilon_4\gamma_4 \}, \\ c_4 &= \frac{1}{\Lambda} \{ -A_1\epsilon_3\gamma_2 - A_2\gamma_5 + (\delta A_4 - A_3)\epsilon_3\gamma_3 + (\rho A_6 - A_5)\epsilon_3\gamma_4 \}, \end{aligned}$$

where Λ is given by (9). Substituting the values of c_0, c_1, c_3 and c_4 together with the notations (5) in (10) and (11), we get the solution (3) and (4). The converse follows by direct computation. This completes the proof. $\square$

Let $\mathcal{X} = C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ equipped with the norm $\|x\| = \sup \{|x(t)| : t \in [0, 1]\}$. The product space $(\mathcal{X} \times \mathcal{X}, \|(x, y)\|)$ is also Banach space endowed with norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 2, we define an operator $\mathcal{K} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ as

$$\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{K}_1(x, y)(t) &= \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds + u_1(t) \kappa_1 \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds \\ &\quad + u_2(t) \left[-\kappa_2 \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds - \kappa_3 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds \right. \\ &\quad + \kappa_3 \delta \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds - \kappa_4 \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds \\ &\quad \left. + \kappa_4 \rho \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds \right] + u_3(t) \kappa_5 \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds, \\ \mathcal{K}_2(x, y)(t) &= \int_0^t \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds + v_1(t) \gamma_1 \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds \\ &\quad + v_2(t) \left[-\gamma_2 \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds - \gamma_3 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds \right. \\ &\quad + \gamma_3 \delta \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds - \gamma_4 \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds \\ &\quad \left. + \gamma_4 \rho \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} \hat{f}_1(s, x(s), y(s)) ds \right] + v_3(t) \gamma_5 \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \hat{f}_2(s, x(s), y(s)) ds, \\ \hat{f}_1(s, x(s), y(s)) &= I_{1-}^{\alpha_1} f_1(s, x(s), y(s)) - \lambda_1 I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} h_1(s, x(s), y(s)), \\ \hat{f}_2(s, x(s), y(s)) &= I_{1-}^{\alpha_2} f_2(s, x(s), y(s)) - \lambda_2 I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} h_2(s, x(s), y(s)). \end{aligned}$$

Note that

$$\begin{aligned} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^q ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} \int_0^u \frac{(u-r)^{q-1}}{\Gamma(q)} dr du ds \\ &\leq \frac{t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+p+1)\Gamma(q+1)}, \\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du ds \leq \frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \end{aligned}$$

where we have used the fact that $u^q \leq 1$, $(1-s)^{\alpha+p} < 1$, $(1-s)^\alpha \leq 1$ for $p, q > 0$, $1 < \alpha \leq 2$.

For computational convenience, we set

$$\begin{aligned}\Omega_1 &= \frac{\Phi_1}{\Gamma(\beta_1 + 1)\Gamma(\alpha_1 + 1)}, & \Omega_3 &= \frac{|\lambda_1|\Phi_1}{\Gamma(\beta_1 + 1)\Gamma(\alpha_1 + p_1 + 1)\Gamma(q_1 + 1)}, \\ \Omega_2 &= \frac{\Phi_2}{\Gamma(\beta_2 + 1)\Gamma(\alpha_2 + 1)}, & \Omega_4 &= \frac{|\lambda_2|\Phi_2}{\Gamma(\beta_2 + 1)\Gamma(\alpha_2 + p_2 + 1)\Gamma(q_2 + 1)}, \\ \Omega_5 &= \frac{\Phi_3}{\Gamma(\beta_1 + 1)\Gamma(\alpha_1 + 1)}, & \Omega_7 &= \frac{|\lambda_1|\Phi_3}{\Gamma(\beta_1 + 1)\Gamma(\alpha_1 + p_1 + 1)\Gamma(q_1 + 1)}, \\ \Omega_6 &= \frac{\Phi_4}{\Gamma(\beta_2 + 1)\Gamma(\alpha_2 + 1)}, & \Omega_8 &= \frac{|\lambda_2|\Phi_4}{\Gamma(\beta_2 + 1)\Gamma(\alpha_2 + p_2 + 1)\Gamma(q_2 + 1)},\end{aligned}\quad (13)$$

where

$$\begin{aligned}\Phi_1 &= [1 + \xi^{\beta_1}(\bar{u}_1|\kappa_1| + \bar{u}_3|\kappa_5|) + \bar{u}_2(|\kappa_3| + |\kappa_4||\rho|\omega^{\beta_1})], & \Phi_2 &= [\bar{u}_2(|\kappa_2|\eta^{\beta_2} + |\kappa_3||\delta|\mu^{\beta_2} + |\kappa_4|)], \\ \Phi_3 &= [\bar{v}_2(|\gamma_2|\xi^{\beta_1} + |\gamma_3| + |\gamma_4||\rho|\omega^{\beta_1})], & \Phi_4 &= [1 + \eta^{\beta_2}(\bar{v}_1|\gamma_1| + \bar{v}_3|\gamma_5|) + \bar{v}_2(|\gamma_4| + |\gamma_3||\delta|\mu^{\beta_2})],\end{aligned}$$

$$\begin{aligned}\bar{u}_1 &= \max_{t \in [0,1]} |u_1(t)|, & \bar{u}_2 &= \max_{t \in [0,1]} |u_2(t)|, & \bar{u}_3 &= \max_{t \in [0,1]} |u_3(t)|, \\ \bar{v}_1 &= \max_{t \in [0,1]} |v_1(t)|, & \bar{v}_2 &= \max_{t \in [0,1]} |v_2(t)|, & \bar{v}_3 &= \max_{t \in [0,1]} |v_3(t)|.\end{aligned}$$

Now we are in a position to prove the existence and uniqueness of solutions to the system (1) by Banach contraction mapping principle.

Theorem 1. Let $f_1, f_2, h_1, h_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that the following conditions hold:

(B₁) There exist $L_1, L_2 > 0$ such that $\forall t \in [0, 1]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2$,

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \leq L_1(|x_1 - x_2| + |y_1 - y_2|),$$

$$|f_2(t, x_1, y_1) - f_2(t, x_2, y_2)| \leq L_2(|x_1 - x_2| + |y_1 - y_2|).$$

(B₂) There exist $K_1, K_2 > 0$ such that $\forall t \in [0, 1]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2$,

$$|h_1(t, x_1, y_1) - h_1(t, x_2, y_2)| \leq K_1(|x_1 - x_2| + |y_1 - y_2|),$$

$$|h_2(t, x_1, y_1) - h_2(t, x_2, y_2)| \leq K_2(|x_1 - x_2| + |y_1 - y_2|).$$

Then the system (1) has a unique solution on $[0, 1]$ provided that

$$\Psi := L_1(\Omega_1 + \Omega_5) + L_2(\Omega_2 + \Omega_6) + K_1(\Omega_3 + \Omega_7) + K_2(\Omega_4 + \Omega_8) < 1, \quad (14)$$

where $\Omega_i, i = 1, 2, \dots, 8$, are defined by (13).

Proof. Let us set

$$r \geq \frac{f_0(\Omega_1 + \Omega_5) + \hat{f}_0(\Omega_2 + \Omega_6) + h_0(\Omega_3 + \Omega_7) + \hat{h}_0(\Omega_4 + \Omega_8)}{1 - \Psi},$$

where $f_0, \hat{f}_0, h_0, \hat{h}_0$ are finite numbers defined by

$$\sup_{t \in [0,1]} |f_1(t, 0, 0)| = f_0, \quad \sup_{t \in [0,1]} |f_2(t, 0, 0)| = \hat{f}_0, \quad \sup_{t \in [0,1]} |h_1(t, 0, 0)| = h_0, \quad \sup_{t \in [0,1]} |h_2(t, 0, 0)| = \hat{h}_0,$$

and Ψ is defined by (14). Next we consider a closed ball $\mathcal{B}_r = \{(x, y) \in \mathcal{X} \times \mathcal{X} : \|(x, y)\| \leq r\}$ and show that $\mathcal{KB}_r \subset \mathcal{B}_r$. Then, in view of the assumption (B_1) , we have

$$\begin{aligned} |f_1(t, x, y)| &= |f_1(t, x, y) - f_1(t, 0, 0) + f_1(t, 0, 0)| \leq |f_1(t, x, y) - f_1(t, 0, 0)| + |f_1(t, 0, 0)| \\ &\leq L_1(|x(t)| + |y(t)|) + f_0 \leq L_1(\|x\| + \|y\|) + f_0 \leq L_1r + f_0. \end{aligned}$$

Similarly, we can find that

$$|f_2(t, x, y)| \leq L_2r + \hat{f}_0, \quad |h_1(t, x, y)| \leq K_1r + h_0, \quad |h_2(t, x, y)| \leq K_2r + \hat{h}_0.$$

Then, for $(x, y) \in \mathcal{B}_r$, we have

$$\begin{aligned} \|\mathcal{K}_1(x, y)\| &\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\hat{f}_1(s, x(s), y(s))| ds \right. \\ &\quad + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\hat{f}_1(s, x(s), y(s))| ds \\ &\quad + |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\hat{f}_2(s, x(s), y(s))| ds \right. \\ &\quad + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\hat{f}_1(s, x(s), y(s))| ds \\ &\quad + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\hat{f}_2(s, x(s), y(s))| ds \\ &\quad + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\hat{f}_2(s, x(s), y(s))| ds \\ &\quad + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\hat{f}_1(s, x(s), y(s))| ds \Big] \\ &\quad \left. + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\hat{f}_1(s, x(s), y(s))| ds \right\} \\ &\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1} (L_1r + f_0) + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} (K_1r + h_0) \right] ds \right. \\ &\quad + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1} (L_1r + f_0) + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} (K_1r + h_0) \right] ds \\ &\quad + |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2} (L_2r + \hat{f}_0) + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} (K_2r + \hat{h}_0) \right] ds \right. \\ &\quad + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1} (L_1r + f_0) + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} (K_1r + h_0) \right] ds \\ &\quad + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2} (L_2r + \hat{f}_0) + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} (K_2r + \hat{h}_0) \right] ds \\ &\quad + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2} (L_2r + \hat{f}_0) + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2} (K_2r + \hat{h}_0) \right] ds \\ &\quad + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1} (L_1r + f_0) + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} (K_1r + h_0) \right] ds \Big] \\ &\quad \left. + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1} (L_1r + f_0) + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1} (K_1r + h_0) \right] ds \right\} \\ &\leq (L_1r + f_0) \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1} (1) ds + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1} (1) ds \right. \end{aligned}$$

$$\begin{aligned}
& +|u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \right] \\
& +|u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \Big\} \\
& + (L_2 r + \hat{f}_0) \sup_{t \in [0,1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds \right. \right. \\
& \left. \left. + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds \right] \right\} \\
& + (K_1 r + h_0) |\lambda_1| \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right. \\
& + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \\
& + |u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right. \\
& \left. + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right] \\
& \left. + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right\} \\
& + (K_2 r + \hat{h}_0) |\lambda_2| \sup_{t \in [0,1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \right. \right. \\
& \left. \left. + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \right] \right\} \\
& \leq \frac{1}{\Gamma(\beta_1+1)} \left[\frac{(L_1 r + f_0)}{\Gamma(\alpha_1+1)} + \frac{(K_1 r + h_0) |\lambda_1|}{\Gamma(\alpha_1+p_1+1) \Gamma(q_1+1)} \right] \Phi_1 \\
& + \frac{1}{\Gamma(\beta_2+1)} \left[\frac{(L_2 r + \hat{f}_0)}{\Gamma(\alpha_2+1)} + \frac{(K_2 r + \hat{h}_0) |\lambda_2|}{\Gamma(\alpha_2+p_2+1) \Gamma(q_2+1)} \right] \Phi_2 \\
& = (L_1 r + f_0) \Omega_1 + (L_2 r + \hat{f}_0) \Omega_2 + (K_1 r + h_0) \Omega_3 + (K_2 r + \hat{h}_0) \Omega_4,
\end{aligned}$$

which implies that

$$\|\mathcal{K}_1(x, y)\| \leq [L_1 \Omega_1 + L_2 \Omega_2 + K_1 \Omega_3 + K_2 \Omega_4] r + f_0 \Omega_1 + \hat{f}_0 \Omega_2 + h_0 \Omega_3 + \hat{h}_0 \Omega_4.$$

Similarly, we can find that

$$\|\mathcal{K}_2(x, y)\| \leq [L_1 \Omega_5 + L_2 \Omega_6 + K_1 \Omega_7 + K_2 \Omega_8] r + f_0 \Omega_5 + \hat{f}_0 \Omega_6 + h_0 \Omega_7 + \hat{h}_0 \Omega_8.$$

Consequently, we get

$$\|\mathcal{K}(x, y)\| \leq \Psi r + (\Omega_1 + \Omega_5) f_0 + (\Omega_2 + \Omega_6) \hat{f}_0 + (\Omega_3 + \Omega_7) h_0 + (\Omega_4 + \Omega_8) \hat{h}_0 < r,$$

which implies that $\mathcal{K}(x, y) \in \mathcal{B}_r$ for any $(x, y) \in \mathcal{B}_r$. Therefore $\mathcal{K}\mathcal{B}_r \subset \mathcal{B}_r$.

Now, we prove that $\mathcal{K}$ is a contraction. Let $(x_1, y_1), (x_2, y_2) \in \mathcal{X} \times \mathcal{X}$ for each $t \in [0, 1]$. Then, by the conditions (B_1) and (B_2) , we get

$$\|\mathcal{K}_1(x_2, y_2) - \mathcal{K}_1(x_1, y_1)\|$$

$$\begin{aligned}
&\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\widehat{f}_1(s, x_2(s), y_2(s)) - \widehat{f}_1(s, x_1(s), y_1(s))| ds \right. \\
&\quad + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\widehat{f}_1(s, x_2(s), y_2(s)) - \widehat{f}_1(s, x_1(s), y_1(s))| ds \\
&\quad + |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\widehat{f}_2(s, x_2(s), y_2(s)) - \widehat{f}_2(s, x_1(s), y_1(s))| ds \right. \\
&\quad + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\widehat{f}_1(s, x_2(s), y_2(s)) - \widehat{f}_1(s, x_1(s), y_1(s))| ds \\
&\quad + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\widehat{f}_2(s, x_2(s), y_2(s)) - \widehat{f}_2(s, x_1(s), y_1(s))| ds \\
&\quad + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} |\widehat{f}_2(s, x_2(s), y_2(s)) - \widehat{f}_2(s, x_1(s), y_1(s))| ds \\
&\quad + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\widehat{f}_1(s, x_2(s), y_2(s)) - \widehat{f}_1(s, x_1(s), y_1(s))| ds \Big] \\
&\quad \left. + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} |\widehat{f}_1(s, x_2(s), y_2(s)) - \widehat{f}_1(s, x_1(s), y_1(s))| ds \right\} \\
&\leq (\|x_2 - x_1\| + \|y_2 - y_1\|) \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) L_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1}(1) I_{0+}^{q_1} K_1 \right] ds \right. \\
&\quad + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) L_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) K_1 \right] ds \\
&\quad + |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) L_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) K_2 \right] ds \right. \\
&\quad + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) L_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) K_1 \right] ds \\
&\quad + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) L_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) K_2 \right] ds \\
&\quad + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) L_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) K_2 \right] ds \\
&\quad + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) L_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) K_1 \right] ds \Big] \\
&\quad \left. + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) L_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) K_1 \right] ds \right\} \\
&\leq L_1 (\|x_2 - x_1\| + \|y_2 - y_1\|) \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \right. \\
&\quad + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds + |u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \right. \\
&\quad + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \Big] + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1) ds \Big\} \\
&\quad + L_2 (\|x_2 - x_1\| + \|y_2 - y_1\|) \sup_{t \in [0,1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds \right. \right. \\
&\quad \left. + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1) ds \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& +K_1(\|x_2 - x_1\| + \|y_2 - y_1\|)|\lambda_1| \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right. \\
& + |u_1(t)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \\
& + |u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right. \\
& + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \left. \right] \\
& + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \left. \right\} \\
& +K_2(\|x_2 - x_1\| + \|y_2 - y_1\|)|\lambda_2| \sup_{t \in [0,1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \right. \right. \\
& + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \left. \right] \left. \right\} \\
& \leq \left\{ \frac{1}{\Gamma(\beta_1+1)} \left[\frac{L_1}{\Gamma(\alpha_1+1)} + \frac{K_1|\lambda_1|}{\Gamma(\alpha_1+p_1+1)\Gamma(q_1+1)} \right] \Phi_1 \right. \\
& + \frac{1}{\Gamma(\beta_2+1)} \left[\frac{L_2}{\Gamma(\alpha_2+1)} + \frac{K_2|\lambda_2|}{\Gamma(\alpha_2+p_2+1)\Gamma(q_2+1)} \right] \Phi_2 \left. \right\} (\|x_2 - x_1\| + \|y_2 - y_1\|),
\end{aligned}$$

which implies that

$$\|\mathcal{K}_1(x_2, y_2) - \mathcal{K}_1(x_1, y_1)\| \leq (L_1\Omega_1 + L_2\Omega_2 + K_1\Omega_3 + K_2\Omega_4)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (15)$$

In a similar manner, one can find that

$$\|\mathcal{K}_2(x_2, y_2) - \mathcal{K}_2(x_1, y_1)\| \leq (L_1\Omega_5 + L_2\Omega_6 + K_1\Omega_7 + K_2\Omega_8)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (16)$$

Using (15) and (16), we obtain

$$\|\mathcal{K}(x_2, y_2) - \mathcal{K}(x_1, y_1)\| \leq \Psi(\|x_2 - x_1\| + \|y_2 - y_1\|).$$

From the above inequality, it follows by the assumption $\Psi < 1$ that $\mathcal{K}$ is a contraction. Hence we deduce by the Banach fixed point theorem that the operator $\mathcal{K}$ has a unique fixed point, which corresponds to a unique solution of system (1). The proof is completed. $\square$

Let us now recall Schaefer's fixed point theorem [42], which plays a key role in proving the next existence result.

Lemma 3 (Schaefer's fixed point Theorem). *Let X be a Banach space. Assume that $T : X \rightarrow X$ is a completely continuous operator and the set $V = \{u \in X | u = \nu Tu; 0 < \nu < 1\}$ is bounded. Then T has a fixed point in X .*

Theorem 2. *Let $f_1, f_2, h_1, h_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions satisfying the condition:*

(B₃) *There exist real constants $a_j, b_j, c_j, d_j \geq 0, j = 0, 1, 2$ and $a_0, b_0, c_0, d_0 \neq 0$ such that, $\forall x_k \in \mathbb{R}, k = 1, 2$,*

$$|f_1(t, x_1, x_2)| \leq a_0 + a_1|x_1| + a_2|x_2|, \quad |f_2(t, x_1, x_2)| \leq b_0 + b_1|x_1| + b_2|x_2|,$$

$$|h_1(t, x_1, x_2)| \leq c_0 + c_1|x_1| + c_2|x_2|, \quad |h_2(t, x_1, x_2)| \leq d_0 + d_1|x_1| + d_2|x_2|.$$

Then the system (1) has at least one solution on $[0, 1]$ if

$$(\Omega_1 + \Omega_5)a_1 + (\Omega_2 + \Omega_6)b_1 + (\Omega_3 + \Omega_7)c_1 + (\Omega_4 + \Omega_8)d_1 < 1,$$

and

$$(\Omega_1 + \Omega_5)a_2 + (\Omega_2 + \Omega_6)b_2 + (\Omega_3 + \Omega_7)c_2 + (\Omega_4 + \Omega_8)d_2 < 1,$$

where $\Omega_i, i = 1, 2, \dots, 8$ are given by (13).

Proof. Observe that continuity of the functions f_1, f_2, h_1, h_2 implies that the operator $\mathcal{K}$ is continuous. Next, we show that the operator $\mathcal{K}$ is uniformly bounded. Let $Q \subset \mathcal{X} \times \mathcal{X}$ be a bounded set. Then, for all $(x, y) \in Q$, there exist constants $M_i > 0, i = 1, 2, 3, 4$ such that $|f_1(t, x(t), y(t))| \leq M_1, |f_2(t, x(t), y(t))| \leq M_2, |h_1(t, x(t), y(t))| \leq M_3, |h_2(t, x(t), y(t))| \leq M_4$. For any $(x, y) \in Q$, we have

$$\begin{aligned} & \|\mathcal{K}_1(x, y)\| \\ \leq & \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1)M_1 + |\lambda_1|I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)M_3 \right] ds \right. \\ & + |u_1(t)||\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1)M_1 + |\lambda_1|I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)M_3 \right] ds \\ & + |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1)M_2 + |\lambda_2|I_{1-}^{\alpha_2+p_2}I_{0+}^{q_2}(1)M_4 \right] ds \right. \\ & + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1)M_1 + |\lambda_1|I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)M_3 \right] ds \\ & + |\kappa_3||\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1)M_2 + |\lambda_2|I_{1-}^{\alpha_2+p_2}I_{0+}^{q_2}(1)M_4 \right] ds \\ & + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1)M_2 + |\lambda_2|I_{1-}^{\alpha_2+p_2}I_{0+}^{q_2}(1)M_4 \right] ds \\ & + |\kappa_4||\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1)M_1 + |\lambda_1|I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)M_3 \right] ds \Big] \\ & \left. + |u_3(t)||\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1)M_1 + |\lambda_1|I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)M_3 \right] ds \right\} \\ \leq & M_1 \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1)ds + |u_1(t)||\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1)ds \right. \\ & + |u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1)ds + |\kappa_4||\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1)ds \right] \\ & \left. + |u_3(t)||\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1}(1)ds \right\} \\ & + M_2 \sup_{t \in [0, 1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1)ds + |\kappa_3||\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1)ds \right. \right. \\ & \left. \left. + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2}(1)ds \right] \right\} \\ & + M_3 |\lambda_1| \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1}(1)I_{0+}^{q_1}ds \right. \\ & \left. + |u_1(t)||\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1}I_{0+}^{q_1}(1)ds \right\} \end{aligned}$$

$$\begin{aligned}
& +|u_2(t)| \left[|\kappa_3| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \right. \\
& + |\kappa_4| |\rho| \int_0^\omega \frac{(\omega-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \Big] \\
& + |u_3(t)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) ds \Big\} \\
& + M_4 |\lambda_2| \sup_{t \in [0,1]} \left\{ |u_2(t)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \right. \right. \\
& + |\kappa_3| |\delta| \int_0^\mu \frac{(\mu-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds + |\kappa_4| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) ds \Big] \Big\} \\
& \leq \frac{1}{\Gamma(\beta_1+1)} \left[\frac{M_1}{\Gamma(\alpha_1+1)} + \frac{M_3 |\lambda_1|}{\Gamma(\alpha_1+p_1+1) \Gamma(q_1+1)} \right] \Phi_1 \\
& + \frac{1}{\Gamma(\beta_2+1)} \left[\frac{M_2}{\Gamma(\alpha_2+1)} + \frac{M_4 |\lambda_2|}{\Gamma(\alpha_2+p_2+1) \Gamma(q_2+1)} \right] \Phi_2 \\
& = M_1 \Omega_1 + M_2 \Omega_2 + M_3 \Omega_3 + M_4 \Omega_4.
\end{aligned}$$

Analogously, we can find that

$$\|\mathcal{K}_2(x, y)\| \leq M_1 \Omega_5 + M_2 \Omega_6 + M_3 \Omega_7 + M_4 \Omega_8.$$

From the foregoing inequalities, it follows that

$$\|\mathcal{K}(x, y)\| \leq (\Omega_1 + \Omega_5) M_1 + (\Omega_2 + \Omega_6) M_2 + (\Omega_3 + \Omega_7) M_3 + (\Omega_4 + \Omega_8) M_4.$$

Thus the operator $\mathcal{K}$ is uniformly bounded.

Next, we show that $\mathcal{K}$ is equicontinuous. For $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
& |\mathcal{K}_1(x(t_2), y(t_2)) - \mathcal{K}_1(x(t_1), y(t_1))| \\
& \leq \left| \int_0^{t_1} \frac{(t_2-s)^{\beta_1-1} - (t_1-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) M_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) M_3 \right] ds \right| \\
& + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) M_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) M_3 \right] ds \right| \\
& + |u_1(t_2) - u_1(t_1)| |\kappa_1| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) M_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) M_3 \right] ds \\
& + |u_2(t_2) - u_2(t_1)| \left[|\kappa_2| \int_0^\eta \frac{(\eta-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) M_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) M_4 \right] ds \right. \\
& + |\kappa_3| \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) M_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) M_4 \right] ds \\
& + |\kappa_4| \int_0^\omega \frac{(\omega-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) M_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) M_4 \right] ds \\
& + |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_2-1}}{\Gamma(\beta_2)} \left[I_{1-}^{\alpha_2}(1) M_2 + |\lambda_2| I_{1-}^{\alpha_2+p_2} I_{0+}^{q_2}(1) M_4 \right] ds \Big] \\
& + |u_3(t_2) - u_3(t_1)| |\kappa_5| \int_0^\xi \frac{(\xi-s)^{\beta_1-1}}{\Gamma(\beta_1)} \left[I_{1-}^{\alpha_1}(1) M_1 + |\lambda_1| I_{1-}^{\alpha_1+p_1} I_{0+}^{q_1}(1) M_3 \right] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\beta_1+1)} \left(\frac{M_1}{\Gamma(\alpha_1+1)} + \frac{M_3|\lambda_1|}{\Gamma(\alpha_1+p_1+1)\Gamma(q_1+1)} \right) \left[2(t_2-t_1)^{\beta_1} + |t_2^{\beta_1} - t_1^{\beta_1}| \right. \\
&\quad + \zeta^{\beta_1} \left\{ \frac{|t_1^{\beta_1} - t_2^{\beta_1}|}{|\Lambda|} \left[\frac{|\kappa_1|}{\Gamma(\beta_1+1)} \right] + \frac{|t_2^{\beta_1+1} - t_1^{\beta_1+1}|}{|\Lambda|} \left[\frac{|\kappa_5|}{\Gamma(\beta_1+2)} \right] \right\} \\
&\quad + \left\{ \frac{|t_2^{\beta_1} - t_1^{\beta_1}|}{|\Lambda|} \left[\frac{|\epsilon_2|}{\Gamma(\beta_1+1)} \right] + \frac{|t_2^{\beta_1+1} - t_1^{\beta_1+1}|}{|\Lambda|} \left[\frac{|\epsilon_1|}{\Gamma(\beta_1+2)} \right] \right\} (|\kappa_3| + |\kappa_4|\rho|\omega^{\beta_1}|) \Bigg] \\
&\quad + \frac{1}{\Gamma(\beta_2+1)} \left(\frac{M_2}{\Gamma(\alpha_2+1)} + \frac{M_4|\lambda_2|}{\Gamma(\alpha_2+p_2+1)\Gamma(q_2+1)} \right) \left[\left\{ \frac{|t_2^{\beta_1} - t_1^{\beta_1}|}{|\Lambda|} \left[\frac{|\epsilon_2|}{\Gamma(\beta_1+1)} \right] \right. \right. \\
&\quad \left. \left. + \frac{|t_2^{\beta_1+1} - t_1^{\beta_1+1}|}{|\Lambda|} \left[\frac{|\epsilon_1|}{\Gamma(\beta_1+2)} \right] \right\} (|\kappa_2|\eta^{\beta_2} + |\kappa_3|\delta|\mu^{\beta_2}| + |\kappa_4|) \right],
\end{aligned}$$

which tends to 0 as $t_1 \rightarrow t_2$ independently of $(x, y) \in Q$.

Analogously, we can obtain

$$\begin{aligned}
&|\mathcal{K}_2(x(t_2), y(t_2)) - \mathcal{K}_2(x(t_1), y(t_1))| \\
&\leq \frac{1}{\Gamma(\beta_2+1)} \left(\frac{M_2}{\Gamma(\alpha_2+1)} + \frac{M_4|\lambda_2|}{\Gamma(\alpha_2+p_2+1)\Gamma(q_2+1)} \right) \left[2(t_2-t_1)^{\beta_2} + (t_2^{\beta_2} - t_1^{\beta_2}) \right. \\
&\quad + \eta^{\beta_2} \left\{ \frac{|\gamma_1||t_2^{\beta_2} - t_1^{\beta_2}|}{|\Lambda|\Gamma(\beta_2+1)} + \frac{|\gamma_5||t_1^{\beta_2+1} - t_2^{\beta_2+1}|}{\Lambda\Gamma(\beta_2+2)} \right\} \\
&\quad + \left\{ \frac{|\epsilon_3||t_2^{\beta_2+1} - t_1^{\beta_2+1}|}{|\Lambda|\Gamma(\beta_2+2)} + \frac{|\epsilon_4||t_2^{\beta_2} - t_1^{\beta_2}|}{|\Lambda|\Gamma(\beta_2+1)} \right\} (|\gamma_4| + |\gamma_3|\delta|\mu^{\beta_2}|) \Bigg] \\
&\quad + \frac{1}{\Gamma(\beta_1+1)} \left(\frac{M_1}{\Gamma(\alpha_1+1)} + \frac{M_3|\lambda_1|}{\Gamma(\alpha_1+p_1+1)\Gamma(q_1+1)} \right) \left[\left\{ \frac{|\epsilon_3||t_2^{\beta_2+1} - t_1^{\beta_2+1}|}{|\Lambda|\Gamma(\beta_2+2)} \right. \right. \\
&\quad \left. \left. + \frac{|\epsilon_4||t_2^{\beta_2} - t_1^{\beta_2}|}{|\Lambda|\Gamma(\beta_2+1)} \right\} (|\gamma_2|\zeta^{\beta_1} + |\gamma_3| + |\gamma_4|\rho|\omega^{\beta_1}|) \right],
\end{aligned}$$

which tends to 0 as $t_1 \rightarrow t_2$ independently of $(x, y) \in Q$. In consequence, the operator $\mathcal{K}$ is equicontinuous. From the foregoing arguments, we deduce that the operator $\mathcal{K}(x, y)$ is completely continuous.

Finally, we show that the set $V = \{(x, y) \in \mathcal{X} \times \mathcal{X} | (x, y) = \nu \mathcal{K}(x, y), 0 \leq \nu \leq 1\}$ is bounded. Let $(x, y) \in V$ such that $(x, y) = \nu \mathcal{K}(x, y), \forall t \in [0, 1]$. Then we have

$$x(t) = \nu \mathcal{K}_1(x, y)(t), \quad y(t) = \nu \mathcal{K}_2(x, y)(t).$$

By the condition (B_3) , it is found that

$$\begin{aligned}
|x(t)| &\leq \Omega_1(a_0 + a_1|x| + a_2|y|) + \Omega_2(b_0 + b_1|x| + b_2|y|) \\
&\quad + \Omega_3(c_0 + c_1|x| + c_2|y|) + \Omega_4(d_0 + d_1|x| + d_2|y|),
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &\leq \Omega_5(a_0 + a_1|x| + a_2|y|) + \Omega_6(b_0 + b_1|x| + b_2|y|) \\
&\quad + \Omega_7(c_0 + c_1|x| + c_2|y|) + \Omega_8(d_0 + d_1|x| + d_2|y|).
\end{aligned}$$

Hence

$$\|x\| \leq \Omega_1 a_0 + \Omega_2 b_0 + \Omega_3 c_0 + \Omega_4 d_0 + (\Omega_1 a_1 + \Omega_2 b_1 + \Omega_3 c_1 + \Omega_4 d_1) \|x\| + (\Omega_1 a_2 + \Omega_2 b_2 + \Omega_3 c_2 + \Omega_4 d_2) \|y\|,$$

and

$$\|y\| \leq \Omega_5 a_0 + \Omega_6 b_0 + \Omega_7 c_0 + \Omega_8 d_0 + (\Omega_5 a_1 + \Omega_6 b_1 + \Omega_7 c_1 + \Omega_8 d_1) \|x\| + (\Omega_5 a_2 + \Omega_6 b_2 + \Omega_7 c_2 + \Omega_8 d_2) \|y\|.$$

Consequently, we get

$$\begin{aligned} \|x\| + \|y\| &\leq (\Omega_1 + \Omega_5) a_0 + (\Omega_2 + \Omega_6) b_0 + (\Omega_3 + \Omega_7) c_0 + (\Omega_4 + \Omega_8) d_0 \\ &\quad + ((\Omega_1 + \Omega_5) a_1 + (\Omega_2 + \Omega_6) b_1 + (\Omega_3 + \Omega_7) c_1 + (\Omega_4 + \Omega_8) d_1) \|x\| \\ &\quad + ((\Omega_1 + \Omega_5) a_2 + (\Omega_2 + \Omega_6) b_2 + (\Omega_3 + \Omega_7) c_2 + (\Omega_4 + \Omega_8) d_2) \|y\|, \end{aligned}$$

which leads to

$$\|(x, y)\| \leq \frac{(\Omega_1 + \Omega_5) a_0 + (\Omega_2 + \Omega_6) b_0 + (\Omega_3 + \Omega_7) c_0 + (\Omega_4 + \Omega_8) d_0}{W_0},$$

where

$$W_0 = \min \left\{ 1 - ((\Omega_1 + \Omega_5) a_1 + (\Omega_2 + \Omega_6) b_1 + (\Omega_3 + \Omega_7) c_1 + (\Omega_4 + \Omega_8) d_1), \right. \\ \left. 1 - ((\Omega_1 + \Omega_5) a_2 + (\Omega_2 + \Omega_6) b_2 + (\Omega_3 + \Omega_7) c_2 + (\Omega_4 + \Omega_8) d_2) \right\}.$$

Therefore, the set V is bounded. Hence, by Lemma 3, the operator $\mathcal{K}$ has at least one fixed point, which is indeed a solution of the system (1) on $[0, 1]$. The theorem is proved. $\square$

4. Example

In this section, we demonstrate the application of Theorem 1 by constructing an example containing a coupled system of mixed integro-fractional differential equations of fixed orders with boundary conditions involving fixed parameters. Precisely, we consider the following system:

$$\begin{cases} {}^C D_{1-}^{5/4} {}^{RL} D_{0+}^{1/2} x(t) + I_{1-}^{1/3} I_{0+}^{1/5} h_1(t, x(t), y(t)) = f_1(t, x(t), y(t)), & t \in J := [0, 1], \\ {}^C D_{1-}^{3/2} {}^{RL} D_{0+}^{1/4} y(t) + I_{1-}^{2/3} I_{0+}^{2/5} h_2(t, x(t), y(t)) = f_2(t, x(t), y(t)), & t \in J := [0, 1], \\ x(0) = x(1/5) = 0, & x(1) = 2y(2/5), \\ y(0) = y(3/5) = 0, & y(1) = 4x(4/5). \end{cases} \quad (17)$$

Here $\alpha_1 = 5/4$, $\alpha_2 = 3/2$, $\beta_1 = 1/2$, $\beta_2 = 1/4$, $p_1 = 1/3$, $p_2 = 2/3$, $q_1 = 1/5$, $q_2 = 2/5$, $\xi = 1/5$, $\mu = 2/5$, $\eta = 3/5$, $\omega = 4/5$, $\delta = 2$, $\rho = 4$, $\lambda_1 = \lambda_2 = 1$ and

$$\begin{aligned} f_1(t, x, y) &= \frac{1}{100} (x + \tan^{-1} x + \sin^2 t) + \frac{1}{2(t^2 + 5)^2} \frac{|y|}{1 + |y|} + \sin^2 t, \\ f_2(t, x, y) &= \frac{1}{5} \left(\frac{1}{\sqrt{(t^2 + 16)}} \frac{|x|}{1 + |x|} + \frac{t^2}{4} \right) + \frac{1}{(t^2 + 40)} \left(\frac{|y|}{1 + |y|} + |y| + \sin^2 t \right), \\ h_1(t, x, y) &= \frac{1}{30} \left(\cos t + \tan^{-1} x + \frac{t^2}{5} \right) + \frac{1}{5(t^2 + 12)} \left(\sin y + t^2 \tan^{-1} y + \frac{(t^2 + 2)^2}{16} \right), \end{aligned}$$

$$h_2(t, x, y) = \frac{1}{10(t^2 + 2)} \tan^{-1} x + \frac{1}{2(t^2 + 1)} \left(\cos^2 t + \frac{1}{20} [y + \sin y] \right).$$

It is easy to verify that the conditions (B_1) and (B_2) are satisfied with $L_1 = 1/50$, $L_2 = 1/20$, $K_1 = 1/30$, $K_2 = 20$. Moreover,

$$\begin{aligned} \bar{u}_1 &= \max_{t \in [0,1]} |u_1(t)| = 3.458562006593686, \quad \bar{v}_1 = \max_{t \in [0,1]} |v_1(t)| = 3.381578108158285, \\ \bar{u}_2 &= \max_{t \in [0,1]} |u_2(t)| = 0.930817096347715, \quad \bar{v}_2 = \max_{t \in [0,1]} |v_2(t)| = 1.050717898844947, \\ \bar{u}_3 &= \max_{t \in [0,1]} |u_3(t)| = 2.305708004395791, \quad \bar{v}_3 = \max_{t \in [0,1]} |v_3(t)| = 2.705262486526629. \end{aligned}$$

Using these values, we find that

$$\begin{aligned} |\Lambda| &\approx 0.326256740502058, \quad \Omega_1 \approx 5.046746159314364, \quad \Omega_2 \approx 1.263627124252848, \\ \Omega_3 &\approx 4.410684839015075, \quad \Omega_4 \approx 0.807322151715547, \quad \Omega_5 \approx 2.273887205116214, \\ \Omega_6 &\approx 4.856648801739063, \quad \Omega_7 \approx 1.987300233582386, \quad \Omega_8 \approx 3.102877490909382. \end{aligned}$$

In consequence, we have $\Psi \approx 0.861202614806036 < 1$, which shows that the condition (14) of Theorem 1 holds true. So it follows by the conclusion of Theorem 1 that the problem (17) has a unique solution on $[0, 1]$.

5. Conclusions

We have presented the criteria for the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations, involving left Caputo and right Riemann–Liouville fractional derivatives of different orders and a pair of nonlinearities, equipped with four-point nonlocal boundary conditions. In order to achieve the desired criteria, we have applied the fixed point theorems due to Banach and Schaefer. Our results are new and enrich the literature on nonlocal boundary value problems of mixed fractional-order coupled integro-differential systems. The work presented in this paper is expected to improve the study carried out in [34,35] as it conveys the idea of introducing a nonlinear forcing term involving the two-sided Riemann–Liouville fractional integrals in addition to the usual nonlinear forcing term.

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