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Convergence Assessment of the Trajectories of a Bioreaction System by Using Asymmetric Truncated Vertex Functions

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Abstract: In several open and closed-loop systems, the trajectories converge to a region instead of an equilibrium point. Identifying the convergence region and proving the asymptotic convergence upon arbitrarily large initial values of the state variables are regarded as important issues. In this work, the convergence of the trajectories of a biological process is determined and proved via truncated functions and Barbalat's Lemma, while a simple and systematic procedure is provided. The state variables of the process asymptotically converge to a compact set instead of an equilibrium point, with asymmetrical bounds of the compact sets. This convergence is rigorously proved by using asymmetric forms with vertex truncation for each state variable and the Barbalat's lemma. This includes the definition of the truncated V_i functions and the arrangement of its time derivative in terms of truncated functions. The proposed truncated function is different from the common one as it accounts for the model nonlinearities and the asymmetry of the vanishment region. The convergence analysis is valid for arbitrarily large initial values of the state variables, and arbitrarily large size of the convergence regions. The positive invariant nature of the convergence regions is proved. Simulations confirm the findings.

Keywords: global stability; asymptotic convergence; Lyapunov-like function; vertex truncation; invariant set

1. Introduction

In several open and closed loop systems, the trajectories converge to a region instead of an equilibrium point. Some examples are: (i) chaotic systems [1–4], (ii) systems that converge to limit cycles [5,6]; (iii) closed loop systems involving plant uncertainties [7–11]. The case of closed loop systems results majorly in adaptive control design for systems with model uncertainties and nonlinearities [8–10].

Identifying the convergence region of these systems and proving the asymptotic convergence upon arbitrarily large initial values of the state variables are regarded as important issues [1,5,12]. This stability analysis can be achieved via the following Lyapunov-function based approaches:

the finite-time Lyapunov theory [8–10], the ultimate bound approach [13–17], and the Lyapunov-like function with vertex truncation approach [18,19]. For these approaches, the size of the target region is not constrained to be small, and cases with no equilibrium points can be considered. The Lyapunov function, its time derivative and the consequent convergence properties are important differences among them. An ideal stability analysis would be the direct extension of the stability analysis commonly used for systems converging to an equilibrium point, to the case of systems converging to a compact set. That is, a radially unbounded Lyapunov function is formulated so that its time derivative is upper bounded by a function that vanishes for the state variables being inside the convergence region, and it is negative otherwise. Then, the Barbalat's Lemma is applied to prove the convergence of the state variables. The advantage of this analysis is its rigor, completeness and clarity. To the author's knowledge, it is only developed in the Lyapunov-like function with vertex truncation approach, which is used for design of adaptive controllers, achieving the convergence of the tracking error to a compact set [11,18–21]. However, it is not well developed for open loop systems.

The finite-time Lyapunov theory is commonly applied for controller design, featuring the convergence of the tracking error of the closed loop system to a small target region within a well-defined time [8–10]. The fundamentals of the finite-time Lyapunov theory were originally given by Theorem 5.2 in [22]. The ultimate bound theory is commonly applied for chaotic systems. The system trajectories converge to attractive invariant sets that are properly identified [13–17]. The fundamentals of the used Lyapunov based theory were originally given by Leonov at the eighties, according to [3,15–17]. In these approaches, the Lyapunov function is formulated so that it appears in the right hand side of the expression of its time derivative. In this way, the Lyapunov function is monotonically decreasing and converges to a compact set, so that the state variables converge to some compact set. The required expression of the time derivative of the Lyapunov function can be obtained in some cases: (i) in open loop systems, e.g., chaotic attractors [13,14]; (ii) in controlled systems, by properly defining the control law [8–10,23]. Nevertheless, it is overly restrictive and overly difficult to obtain in other open loop systems.

Hence, a less restrictive approach is needed for proving the convergence of open loop systems to compact sets. To this end, in this work we prove the stability of a system comprising three differential equations, with a disturbance that induces the system to dwell around an equilibrium point, by proposing an extension of the Lyapunov-like function with vertex truncation approach. To the author's knowledge, this is new to the current literature. This system arises from an open loop bioreaction model. The main contributions of this study are: (i) the asymptotic convergence of each state variable to a compact set of asymmetrical bounds is proved, using truncated forms and the Barbalat's lemma; (ii) we propose a truncated form that is different to the common quadratic truncated form, as it involves the nonlinear reaction rate terms of the model and an asymmetrical vanishment region; (iii) the proof of asymptotic convergence holds for arbitrarily large initial values of the state variables, and arbitrarily large size of the convergence region; (iv) the invariance nature of the convergence sets is proved on the basis of the truncated forms.

The organization of the work is as follows. Section 2 presents the preliminary mathematical definitions (Section 2.1) an the model of the system (Section 2.2), expressing it in terms of its difference with respect to equilibrium conditions. Section 3 presents the main results of the stability analysis of a three dimension model with external disturbance. Section 4 presents the Lyapunov-based stability analysis of two simplified models. Section 4.1 considers a three dimension model with no external disturbances, whereas Section 4.2 considers a one-dimension model with external disturbance. Section 5 presents the detailed stability analysis of a three dimension model with external disturbance. In Section 6 a simulation example is presented. In Section 7 the conclusions are drawn.

2. Preliminary Definitions and Model Description

2.1. Preliminary Definitions

In this subsection, some mathematical expressions and terms used throughout this study are defined.

Compact set. A compact set $\Omega \subset \mathbb{R}^r$ is defined as $\Omega = \{(\bullet) : k_{l,i} \leq (\bullet)_i \leq k_{u,i},\ \ i = 1, \cdots, r\}$, being $k_{l,i}$, $k_{u,i}$ constant real numbers, and r the size of $(\bullet)$ [8,24–26].

Boundedness. A scalar signal $(\bullet)$ is bounded if there exists a constant $\alpha > 0$ such that $|(\bullet)| < \alpha$ for all $t \geq t_o$ [27].

Asymptotic convergence. The signal $(\bullet)$ converges asymptotically to the region Ω, if $(\bullet)$ converges to Ω as $t \to \infty$ [28–30].

Remaining in a region. The signal $(\bullet)$ remains in a region Ω for $t \geq t_k$, $t_k > t_o$ if $(\bullet) \in \Omega$ for all $t \geq t_k$ [24,31,32].

The term 'region' corresponds to a set.

2.2. Model Description

We consider ammonification, nitrification, plant uptake and denitrification as the primary nitrogen removal and formation pathways. Ammonium is converted to nitrite in one step, whereas NO_2^- and NO_3^- are produced by nitrification and are consumed by denitrification [33,34]. Thus, the mass balance for nitrogen concentration across a single CSTR gives:

$$\frac{dON}{dt} = \frac{1}{\tau_{in}}ON_{in} - \frac{1}{\tau}ON - r_a \tag{1}$$

$$\frac{dNH_4}{dt} = \frac{1}{\tau_{in}}NH_{4,in} - \frac{1}{\tau}NH_4 + r_a - r_n - r_p \tag{2}$$

$$\frac{d(NO_2^- + NO_3^-)}{dt} = \frac{1}{\tau_{in}}(NO_2^- + NO_3^-)_{in} - \frac{1}{\tau}(NO_2^- + NO_3^-) + r_n - r_d \tag{3}$$

$$r_a = k_a ON,\ \ r_n = k_n\frac{NH_4}{k_{AN} + NH_4},\ \ r_d = k_d\frac{(NO_2^- + NO_3^-)}{K_{ds} + (NO_2^- + NO_3^-)},\ \ r_p = k_p \tag{4}$$

$$\tau_{in} = \frac{V}{Q_{in}},\ \ \tau = \frac{V}{Q_{out}}, \tag{5}$$

where ON is the concentration of organic nitrogen, and ON_{in} is its inflow concentration; NH_4 is the $NH_4^+ - N$ concentration, and $NH_{4,in}$ is its inflow concentration; $(NO_2^- + NO_3^-)$ is the concentration of nitrites plus nitrates, and $(NO_2^- + NO_3^-)_{in}$ is its inflow concentration. In addition, r_a is the ammonification rate, r_n is the nitrification rate, r_p is the plant uptake rate, r_d is the denitrification rate; Q_{in} is the inlet flowrate, Q_{out} is the outlet flow rate, V is the water volume. The effect of pH, temperature and dissolved oxygen are not considered, in order to facilitate the dynamic analysis. We use the following notation:

$$X_1 = ON,\ \ X_2 = NH_4,\ \ X_3 = NO_2^- + NO_3^-.$$

Thus, models (1) to (3) with functions (4) to (5) is rewritten as:

$$\frac{dX_1}{dt} = \frac{1}{\tau_{in}}X_{1,in} - \frac{1}{\tau}X_1 - k_a X_1 \tag{6}$$

$$\frac{dX_2}{dt} = \frac{1}{\tau_{in}}X_{2,in} - \frac{1}{\tau}X_2 + k_a X_1 - k_n\frac{X_2}{k_{AN} + X_2} - k_p \tag{7}$$

$$\frac{dX_3}{dt} = \frac{1}{\tau_{in}}X_{3,in} - \frac{1}{\tau}X_3 + k_n\frac{X_2}{k_{AN} + X_2} - k_d\frac{X_3}{K_{ds} + X_3}, \tag{8}$$

subject to the following features:

Characteristic 1. *τ_{in}, τ, k_a, k_n, k_{AN}, K_{ds}, k_p are constant and positive.*

Characteristic 2. *$X_{2,in}$, $X_{3,in}$ are constant and positive.*

Characteristic 3. *$X_{1,in}$ varies according to $X_{1,in} = X^p_{1,in} + \delta_0$, where $X^p_{1,in}$ is constant and positive, whereas δ_0 is time varying and satisfies: $max\{\delta_o\} > 0$; $min\{\delta_o\} < 0$; $min\{\delta_o\} = -max\{\delta_o\}$.*

Characteristic 4. *X_1, X_2, X_3 remain in the region $\Omega^\dagger_{123} = \{(X_1, X_2, X_3) \in \mathbb{R}^3 |\ X_1 > 0,\ X_2 > 0,\ X_3 > 0\}$.*

Now, we rewrite the model in terms of the equilibrium condition corresponding to $X_{1,in} = X^p_{1,in}$. Subtracting the equilibrium condition from Equation (6), yields

$$\frac{d\bar{X}_1}{dt} = -k_1\bar{X}_1 + \delta_1 \tag{9}$$

$$\bar{X}_1 = X_1 - X_1^{eq},\ k_1 = \frac{1}{\tau} + k_a, \tag{10}$$

$$\delta_1 = \frac{\delta_0}{\tau_{in}},\ min\{\delta_1\} < 0,\ max\{\delta_1\} > 0,\ min\{\delta_1\} = -max\{\delta_1\}. \tag{11}$$

Equation (11) follows from Characteristics 1 and 3. Subtracting the equilibrium condition from Equation (7), yields

$$\frac{dX_2}{dt} = k_a\bar{X}_1 - \bar{g}_2 \tag{12}$$

$$\bar{X}_2 = X_2 - X_2^{eq} \tag{13}$$

$$\bar{g}_2(\bar{X}_2) = \frac{1}{\tau}\bar{X}_2 + k_n\frac{\bar{X}_2 + X_2^{eq}}{k_{AN} + \bar{X}_2 + X_2^{eq}} - k_n\frac{X_2^{eq}}{k_{AN} + X_2^{eq}}. \tag{14}$$

Subtracting the equilibrium condition from Equation (8), yields

$$\frac{d\bar{X}_3}{dt} = -\bar{g}_3 + \bar{g}_{2b} \tag{15}$$

$$\bar{X}_3 = X_3 - X_3^{eq} \tag{16}$$

$$\bar{g}_{2b}(\bar{X}_2) = k_n\frac{\bar{X}_2 + X_2^{eq}}{k_{AN} + \bar{X}_2 + X_2^{eq}} - k_n\frac{X_2^{eq}}{k_{AN} + X_2^{eq}} \tag{17}$$

$$\bar{g}_3(\bar{X}_3) = \frac{1}{\tau}\bar{X}_3 + k_d\frac{\bar{X}_3 + X_3^{eq}}{K_{ds} + \bar{X}_3 + X_3^{eq}} - k_d\frac{X_3^{eq}}{K_{ds} + X_3^{eq}}. \tag{18}$$

The following properties hold:

$$\bar{g}_2|_{\bar{X}_2=0} = 0,\ \bar{g}_{2b}|_{\bar{X}_2=0} = 0,\ \bar{g}_3|_{\bar{X}_3=0} = 0$$
$$\frac{d\bar{g}_2}{d\bar{X}_2} > 0,\ \frac{d\bar{g}_{2b}}{d\bar{X}_2} > 0,\ \frac{d\bar{g}_3}{d\bar{X}_3} > 0$$

Remark 1. *Characteristic 4 implies that $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ remain in the region*

$$\bar{R}_{123} = \{(\bar{X}_1, \bar{X}_2, \bar{X}_3) \in (-X_1^{eq},\ \infty) \times (-X_2^{eq},\ \infty) \times (-X_3^{eq},\ \infty)\}. \tag{19}$$

3. Main Results

The stability analysis for a three dimension model with external disturbance includes: (i) definition of the truncated functions V_i, what involves the choice of its gradient and the definition of the convergence regions Ω_i; (ii) determination of the time derivatives of the V_i functions, what involves arranging the $\dot{V}_i$ expressions in terms of g_{it} functions; and (iii) determination of the boundedness, convergence and invariance properties of the state variables. The detailed procedure is presented in Section 5, whereas the main results are presented at what follows.

The gradient of the V_1 function is chosen to be:

$$\frac{dV_1}{d\bar{X}_1} = g_{1t}$$

where g_{1t} is defined as

$$g_{1t} = \begin{cases} \bar{X}_1 - \frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ 0 & \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ \bar{X}_1 + \frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases} \tag{20}$$

$$\begin{aligned} &\text{where} \\ &\bar{X}_1^{*a} := -\frac{1}{k_1} max\{|\delta_1|\} \\ &\bar{X}_1^{*b} := \frac{1}{k_1} max\{|\delta_1|\} \\ &\bar{X}_1^{*a} < 0,\ \bar{X}_1^{*b} > 0. \end{aligned}$$

The main properties of g_{1t} are:

$$\begin{aligned} &Pi)\ g_{1t}\left(\bar{X}_1 + (-\delta_1)/k_1\right) \geq g_{1t}^2 \\ &Pii)\ \ g_{1t}\ \text{ is continuous with respect to }\ \bar{X}_1 \\ &Piii)\ g_{1t} = 0 \text{ for }\ \bar{X}_1 \in [\bar{X}_1^{*a}\ \ \bar{X}_1^{*b}], \\ &\qquad g_{1t} > 0 \text{ for } \bar{X}_1 > \bar{X}_1^{*b},\ \text{and } g_{1t} < 0 \text{ for } \bar{X}_1 < \bar{X}_1^{*a} \\ &Piv)\ \text{if } g_{1t} \to 0 \text{ as } t \to \infty,\ \text{then } \bar{X}_1 \to \Omega_1, \\ &\qquad \Omega_1 = \left\{\bar{X}_1 : \bar{X}_1^{*a} \leq \bar{X}_1 \leq \bar{X}_1^{*b}\right\}. \end{aligned} \tag{21}$$

Definition of the V_1 function:

$$V_1(\bar{X}_1) = \begin{cases} \int_{\bar{X}_1^{*b}}^{\bar{X}_1} \left(x - \dfrac{1}{k_1} max\left\{|\delta_1|\right\}\right) dx & \text{for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ 0 & \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ \int_{\bar{X}_1^{*a}}^{\bar{X}_1} \left(x + \dfrac{1}{k_1} max\left\{|\delta_1|\right\}\right) dx & \text{for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases}.$$

whose main properties are

$$\begin{aligned} &V_1 > 0 \text{ for } \bar{X}_1 > \bar{X}_1^{*b} \\ &V_1 > 0 \text{ for } \bar{X}_1 < \bar{X}_1^{*a} \\ &V_1 = 0 \text{ for } \bar{X}_1 \in [\bar{X}_1^{*a},\ \bar{X}_1^{*b}]. \end{aligned}$$

The time derivative of V_1 is:

$$\frac{dV_1}{dt} \leq -k_1 g_{1t}^2 \leq 0$$

By applying the Barbalat's lemma, one obtains that $\bar{X}_1$ converges asymptotically to Ω_1.

The gradient of the V_2 function is chosen to be:

$$\frac{dV_2}{d\bar{X}_2} = g_{2t}$$

where g_{2t} is defined as

$$g_{2t} = \begin{cases} \bar{g}_2 - k_a \frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ 0 & \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ \bar{g}_2 + k_a \frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_2 \leq \bar{X}_2^{*a} \end{cases} \tag{22}$$

$$\begin{aligned} &\text{where} \\ &\bar{X}_2^{*a} := \left\{ \bar{X}_2 :\ \bar{g}_2 + k_a \frac{1}{k_1} max\left\{|\delta_1|\right\} = 0 \right\} \\ &\bar{X}_2^{*b} := \left\{ \bar{X}_2 :\ \bar{g}_2 - k_a \frac{1}{k_1} max\left\{|\delta_1|\right\} = 0 \right\} \\ &\bar{X}_2^{*a} < 0,\ \bar{X}_2^{*b} > 0 \end{aligned}$$

The main properties of g_{2t} are:

$$\begin{aligned} &Pi)\ g_{2t}(\bar{g}_2 + k_a d_1) \geq g_{2t}^2 \\ &Pii)\ g_{2t} \text{ is continuous with respect to } \bar{X}_2 \\ &Piii)\ g_{2t} = 0 \text{ for } \bar{X}_2 \in [\bar{X}_2^{*a}\ \ \bar{X}_2^{*b}], \\ &\qquad g_{2t} > 0 \text{ for } \bar{X}_2 > \bar{X}_2^{*b}, \text{ and } g_{2t} < 0 \text{ for } \bar{X}_2 < \bar{X}_2^{*a} \\ &Piv)\ \text{if } g_{2t} \to 0 \text{ as } t \to \infty, \text{ then } \bar{X}_2 \to \Omega_2, \\ &\quad \Omega_2 = \left\{ \bar{X}_2 : \bar{X}_2^{*a} \leq \bar{X}_2 \leq \bar{X}_2^{*b} \right\}. \end{aligned} \tag{23}$$

The function V_2 is defined as:

$$V_2(\bar{X}_2) = \begin{cases} \int_{\bar{X}_2^{*b}}^{\bar{X}_2} \left(\bar{g}_2(x) - \frac{k_a}{k_1} max\left\{|\delta_1|\right\} \right) dx & \text{for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ 0 & \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ \int_{\bar{X}_2^{*a}}^{\bar{X}_2} \left(\bar{g}_2(x) + \frac{k_a}{k_1} max\left\{|\delta_1|\right\} \right) dx & \text{for } \bar{X}_2 \leq \bar{X}_2^{*a} \end{cases} .$$

and its main properties are:

$$\begin{aligned} &V_2 > 0 \text{ for } \bar{X}_2 > \bar{X}_2^{*b} \\ &V_2 > 0 \text{ for } \bar{X}_2 < \bar{X}_2^{*a} \\ &V_2 = 0 \text{ for } \bar{X}_2 \in [\bar{X}_2^{*a},\ \bar{X}_2^{*b}]. \end{aligned}$$

The linear combination of $\dot{V}_1$ and $\dot{V}_2$ gives:

$$\frac{dV_1}{dt} + \frac{d}{dt}\left(4\frac{\alpha_2 k_1}{k_a^2}V_2\right) \le -4\frac{\alpha_2^2 k_1}{k_a^2}\left(g_{2t} - \frac{k_a}{2\alpha_2}g_{1t}\right)^2 - 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}g_{2t}^2 \le 0.$$

By applying the Barbalat's Lemma, one obtains that g_{2t}^2 converges asymptotically to zero, and $\bar{X}_2$ converges asymptotically to Ω_2 (23).

The gradient of the V_3 function is chosen to be:

$$\frac{dV_3}{d\bar{X}_3} = g_{3t},$$

where g_{3t} is defined as:

$$g_{3t} = \begin{cases} \bar{g}_3 - \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} \text{ for } \bar{X}_3 \ge \bar{X}_3^{*b} \\ 0 \qquad \text{for } \bar{X}_3 \in (\bar{X}_3^{*a},\ \bar{X}_3^{*b}) \\ \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} \text{ for } \bar{X}_3 \le \bar{X}_3^{*a} \end{cases}. \tag{24}$$

where

$$\bar{X}_3^{*a} := \{\bar{X}_3 : \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} = 0\}$$
$$\bar{X}_3^{*b} := \{\bar{X}_3 : \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} = 0\}$$

$$\bar{X}_3^{*a} < 0,\ \bar{X}_3^{*b} > 0.$$

$$\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} > 0,$$
$$\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} < 0.$$

The main properties of g_{3t} are:

$$\begin{aligned} &Pi)\ g_{3t}(\bar{g}_3 + d_{2b}) \ge g_{3t}^2 \\ &Pii)\ g_{3t} \text{ is continuous with respect to } \bar{X}_3 \\ &Piii)\ g_{3t} = 0 \text{ for } \ \bar{X}_3 \in [\bar{X}_3^{*a} \quad \bar{X}_3^{*b}], \\ &\qquad g_{3t} > 0 \text{ for } \bar{X}_3 > \bar{X}_3^{*b}, \text{ and } g_{3t} < 0 \text{ for } \bar{X}_3 < \bar{X}_3^{*a}. \\ &Piv)\ \text{if } g_{3t} \to 0 \text{ as } t \to \infty, \text{ then } \bar{X}_3 \to \Omega_3, \\ &\qquad \Omega_3 = \left\{\bar{X}_3 : \bar{X}_3^{*a} \le \bar{X}_3 \le \bar{X}_3^{*b}\right\}. \end{aligned} \tag{25}$$

The definition of the function V_3 is:

$$V_3(\bar{X}_3) = \begin{cases} \int_{\bar{X}_3^{*b}}^{\bar{X}_3}\left(\bar{g}_3(x) - \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}}\right)dx \text{ for } \bar{X}_3 \ge \bar{X}_3^{*b} \\ 0 \qquad \text{for } \bar{X}_3 \in (\bar{X}_3^{*a},\ \bar{X}_3^{*b}) \\ \int_{\bar{X}_3^{*a}}^{\bar{X}_3}\left(\bar{g}_3(x) + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}}\right)dx \text{ for } \bar{X}_3 \le \bar{X}_3^{*a} \end{cases}.$$

and its main properties are:

$$\begin{aligned} &V_3 > 0 \text{ for } \bar{X}_3 > \bar{X}_3^{*b} \\ &V_3 > 0 \text{ for } \bar{X}_3 < \bar{X}_3^{*a} \\ &V_3 = 0 \text{ for } \bar{X}_3 \in [\bar{X}_3^{*a},\ \bar{X}_3^{*b}]. \end{aligned}$$

The linear combination of $\dot{V}_1$, $\dot{V}_2$ and $\dot{V}_2$ gives:

$$\begin{aligned} &\tfrac{dV_1}{dt} + \tfrac{d}{dt}\left(\tfrac{4\alpha_2 k_1}{k_a^2} V_2\right) + \tfrac{d}{dt}\left(16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2} V_3\right) \\ &\leq -\tfrac{4\alpha_2^2 k_1}{k_a^2}\left(g_{2t} - \tfrac{k_a}{2\alpha_2} g_{1t}\right)^2 + (-1)\left(16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}\right)\left(\sqrt{\alpha_3} g_{3t} - \tfrac{1}{2\alpha_3^{1/2}} g_{2bt}\right)^2 \cdot \\ &\quad +(-1)16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}(1-\alpha_3) g_{3t}^2 \leq 0 \end{aligned}$$

By applying the Barbalat's lemma, one obtains that g_{3t}^2 convergences asymptotically to zero and $\bar{X}_3$ to Ω_3 (25).

Proposition 1 (Boundedness)**.** *Consider the system* (6)–(8) *subject to Characteristics 1 to 4, and signals* $\bar{X}_1$ (10), g_{1t} (20)*;* $\bar{g}_2$ (14), $\bar{X}_2$ (13), g_{2t} (22)*;* $\bar{g}_3$ (18), $\bar{X}_3$ (16), g_{3t} (24)*. All these signals are bounded for* $\bar{X}_1$*,* $\bar{X}_2$*,* $\bar{X}_3$ *remaining in* $\bar{R}_{123}$*.*

Proposition 2 (Convergence)**.** *Consider the system* (6)–(8) *subject to Characteristics 1 to 4, and signals* $\bar{X}_1$ (10), g_{1t} (20)*;* $\bar{g}_2$ (14), $\bar{X}_2$ (13), g_{2t} (22)*;* $\bar{g}_3$ (18), $\bar{X}_3$ (16), g_{3t} (24)*.* $\bar{X}_1$ *converges asymptotically to* Ω_1 (21), $\bar{X}_2$ *converges asymptotically to* Ω_2 (23) *and* $\bar{X}_3$ *converges asymptotically to* Ω_3 (25)*.*

Proposition 3 (Invariance)**.** *Consider the system* (6)–(8) *subject to Characteristics 1 to 4, and signals* $\bar{X}_1$ (10), g_{1t} (20)*;* $\bar{g}_2$ (14), $\bar{X}_2$ (13), g_{2t} (22)*;* $\bar{g}_3$ (18), $\bar{X}_3$ (16), g_{3t} (24)*, and the sets* Ω_1 (21), Ω_2 (23), Ω_3 (25)*. Let*

$$\Omega_{12} = \Omega_1 \cup \Omega_2,\ \ \Omega_{123} = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

The sets Ω_1*,* Ω_{12}*,* Ω_{123} *are positively invariant.*

The proof of Proposition 1 is presented in Section 5.4, the proof of Proposition 2 is presented in Section 5.5, and the proof of Proposition 3 is presented in Section 5.6.

Remark 2. *The proposed* V_1*,* V_2*,* V_3 *functions allow to develop a rigorous and complete proof for the asymptotic convergence of* $\bar{X}_1$*,* $\bar{X}_2$*,* $\bar{X}_3$ *to the compact sets* Ω_1*,* Ω_2 *and* Ω_3*, respectively, via the Barbalat's lemma, taking into account the nonlinear terms of the model and the asymmetry of* Ω_1*,* Ω_2 *and* Ω_3*. To this end, the* V_1*,* V_2 *and* V_3 *functions involve the nonlinear model terms* $\bar{g}_2(\bar{X}_2)$ *and* $\bar{g}_3(\bar{X}_3)$*, and exhibit asymmetrical vanishment regions* Ω_1*,* Ω_2 *and* Ω_3*. Consequently, the linear combinations of the* $\dot{V}_1$*,* $\dot{V}_2$ *and* $\dot{V}_3$ *expressions involve the* $-kg_{1t}^2$*,* $-kg_{2t}^2$*,* $-kg_{3t}^2$ *terms, which vanish for* $\bar{X}_1 \in \Omega_1$*,* $\bar{X}_2 \in \Omega_2$ *and* $\bar{X}_3 \in \Omega_3$*, respectively; then the Barbalat's lemma can be applied in order to prove asymptotic convergence.*

The main differences of the functions $V_1(\bar{X}_1)$*,* $V_2(\bar{X}_2)$*,* $V_3(\bar{X}_3)$ *with respect to the common truncated quadratic form (e.g., [11,18]), are: (i) they involve the nonlinear asymmetrical functions* $\bar{g}_i(\bar{X}_i)$*; (ii) the vanishment regions* Ω_i *are asymmetrical, what renders* $V_i(\bar{X}_i)$ *asymmetrical.*

The proof of asymptotic convergence is valid for: i) arbitrarily large but bounded positive initial values of $\bar{X}_1$*,* $\bar{X}_2$*,* $\bar{X}_3$*; ii) arbitrarily large but bounded size of the convergence regions: the sizes of* Ω_1 (21), Ω_2 (23), Ω_3 (25) *depend on the bounds of* δ_o*, so that they can be arbitrarily large.*

Remark 3. *The proposed* V_1*,* V_2 *and* V_3 *functions allow to develop a rigorous proof of positive invariance of the convergence sets* Ω_1*,* Ω_{12}*,* Ω_{123}*. To this end, the characteristics of the* $\dot{V}_1$*,* $\dot{V}_2$*,* $\dot{V}_3$ *expressions allow to obtain:* $\dot{V}_1 \leq 0$ *for* $\bar{X}_1 \in \Omega_1$*;* $\dot{V}_2 \leq 0$ *for* $\bar{X}_1 \in \Omega_1$ *and* $\bar{X}_2 \in \Omega_2$*; and* $\dot{V}_3 \leq 0$ *for* $\bar{X}_2 \in \Omega_2$ *and* $\bar{X}_3 \in \Omega_3$*.*

4. Preliminary Results: Stability Analysis for Simplified Models

In this section, the asymptotic convergence of two simple systems is determined by using Lyapunov-like functions and functions with vertex truncation. The purpose is to provide the basic procedures of the stability analysis that will be developed later for a three dimension model with external disturbance. Section 4.1 considers a three-dimension system with no external disturbance, whereas Section 4.2 considers a one-dimensional system with an external disturbance. Truncated forms are only used in Section 4.2.

4.1. Three-Dimension Model with No External Disturbance

In this section, we determine the asymptotic convergence of the state variables of a three-dimension model with no external disturbances. Consider the model (9) to (18). In absence of disturbance, we have $\delta_0 = 0$, so that $\delta_1 = 0$ and Equation (9) becomes:

$$\frac{d}{dt}\bar{X}_1 = -k_1\bar{X}_1$$

The time derivative of the function V_1 satisfies:

$$\frac{dV_1}{dt} = \frac{dV_1}{d\bar{X}_1}\frac{d\bar{X}_1}{dt}$$

Combining the above expressions, yields:

$$\frac{dV_1}{dt} = \frac{dV_1}{d\bar{X}_1}(-k_1)\left(\bar{X}_1\right) \tag{26}$$

We impose the following condition on V_1:

$$\frac{dV_1}{d\bar{X}_1} = \bar{X}_1 \tag{27}$$

so that the definition of V_1 is:

$$V_1(\bar{X}_1) = \int_0^{\bar{X}_1}(x)dx$$

whose main properties are:

$$V_1 > 0 \text{ for } \bar{X}_1 \neq 0$$
$$V_1 = 0 \text{ for } \bar{X}_1 = 0$$

Combining Equations (26) and (27), yields

$$\frac{dV_1}{dt} = (-k_1)\bar{X}_1^2 \tag{28}$$

This implies the asymptotic convergence of $\bar{X}_1$ to zero, what is concluded by using the Barbalat's Lemma on $\bar{X}_1^2$.

The time derivative of the function V_2 satisfies

$$\frac{dV_2}{dt} = \frac{dV_2}{d\bar{X}_2}\frac{d\bar{X}_2}{dt}$$

Combining with Equation (12), yields

$$\frac{dV_2}{dt} = \frac{dV_2}{d\bar{X}_2}(-\bar{g}_2 + k_a\bar{X}_1). \tag{29}$$

We impose the following condition on V_2:

$$\frac{dV_2}{d\bar{X}_2} = \bar{g}_2. \tag{30}$$

On the basis of this condition, the definition of the function V_2 is:

$$V_2(\bar{X}_2) = \int_0^{\bar{X}_2} (\bar{g}_2(x))dx$$

Its main properties are:

$$V_2 > 0 \text{ for } \bar{X}_2 \neq 0$$
$$V_2 = 0 \text{ for } \bar{X}_2 = 0.$$

Combining Equations (29) and (30), yields:

$$\frac{dV_2}{dt} = -\bar{g}_2^2 + k_a\bar{X}_1\bar{g}_2 \tag{31}$$

We consider the constant α_2, that satisfies

$$\alpha_2 \in (0,\ 1)$$

Factorizing (31), arranging and multiplying by $4\alpha_2 k_1/k_a^2$, yields

$$\frac{4\alpha_2 k_1}{k_a^2}\frac{dV_2}{dt} \leq -4\frac{\alpha_2^2 k_1}{k_a^2}\left(\bar{g}_2 - \frac{k_a}{2\alpha_2}\bar{X}_1\right)^2 + k_1\bar{X}_1^2 - 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}\bar{g}_2^2.$$

Adding this and Equation (28), yields

$$\frac{dV_1}{dt} + \frac{d}{dt}\left(4\frac{\alpha_2 k_1}{k_a^2}V_2\right) \leq -4\frac{\alpha_2^2 k_1}{k_a^2}\left(\bar{g}_2 - \frac{k_a}{2\alpha_2}\bar{X}_1\right)^2 - 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}\bar{g}_2^2 \leq 0. \tag{32}$$

This implies the asymptotic convergence of $\bar{g}_2$ to zero, what follows by using the Barbalat's lemma on $\bar{g}_2^2$. Consequently, $\bar{X}_2$ converges asymptotically to zero.

The time derivative of the function V_3 satisfies:

$$\frac{dV_3}{dt} = \frac{dV_3}{d\bar{X}_3}\frac{d\bar{X}_3}{dt}$$

Combining with Equation (15), yields:

$$\frac{dV_3}{dt} = \frac{dV_3}{d\bar{X}_3}\left[-(\bar{g}_3) + \bar{g}_{2b}\right]. \tag{33}$$

We impose the following condition on V_3:

$$\frac{dV_3}{d\bar{X}_3} = \bar{g}_3. \tag{34}$$

so that the definition of the function V_3 is:

$$V_3(\bar{X}_3) = \int_0^{\bar{X}_3} (\bar{g}_3(x))dx.$$

and its main properties are:

$$V_3 > 0 \text{ for } \bar{X}_3 \neq 0$$
$$V_3 = 0 \text{ for } \bar{X}_3 = 0.$$

Combining (33) and (34), yields

$$\frac{dV_3}{dt} = (-1)\bar{g}_3^2 + \bar{g}_3\bar{g}_{2b}. \tag{35}$$

From Equations (14) and (17), it follows that:

$$|\bar{g}_{2b}| \leq |\bar{g}_2| \tag{36}$$

We consider the constant α_3, that satisfies:

$$\alpha_3 \in (0,\ \ 1).$$

Factorizing (35), multiplying by $16\alpha_2(1-\alpha_2)\alpha_3 k_1/k_a^2$ and using property (36), yields:

$$16\frac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}\frac{dV_3}{dt} \leq (-1)16\frac{\alpha_2(1-\alpha_2)\alpha_3^2 k_1}{k_a^2}\left(\bar{g}_3 - \frac{1}{2\alpha_3}\bar{g}_{2b}\right)^2 + 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}\bar{g}_2^2$$
$$+(-1)16\frac{\alpha_2(1-\alpha_2)\alpha_3(1-\alpha_3)k_1}{k_a^2}\bar{g}_3^2.$$

Adding this and Equation (32), yields:

$$\begin{aligned}
&\frac{dV_1}{dt} + \frac{4\alpha_2 k_1}{k_a^2}\frac{dV_2}{dt} + \frac{16\alpha_2 k_1(1-\alpha_2)\alpha_3}{k_a^2}\frac{dV_3}{dt} \\
&\leq -\frac{4\alpha_2^2 k_1}{k_a^2}\left(\bar{g}_2 - \frac{k_a}{2\alpha_2}\bar{X}_1\right)^2 - \frac{16\alpha_2 k_1(1-\alpha_2)\alpha_3^2}{k_a^2}\left(\bar{g}_3 - \frac{1}{2\alpha_3}\bar{g}_{2b}\right)^2 \\
&\quad - \frac{16\alpha_2 k_1(1-\alpha_2)\alpha_3(1-\alpha_3)}{k_a^2}\bar{g}_3^2.
\end{aligned}$$

This implies the asymptotic convergence of $\bar{g}_3$ to zero, what is concluded by using the Barbalat's lemma on $\bar{g}_3^2$. Consequently, $\bar{X}_3$ converges asymptotically to zero.

4.2. One-Dimension Model with External Disturbance

In this section, we determine and prove the asymptotic convergence of the state variable of a one-dimension system to a compact set of asymmetrical size. This stability analysis is based on the robust adaptive controller design that involves truncated forms (see [11,18]). In that approach, the Lyapunov function comprises a truncated quadratic form for the convergent state variable, and quadratic forms for other closed loop states. The truncated form exhibits a vanishment for values of the convergent state variable inside the convergence region. An early version of this type of functions is reported by [35], and later variants are reported by [11,18,19]. The time derivative of the Lyapunov function is an inequality in terms of the truncated quadratic form. The convergence of the convergent state variable is deduced by using the Barbalat's Lemma, although the convergence time is not usually well-defined [11,21,36]. In this section, we apply the aforementioned approach to a one-dimension model with an external disturbance whose bounds are asymmetrical. To that end we propose a truncated form involving the nonlinear reaction rate terms and an asymmetrical vanishment region, instead of using the common truncated quadratic form.

Consider the system:

$$\frac{d\bar{X}}{dt} = -k\bar{g} + \delta = -k\left(\bar{g} - \frac{\delta}{k}\right), \tag{37}$$

where k is constant and positive; δ is a time varying disturbance, satisfying $max\{\delta\} > 0$, $min\{\delta\} < 0$; and $\bar{g}$ is a function of $\bar{X}$ that satisfies $\bar{g}|_{\bar{X}=0} = 0$ and $d\bar{g}/d\bar{X} > 0$. $\bar{X}$ is defined in the region

$$\bar{R}^{\dagger} = \{\bar{X} \in (0,\ \infty)\}. \tag{38}$$

Remark 4. *The bounds of $-\delta/k$ are asymmetrical, that is $min\{-\delta/k\} \neq (-1)max\{-\delta/k\}$, what implies that $\bar{X}$ converges to a compact set of asymmetrical bounds.*

The time derivative of the function V satisfies:

$$\frac{dV}{dt} = \frac{dV}{d\bar{X}}\frac{d\bar{X}}{dt}.$$

Combining this with Equation (37), yields

$$\begin{aligned} \frac{dV}{dt} &= -k\frac{dV}{d\bar{X}}\left(\bar{g} + d\right) \\ d &= -\frac{\delta}{k}. \end{aligned} \tag{39}$$

where d is a disturbance-like term satisfying $max\{d\} > 0$, $min\{d\} < 0$. We impose the following condition on the function V:

$$\frac{dV}{d\bar{X}} = g_t. \tag{40}$$

where g_t is a truncated function that allows to prove the convergence of $\bar{X}$. To generate a proper expression of dV/dt, we require g_t to fulfill the following:

$$\begin{aligned} &-g_t(\bar{g} + d) = 0 \text{ for } \bar{X} \in [\bar{X}^{*a}\ \ \bar{X}^{*b}] \\ &-g_t(\bar{g} + d) < 0 \text{ for } \bar{X} < \bar{X}^{*a} \\ &-g_t(\bar{g} + d) < 0 \text{ for } \bar{X} > \bar{X}^{*b}. \end{aligned} \tag{41}$$

For the case $\bar{X} < \bar{X}^{*a}$, we have $g_t < 0$, therefore $\bar{X}^{*a}$ must be chosen such that $\bar{g} + d < 0$ for $\bar{X} < \bar{X}^{*a}$. This implies $\bar{g} < -max\{d\} < 0$ for $\bar{X} < \bar{X}^{*a}$. Thus, we choose

$$\begin{aligned} &g_t = \bar{g} + max\{d\} \text{ for } \bar{X} \leq \bar{X}^{*a} \\ &\bar{X}^{*a} = \{\bar{X} : \bar{g} + max\{d\} = 0\} \\ &\text{where} \\ &max\{d\} > 0,\ \bar{X}^{*a} < 0. \end{aligned} \tag{42}$$

For the case $\bar{X} > \bar{X}^{*b}$ we have $g_t > 0$. Therefore, $\bar{X}^{*b}$ must be chosen such that $\bar{g} + d > 0$ for $\bar{X} > \bar{X}^{*b}$. This implies $\bar{g} > -min\{d\} > 0$ for $\bar{X} > \bar{X}^{*b}$. Therefore, we choose:

$$\begin{aligned}&g_t = \bar{g} + min\{d\} \text{ for } \bar{X} \geq \bar{X}^{*b}\\&\bar{X}^{*b} = \{\bar{X} : \bar{g} + min\{d\} = 0\}\\&\text{where}\\&min\{d\} < 0,\ \bar{X}^{*b} > 0\end{aligned} \tag{43}$$

Combining Equations (42) and (43), yields

$$g_t = \begin{cases} \bar{g} + min\{d\} \text{ for } \bar{X} \geq \bar{X}^{*b} \\ 0 \quad \text{for } \bar{X} \in (\bar{X}^{*a},\ \bar{X}^{*b}) \\ \bar{g} + max\{d\} \text{ for } \bar{X} \leq \bar{X}^{*a} \end{cases} \tag{44}$$

$$\begin{aligned}&\text{where}\\&\bar{X}^{*a} = \{\bar{X} : \bar{g} + max\{d\} = 0\},\ \bar{X}^{*a} < 0\\&\bar{X}^{*b} = \{\bar{X} : \bar{g} + min\{d\} = 0\},\ \bar{X}^{*b} > 0.\end{aligned}$$

The main properties of g_t are:

$$Pi)\ g_t\,(\bar{g} + d) \geq g_t^2 \tag{45}$$

$$\begin{aligned}&Pii)\ g_t \text{ is continuous with respect to } \bar{X}\\&Piii)\ g_t = 0 \text{ for } \bar{X} \in [\bar{X}^{*a},\ \bar{X}^{*b}],\\&\quad g_t > 0 \text{ for } \bar{X} > \bar{X}^{*b}, \text{ and } g_t < 0 \text{ for } \bar{X} < \bar{X}^{*a}.\end{aligned} \tag{46}$$

These properties imply that requirements (41) are fulfilled, and also

$$\text{if } g_t \to 0 \text{ as } t \to \infty, \text{ then } \bar{X} \to \Omega \tag{47}$$

$$\Omega = \left\{\bar{X} : \bar{X}^{*a} \leq \bar{X} \leq \bar{X}^{*b}\right\}. \tag{48}$$

On the basis of conditions (40) and (44), the definition of the function V is:

$$V(\bar{X}) = \begin{cases} \int_{\bar{X}^{*b}}^{\bar{X}} (\bar{g} + min\{d\})dx \text{ for } \bar{X} \geq \bar{X}^{*b} \\ 0 \quad \text{for } \bar{X} \in (\bar{X}^{*a},\ \bar{X}^{*b}) \\ \int_{\bar{X}^{*a}}^{\bar{X}} (\bar{g} + max\{d\})dx \text{ for } \bar{X} \leq \bar{X}^{*a} \end{cases}. \tag{49}$$

whose main properties are:

$$\begin{aligned}&V > 0 \text{ for } \bar{X} > \bar{X}^{*b},\ \ V > 0 \text{ for } \bar{X} < \bar{X}^{*a},\\&V = 0 \text{ for } \bar{X} \in \Omega.\end{aligned} \tag{50}$$

Remark 5. *The function V is not a Lyapunov function in the context of the definition used by [35] (p. 61), the main reason is that it is not positive definite, what is due to the truncation.*

Remark 6. *The main differences of the function $V(\bar{X})$ with respect to common truncated quadratic form (e.g., [11,18]) are: (i) it involves the nonlinear asymmetrical function $\bar{g}(\bar{X})$ which is a nonlinearity of the model; and (ii) the vanishment region Ω is asymmetrical, as $|\bar{X}^{*a}| \neq |\bar{X}^{*b}|$, what renders $V(\bar{X})$ asymmetrical. This structure allows us to develop a rigorous convergence proof, taking into account the nonlinear terms of the model and the asymmetry of the convergence set.*

Substituting (40) into (39), yields

$$\frac{dV}{dt} = g_t(-k)\left(\bar{g} + d\right),$$

using property (45), yields

$$\frac{dV}{dt} \leq -kg_t^2. \tag{51}$$

Integrating, yields

$$V + \int_{to}^{t} kg_t^2 d\tau \leq V(\bar{X}(t_o)).$$

In view of properties (50), we have

$$V \leq V(\bar{X}(t_o)), \;\; k\int_{to}^{t} g_t^2 d\tau \leq V(\bar{X}(t_o)).$$

This implies the asymptotic convergence of g_t^2 to zero, what can be proved by using the Barbalat's Lemma [21,36] and properties (46) and (47). Consequently, $\bar{X}$ converges asymptotically to Ω (48).

Remark 7. *Due to the condition* (40) *and the definition of* g_t (44), V (49) *exhibits vertex truncation, and the time derivative* $\dot{V}$ *can be expressed in terms of the truncated quadratic form* g_t^2, *see Equation* (51). *This allows to prove the asymptotic convergence of* $\bar{X}$. V (49) *and* g_t (44) *have a common vanishment for* $\bar{X} \in \Omega$ (48), *being the bounds of* Ω *asymmetrical.*

Remark 8. *The validity of the proof of asymptotic convergence of* $\bar{X}$ *is not disrupted by the following facts: (i)* $\bar{X}$ *is defined in the region* $\bar{R}^{\dagger}$ (38), *so that its initial value* $\bar{X}(t_o)$ *can take arbitrarily large positive values; (ii) since* δ *can be arbitrarily large, then the size of the convergence region* Ω (48) *can be arbitrarily large.*

5. Stability Analysis for the Case of Three Dimension Model with External Disturbance

In this section, the asymptotic convergence of a three dimension system with an external disturbance is determined by using functions with vertex truncation. The procedure is based on Section 4: (i) the dependence of the V_i functions on the state variables and the addition of the $\dot{V}_i$ expressions so as to obtain a non-positive nature is based on Section 4.1; (ii) the incorporation of truncation in the definition of the V_i functions and the arrangement of $\dot{V}_i$'s in terms of truncated forms is based on Section 4.2.

5.1. Stability Analysis for $\bar{X}_1$

Recall the differential equation for $\bar{X}_1$, that is, Equation (9). The time derivative of the function V_1 satisfies:

$$\frac{dV_1}{dt} = \frac{dV_1}{d\bar{X}_1}\frac{d\bar{X}_1}{dt}$$

Substituting the $\dot{\bar{X}}_1$ expression (9) and arranging, yields

$$\frac{dV_1}{dt} = -k_1\frac{dV_1}{d\bar{X}_1}\left(\bar{X}_1 + \frac{(-\delta_1)}{k_1}\right). \tag{52}$$

In view of characteristic 1 and Equation (10), k_1 is constant and positive. In view of (11), one further obtains $max\{-\delta_1/k_1\} = (1/k_1)max\{\delta_1\}$, $min\{-\delta_1/k_1\} = -(1/k_1)max\{\delta_1\}$. Thus, in view of the $-\delta_1/k_1$ term, we impose the following condition on V_1:

$$\frac{dV_1}{d\bar{X}_1} = g_{1t}, \tag{53}$$

where g_{1t} is a truncated function. On the basis of the procedure used in Section 4.2, we define it as

$$g_{1t} = \begin{cases} \bar{X}_1 - \frac{1}{k_1} max\{|\delta_1|\} & \text{for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ 0 & \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ \bar{X}_1 + \frac{1}{k_1} max\{|\delta_1|\} & \text{for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases}, \tag{54}$$

where $\bar{X}_1^{*a}$, $\bar{X}_1^{*b}$ are defined as:

$$\bar{X}_1^{*a} = -\frac{1}{k_1} max\{|\delta_1|\} \tag{55}$$

$$\bar{X}_1^{*b} = \frac{1}{k_1} max\{|\delta_1|\} \tag{56}$$

with

$$\bar{X}_1^{*a} < 0,\ \bar{X}_1^{*b} > 0,$$

and the main properties of g_{1t} are:

$$Pi)\ g_{1t}\,(\bar{X}_1 + (-\delta_1)/k_1) \geq g_{1t}^2 \tag{57}$$

$$Pii)\ g_{1t} \text{ is continuous with respect to } \bar{X}_1 \tag{58}$$

$$\begin{aligned} Piii)\ & g_{1t} = 0 \text{ for } \bar{X}_1 \in [\bar{X}_1^{*a}\ \ \bar{X}_1^{*b}], \\ & g_{1t} > 0 \text{ for } \bar{X}_1 > \bar{X}_1^{*b}, \text{ and } g_{1t} < 0 \text{ for } \bar{X}_1 < \bar{X}_1^{*a} \end{aligned} \tag{59}$$

$$Piv)\ \text{if } g_{1t} \to 0 \text{ as } t \to \infty, \text{ then } \bar{X}_1 \to \Omega_1, \tag{60}$$

$$\Omega_1 = \left\{\bar{X}_1 : \bar{X}_1^{*a} \leq \bar{X}_1 \leq \bar{X}_1^{*b}\right\}. \tag{61}$$

On the basis of condition (53) and definition (54), the definition of the function V_1 is:

$$V_1(\bar{X}_1) = \begin{cases} \int_{\bar{X}_1^{*b}}^{\bar{X}_1} \left(x - \frac{1}{k_1} max\{|\delta_1|\}\right) dx & \text{for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ 0 & \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ \int_{\bar{X}_1^{*a}}^{\bar{X}_1} \left(x + \frac{1}{k_1} max\{|\delta_1|\}\right) dx & \text{for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases}, \tag{62}$$

with properties

$$V_1 > 0 \text{ for } \bar{X}_1 > \bar{X}_1^{*b}$$
$$V_1 > 0 \text{ for } \bar{X}_1 < \bar{X}_1^{*a}$$
$$V_1 = 0 \text{ for } \bar{X}_1 \in [\bar{X}_1^{*a},\ \bar{X}_1^{*b}].$$

Substituting (53) into (52), yields:

$$\frac{dV_1}{dt} = g_{1t}\left[(-k_1)\left(\bar{X}_1 + \frac{(-\delta_1)}{k_1}\right)\right]$$

Using Property (57) yields:

$$\frac{dV_1}{dt} \leq -k_1 g_{1t}^2 \leq 0. \tag{63}$$

This implies the asymptotic convergence of g_{1t}^2 to zero, and $\bar{X}_1$ to Ω_1 (61), as stated by Proposition 2. This is concluded by using the Barbalat's Lemma [21,36].

Remark 9. *Due to condition* (53) *and definition of* g_{1t} (54)*,* V_1 (62) *exhibits vertex truncation and* $\dot{V}_1$ *can be expressed in terms of the truncated quadratic form* g_{1t}^2*, see Equation* (63)*. This allows to prove the asymptotic convergence of* $\bar{X}_1$*.*

Remark 10. *The validity of the proof of asymptotic convergence of* $\bar{X}_1$ *is not disrupted by the following facts: (i)* $\bar{X}_1$ *is defined in the region* $\bar{R}_{123}$*, according to Remark 1, so that its initial value can take arbitrarily large positive values; (ii) since* δ_o *can be arbitrarily large, then* δ_1 (11) *and the size of* Ω_1 (61) *can be arbitrarily large.*

5.2. Stability Analysis for $\bar{X}_2$

Since Equation (12) involves the term $\bar{X}_1$, we need to express $\bar{X}_1$ in terms of g_{1t} (54), which converges to zero as was already shown. Let

$$d_1 = g_{1t} - \bar{X}_1. \tag{64}$$

Therefore, X_1 can be expressed in terms of d_1:

$$\bar{X}_1 = g_{1t} - d_1.$$

Substituting into Equation (12) and arranging, yields

$$\frac{d}{dt}\bar{X}_2 = -(\bar{g}_2 + k_a d_1) + k_a g_{1t}, \tag{65}$$

where $\bar{g}_2(\bar{X}_2)$ is defined in Equation (14). The time derivative of the function V_2 satisfies:

$$\frac{dV_2}{dt} = \frac{dV_2}{d\bar{X}_2}\frac{d\bar{X}_2}{dt}.$$

Combining with Equation (65) yields

$$\frac{dV_2}{dt} = -\frac{dV_2}{d\bar{X}_2}\left[(\bar{g}_2 + k_a d_1) - k_a g_{1t}\right]. \tag{66}$$

Substituting (54) into (64) gives

$$d_1 = \begin{cases} -\frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ -\bar{X}_1 & \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ +\frac{1}{k_1} max\left\{|\delta_1|\right\} & \text{for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases}$$

since k_a and k_1 are positive and constant, then $max\{k_a d_1\} = (k_a/k_1) max\{\delta_1\} > 0$, $min\{k_a d_1\} = -(k_a/k_1) max\{\delta_1\} < 0$. In view of the $k_a d_1$ term appearing in Equation (66), we impose the following condition on V_2:

$$\frac{dV_2}{d\bar{X}_2} = g_{2t}, \tag{67}$$

where g_{2t} is a truncated function, that we define as

$$g_{2t} = \begin{cases} \bar{g}_2 - k_a \frac{1}{k_1} max\{|\delta_1|\} & \text{for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ 0 & \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ \bar{g}_2 + k_a \frac{1}{k_1} max\{|\delta_1|\} & \text{for } \bar{X}_2 \leq \bar{X}_2^{*a}, \end{cases} \tag{68}$$

where $\bar{X}_2^{*a}$, $\bar{X}_2^{*b}$ are defined as:

$$\bar{X}_2^{*a} = \left\{ \bar{X}_2 :\ \bar{g}_2 + k_a \frac{1}{k_1} max\{|\delta_1|\} = 0 \right\} \tag{69}$$

$$\bar{X}_2^{*b} = \left\{ \bar{X}_2 :\ \bar{g}_2 - k_a \frac{1}{k_1} max\{|\delta_1|\} = 0 \right\} \tag{70}$$

where

$$\bar{X}_2^{*a} < 0,\ \bar{X}_2^{*b} > 0.$$

The main properties of g_{2t} are:

$$Pi)\ g_{2t}(\bar{g}_2 + k_a d_1) \geq g_{2t}^2 \tag{71}$$

$$Pii)\ g_{2t} \text{ is continuous with respect to } \bar{X}_2$$

$$\begin{aligned} Piii)\ & g_{2t} = 0 \text{ for } \bar{X}_2 \in [\bar{X}_2^{*a}\ \ \bar{X}_2^{*b}], \\ & g_{2t} > 0 \text{ for } \bar{X}_2 > \bar{X}_2^{*b}, \text{ and } g_{2t} < 0 \text{ for } \bar{X}_2 < \bar{X}_2^{*a} \end{aligned} \tag{72}$$

$$Piv)\ \text{if } g_{2t} \to 0 \text{ as } t \to \infty, \text{ then } \bar{X}_2 \to \Omega_2, \tag{73}$$

$$\Omega_2 = \left\{ \bar{X}_2 : \bar{X}_2^{*a} \leq \bar{X}_2 \leq \bar{X}_2^{*b} \right\}. \tag{74}$$

On the basis of conditions (67) and (68), the definition of the function V_2 is:

$$V_2(\bar{X}_2) = \begin{cases} \int_{\bar{X}_2^{*b}}^{\bar{X}_2} \left(\bar{g}_2(x) - \frac{k_a}{k_1} max\{|\delta_1|\} \right) dx & \text{for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ 0 & \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ \int_{\bar{X}_2^{*a}}^{\bar{X}_2} \left(\bar{g}_2(x) + \frac{k_a}{k_1} max\{|\delta_1|\} \right) dx & \text{for } \bar{X}_2 \leq \bar{X}_2^{*a} \end{cases}, \tag{75}$$

where $\bar{g}_2(\bar{X}_2)$ is defined in Equation (14). V_2 exhibits the properties

$$\begin{aligned} & V_2 > 0 \text{ for } \bar{X}_2 > \bar{X}_2^{*b} \\ & V_2 > 0 \text{ for } \bar{X}_2 < \bar{X}_2^{*a} \\ & V_2 = 0 \text{ for } \bar{X}_2 \in [\bar{X}_2^{*a},\ \bar{X}_2^{*b}] \end{aligned}$$

Combining Equations (66) and (67), yields:

$$\frac{dV_2}{dt} = -g_{2t}(\bar{g}_2 + k_a d_1) + k_a g_{2t} g_{1t}.$$

Using Property (71), yields

$$\frac{dV_2}{dt} \leq -g_{2t}^2 + k_a g_{2t} g_{1t}. \tag{76}$$

In view of the term $k_a g_{2t} g_{1t}$, it is necessary to factorize and add the above expression with $\dot{V}_1$. We consider the constant α_2, that satisfies

$$\alpha_2 \in (0,\ 1).$$

Factorizing the right hand side of (76), arranging and multiplying by $4\alpha_2 k_1/k_a^2$, yields

$$\frac{4\alpha_2 k_1}{k_a^2}\frac{dV_2}{dt} \leq -4\frac{\alpha_2^2 k_1}{k_a^2}\left(g_{2t} - \frac{k_a}{2\alpha_2}g_{1t}\right)^2 - 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}g_{2t}^2 + k_1 g_{1t}^2. \tag{77}$$

Adding this and Equation (63), yields

$$\frac{dV_1}{dt} + \frac{d}{dt}\left(4\frac{\alpha_2 k_1}{k_a^2}V_2\right) \leq -4\frac{\alpha_2^2 k_1}{k_a^2}\left(g_{2t} - \frac{k_a}{2\alpha_2}g_{1t}\right)^2 - 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}g_{2t}^2 \leq 0. \tag{78}$$

This implies the asymptotic convergence of g_{2t}^2 to zero, and $\bar{X}_2$ to Ω_2 (74), as stated by Proposition 2.

Remark 11. *Due to condition* (67) *and definition of* g_{2t} (68), V_2 *exhibits vertex truncation, and the addition of* $\dot{V}_1$ *and* $\dot{V}_2$ *can be expressed in terms of the truncated quadratic form* g_{2t}^2, *see Equation* (78). *This allows to prove the asymptotic convergence of* $\bar{X}_2$.

Remark 12. *The validity of the proof of asymptotic convergence of* $\bar{X}_2$ *is not disrupted by the following facts: (i)* $\bar{X}_2$ *is defined in the region* $\bar{R}_{123}$, *according to Remark 1, so that its initial value can take arbitrarily large positive values; (ii) since* δ_o *can be arbitrarily large, then* δ_1 (11) *and the size of* Ω_2 (74) *can be arbitrarily large.*

5.3. Stability Analysis for $\bar{X}_3$

Recall that in Equation (15) the term $\bar{g}_{2b}$ is function of $\bar{X}_2$, being $\bar{g}_{2b}$ defined in Equation (17). Since $\bar{X}_2$ converges to Ω_2 (74), then $\bar{g}_{2b}$ converges to a compact set satisfying

$$\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} \leq \bar{g}_{2b} \leq \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}}.$$

where

$$\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} > 0, \tag{79}$$

$$\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} < 0, \tag{80}$$

and $\bar{X}_2^{*a}$, $\bar{X}_2^{*b}$ were defined in Equations (69) and (70). Thus, we express $\bar{g}_{2b}$ in terms of the truncated function g_{2bt}, defined as:

$$g_{2bt} = \begin{cases} \bar{g}_{2b} - \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} & \text{for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ 0 & \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ \bar{g}_{2b} + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} & \text{for } \bar{X}_2 \leq \bar{X}_2^{*a}. \end{cases} \tag{81}$$

The main properties of g_{2bt} are:

$$\begin{aligned} &Pi)\ g_{2bt} = 0 \text{ for } \bar{X}_2 \in \left[\bar{X}_2^{*a}\quad \bar{X}_2^{*b}\right], \\ &\quad g_{2bt} > 0 \text{ for } \bar{X}_2 > \bar{X}_2^{*b}, \text{ and } g_{2bt} < 0 \text{ for } \bar{X}_2 < \bar{X}_2^{*a} \end{aligned} \tag{82}$$

$$Pii)\ |g_{2bt}| \leq |g_{2t}|. \tag{83}$$

In Equation (15), the term $\bar{g}_{2b}$ must be expressed in terms of g_{2bt}. Let

$$d_{2b} = g_{2bt} - \bar{g}_{2b}. \tag{84}$$

Therefore, $\bar{g}_{2b}$ can be expressed in terms of d_{2b} and g_{2bt}:

$$\bar{g}_{2b} = g_{2bt} - d_{2b}.$$

substituting this into Equation (15) and arranging, yields

$$\frac{d}{dt}\bar{X}_3 = -(\bar{g}_3 + d_{2b}) + g_{2bt}, \tag{85}$$

where $\bar{g}_3$ is defined in Equation (18). The time derivative of the function V_3 satisfies:

$$\frac{dV_3}{dt} = \frac{dV_3}{d\bar{X}_3}\frac{d\bar{X}_3}{dt}.$$

Combining with Equation (85), yields:

$$\frac{dV_3}{dt} = -\frac{dV_3}{d\bar{X}_3}\left[(\bar{g}_3 + d_{2b}) - g_{2bt}\right] \tag{86}$$

where d_{2b} is a disturbance-like term. Substituting (81) into (84) gives:

$$d_{2b} = \begin{cases} -\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} \text{ for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ -\bar{g}_{2b} \text{ for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ +(-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} \text{ for } \bar{X}_2 \leq \bar{X}_2^{*a} \end{cases}$$

Therefore, $max\{d_{2b}\} = (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} > 0$, $min\{d_{2b}\} = (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} < 0$. In view of the d_{2b} term appearing in Equation (86), we impose the following condition on V_3:

$$\frac{dV_3}{d\bar{X}_3} = g_{3t}, \tag{87}$$

where g_{3t} is a truncated function, that we define as:

$$g_{3t} = \begin{cases} \bar{g}_3 - \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} \text{ for } \bar{X}_3 \geq \bar{X}_3^{*b} \\ 0 \quad \text{ for } \bar{X}_3 \in (\bar{X}_3^{*a},\ \bar{X}_3^{*b}) \\ \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} \text{ for } \bar{X}_3 \leq \bar{X}_3^{*a} \end{cases}, \tag{88}$$

where $\bar{g}_{2b}$ satisfies properties (79) and (80), and $\bar{X}_3^{*a}$, $\bar{X}_3^{*b}$ are defined as:

$$\bar{X}_3^{*a} = \{\bar{X}_3 : \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}} = 0\} \tag{89}$$

$$\bar{X}_3^{*b} = \{\bar{X}_3 : \bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}} = 0\}, \tag{90}$$

where

$$\bar{X}_3^{*a} < 0,\ \bar{X}_3^{*b} > 0.$$

The main properties of g_{3t} are:

$$Pi)\ g_{3t}(\bar{g}_3 + d_{2b}) \geq g_{3t}^2 \tag{91}$$

$$Pii)\ g_{3t} \text{ is continuous with respect to } \bar{X}_3$$

$$\begin{aligned} Piii)\ g_{3t} = 0 \text{ for } \ \bar{X}_3 \in [\bar{X}_3^{*a} \ \ \bar{X}_3^{*b}], \\ g_{3t} > 0 \text{ for } \bar{X}_3 > \bar{X}_3^{*b}, \text{ and } g_{3t} < 0 \text{ for } \bar{X}_3 < \bar{X}_3^{*a} \end{aligned} \tag{92}$$

$$Piv) \text{ if } g_{3t} \to 0 \text{ as } t \to \infty, \text{ then } \bar{X}_3 \to \Omega_3, \tag{93}$$

$$\Omega_3 = \left\{ \bar{X}_3 : \bar{X}_3^{*a} \leq \bar{X}_3 \leq \bar{X}_3^{*b} \right\}. \tag{94}$$

On the basis of conditions (87) and (88), the definition of the function V_3 is:

$$V_3(\bar{X}_3) = \begin{cases} \int_{\bar{X}_3^{*b}}^{\bar{X}_3} \left(\bar{g}_3(x) - \bar{g}_{2b}|_{\bar{X}_2 = \bar{X}_2^{*b}} \right) dx \text{ for } \bar{X}_3 \geq \bar{X}_3^{*b} \\ \quad 0 \qquad \text{for } \bar{X}_3 \in (\bar{X}_3^{*a}, \ \bar{X}_3^{*b}) \\ \int_{\bar{X}_3^{*a}}^{\bar{X}_3} \left(\bar{g}_3(x) + (-1)\bar{g}_{2b}|_{\bar{X}_2 = \bar{X}_2^{*a}} \right) dx \text{ for } \bar{X}_3 \leq \bar{X}_3^{*a} \end{cases}, \tag{95}$$

where $\bar{g}_3(\bar{X}_3)$ is defined in Equation (18). V_3 exhibits the properties

$$\begin{aligned} &V_3 > 0 \text{ for } \bar{X}_3 > \bar{X}_3^{*b} \\ &V_3 > 0 \text{ for } \bar{X}_3 < \bar{X}_3^{*a} \\ &V_3 = 0 \text{ for } \bar{X}_3 \in [\bar{X}_3^{*a}, \ \bar{X}_3^{*b}] \end{aligned}$$

Combining (86) and (87), yields

$$\frac{dV_3}{dt} = (-1)g_{3t}(\bar{g}_3 + d_{2b}) + g_{3t}g_{2bt}.$$

Using property (91), yields

$$\frac{dV_3}{dt} \leq -g_{3t}^2 + g_{3t}g_{2bt}. \tag{96}$$

In view of the term $g_{3t}g_{2bt}$, we need to factorize the $-g_{3t}^2 + g_{3t}g_{2bt}$ term and to add the equations for dV_1/dt, dV_2/dt, dV_3/dt. We consider the constant α_3 that satisfies

$$\alpha_3 \in (0, \ 1)$$

By using α_3, the term $-g_{3t}^2 + g_{3t}g_{2bt}$ can be rewritten as $-\alpha_3 g_{3t}^2 + g_{3t}g_{2bt} - (1-\alpha_3)g_{3t}^2$. In turn, the term $-\alpha_3 g_{3t}^2 + g_{3t}g_{2bt}$ can be factorized as

$$-\alpha_3 g_{3t}^2 + g_{3t}g_{2bt} = -\left(\sqrt{\alpha_3} g_{3t} - \frac{1}{2\alpha_3^{1/2}} g_{2bt} \right)^2 + \left(\frac{1}{2\alpha_3^{1/2}} g_{2bt} \right)^2.$$

Using this property, Equation (96) can be expressed as:

$$\frac{dV_3}{dt} \leq -\left(\sqrt{\alpha_3} g_{3t} - \frac{1}{2\alpha_3^{1/2}} g_{2bt} \right)^2 + \frac{1}{4\alpha_3} g_{2bt}^2 - (1-\alpha_3) g_{3t}^2$$

Multiplying by $16\alpha_2(1-\alpha_2)\alpha_3 k_1/k_a^2$ and using property (83) on the g_{2bt}^2 term, yields

$$16\frac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}\frac{dV_3}{dt} \le (-1)16\frac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}\left(\sqrt{\alpha_3}g_{3t} - \frac{1}{2\alpha_3^{1/2}}g_{2bt}\right)^2 + 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}g_{2t}^2$$
$$+(-1)16\frac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}(1-\alpha_3)g_{3t}^2.$$

Adding this and Equation (78), yields

$$\begin{aligned} &\tfrac{dV_1}{dt} + \tfrac{d}{dt}\left(\tfrac{4\alpha_2 k_1}{k_a^2}V_2\right) + \tfrac{d}{dt}\left(16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}V_3\right) \\ &\le -\tfrac{4\alpha_2^2 k_1}{k_a^2}\left(g_{2t} - \tfrac{k_a}{2\alpha_2}g_{1t}\right)^2 + (-1)\left(16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}\right)\left(\sqrt{\alpha_3}g_{3t} - \tfrac{1}{2\alpha_3^{1/2}}g_{2bt}\right)^2 \\ &\quad +(-1)16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}(1-\alpha_3)g_{3t}^2 \le 0 \end{aligned} \tag{97}$$

This implies the asymptotic convergence of g_{3t}^2 to zero and $\bar{X}_3$ to Ω_3 (94), as stated in Proposition 2.

Remark 13. *Due to condition (87) and definition of g_{3t} (88), V_3 exhibits vertex truncation, and the addition of $\dot{V}_1$, $\dot{V}_2$, $\dot{V}_3$ can be expressed in terms of the truncated form g_{3t}^2, see Equation (97). This allows to prove the asymptotic convergence of $\bar{X}_3$.*

Remark 14. *The validity of the proof of asymptotic convergence of $\bar{X}_3$ is not disrupted by the following facts: (i) $\bar{X}_3$ is defined in the region $\bar{R}_{123}$, according to remark 1, so that its initial value can take arbitrarily large positive values; (ii) since δ_o can be arbitrarily large, then δ_1 (11) and the size of Ω_3 (94) can be arbitrarily large.*

5.4. Boundedness Analysis (Proof of Proposition 1)

To prove the boundedness of $\bar{X}_1$, we begin by arranging and integrating (63), what yields

$$V_1 + \int_{to}^{t} k_1 g_{1t}^2 d\tau \le V_1(\bar{X}_1(t_0)). \tag{98}$$

Therefore, $V_1 \in L_\infty$. This and Equation (62) imply $\bar{X}_1 \in L_\infty$. From (54) it follows that $g_{1t} \in L_\infty$.
To prove the boundedness of $\bar{X}_2$, we begin by arranging and integrating (78), what yields

$$V_1 + 4\frac{\alpha_2 k_1}{k_a^2}V_2 + 4\frac{\alpha_2(1-\alpha_2)k_1}{k_a^2}\int_{to}^{t} g_{2t}^2 dt \le V_1(\bar{X}_1(t_0)) + 4\frac{\alpha_2 k_1}{k_a^2}V_2(\bar{X}_2(t_0)). \tag{99}$$

Therefore, $V_2 \in L_\infty$. This and Equation (75) imply $\bar{g}_2 \in L_\infty$; hence, $\bar{X}_2 \in L_\infty$, from (14). From (68) it follows that $g_{2t} \in L_\infty$.

To prove the boundedness of $\bar{X}_3$, we begin by arranging and integrating (97), what yields

$$\begin{aligned} &V_1 + \tfrac{4\alpha_2 k_1}{k_a^2}V_2 + 16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}V_3 + 16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1(1-\alpha_3)}{k_a^2}\int_{to}^{t}\left(g_{3t}^2\right)dt \\ &\le V_1(\bar{X}_1(t_0)) + \tfrac{4\alpha_2 k_1}{k_a^2}V_2(\bar{X}_2(t_0)) + 16\tfrac{\alpha_2(1-\alpha_2)\alpha_3 k_1}{k_a^2}V_3(\bar{X}_3(t_0)). \end{aligned} \tag{100}$$

Therefore, $V_3 \in L_\infty$. This and Equation (95) imply $\bar{g}_3 \in L_\infty$; hence, $\bar{X}_3 \in L_\infty$, from (18). From (88) it follows that $g_{3t} \in L_\infty$.

5.5. Convergence Analysis (Proof of Proposition 2)

From (98) it follows that $g_{1t}^2 \in L_1$. It is necessary to prove that $g_{1t}^2 \in L_\infty$ and $d(g_{1t}^2)/dt \in L_\infty$ to apply Barbalat's Lemma. Recall that $g_{1t} \in L_\infty$ according to Proposition 1, hence $g_{1t}^2 \in L_\infty$.

Differentiating g_{1t}^2 with respect to time, using (54), yields:

$$\frac{d(g_{1t}^2)}{dt} = \frac{d(g_{1t}^2)}{d\bar{X}_1}\frac{d\bar{X}_1}{dt}, \tag{101}$$

where

$$\frac{d(g_{1t}^2)}{d\bar{X}_1} = \begin{cases} 2(\bar{X}_1 - \frac{1}{k_1}max\,\{|\delta_1|\}) \text{ for } \bar{X}_1 \geq \bar{X}_1^{*b} \\ \qquad 0 \qquad\qquad\qquad \text{for } \bar{X}_1 \in (\bar{X}_1^{*a},\ \bar{X}_1^{*b}) \\ 2(\bar{X}_1 + \frac{1}{k_1}max\,\{-\delta_1/k_1\}) \text{ for } \bar{X}_1 \leq \bar{X}_1^{*a} \end{cases}. \tag{102}$$

Therefore

$$\frac{d(g_{1t}^2)}{d\bar{X}_1} = 0 \text{ for } \bar{X}_1 = \bar{X}_1^{*a} \text{ and for } \bar{X}_1 = \bar{X}_1^{*b}$$

Thus, it follows from (101) that $d(g_{1t}^2)/d\bar{X}_1$ is well-defined and continuous with respect to $\bar{X}_1$. Recall that $\bar{X}_1 \in L_\infty$, $\bar{g}_1 \in L_\infty$ according to Proposition 1. This and Equation (102) lead to $d(g_{1t}^2)/d\bar{X}_1 \in L_\infty$.

Since $dg_{1t}^2/d\bar{X}_1$, $d\bar{X}_1/dt$ are bounded, it follows from (101) that $d(g_{1t}^2)/dt$ is bounded. So far we have proved that $g_{1t}^2 \in L_1$, $g_{1t}^2 \in L_\infty$ and $d(g_{1t}^2)/dt \in L_\infty$. Thus, applying Barbalat's lemma [27], yields $\lim_{t\to\infty}(g_{1t}^2) = 0$. Hence, according to properties (59) and (60), $\bar{X}_1$ converges asymptotically to Ω_1.

From (99) it follows that $(g_{2t}^2) \in L_1$. It is necessary to prove that $(g_{2t}^2) \in L_\infty$ and $d(g_{2t}^2)/dt \in L_\infty$ to apply Barbalat's lemma. Recall that $g_{2t} \in L_\infty$ according to Proposition 1, hence $g_{2t}^2 \in L_\infty$. Differentiating g_{2t}^2 with respect to time, using (68), yields:

$$\frac{d(g_{2t}^2)}{dt} = \frac{d(g_{2t}^2)}{d\bar{X}_2}\frac{d\bar{X}_2}{dt}, \tag{103}$$

where

$$\frac{d(g_{2t}^2)}{d\bar{X}_2} = \begin{cases} 2(\bar{g}_2 - \frac{k_a}{k_1}max\,\{|\delta_1|\})\frac{d\bar{g}_2}{d\bar{X}_2} \text{ for } \bar{X}_2 \geq \bar{X}_2^{*b} \\ \qquad 0 \qquad\qquad\qquad\qquad \text{for } \bar{X}_2 \in (\bar{X}_2^{*a},\ \bar{X}_2^{*b}) \\ 2(\bar{g}_2 + \frac{k_a}{k_1}max\,\{|\delta_1|\})\frac{d\bar{g}_2}{d\bar{X}_2} \text{ for } \bar{X}_2 \leq \bar{X}_2^{*a} \end{cases} \tag{104}$$

$$\frac{d\bar{g}_2}{d\bar{X}_2} = \frac{1}{\tau_{out}} + k_n\frac{k_{AN}}{(k_{AN} + \bar{X}_2 + X_2^{eq})^2}. \tag{105}$$

Therefore,

$$\frac{d(g_{2t}^2)}{d\bar{X}_2} = 0 \text{ for } \bar{X}_2 = \bar{X}_2^{*a} \text{ and for } \bar{X}_2 = \bar{X}_2^{*b}.$$

Thus, $d(g_{2t}^2)/d\bar{X}_2$ is well-defined and continuous with respect to $\bar{X}_2$. Recall that $\bar{g}_2 \in L_\infty$, $\bar{X}_2 \in L_\infty$ according to Proposition 1. This and Equations (104) and (105) lead to $d(g_{2t}^2)/d\bar{X}_2 \in L_\infty$.

Since $d(g_{2t}^2)/d\bar{X}_2$, $d\bar{X}_2/dt$ are bounded, it follows from (103) that $d(g_{2t}^2)/dt$ is bounded. So far we have proved that $g_{2t}^2 \in L_1$, $g_{2t}^2 \in L_\infty$ and $d(g_{2t}^2)/dt \in L_\infty$. Thus, applying Barbalat's lemma [27], yields $\lim_{t\to\infty} g_{2t}^2 = 0$. Hence, according to properties (72) and (73), $\bar{X}_2$ converges asymptotically to Ω_2.

From (100) it follows that $g_{3t}^2 \in L_1$. It is necessary to prove that $g_{3t}^2 \in L_\infty$ and $d(g_{3t}^2)/dt \in L_\infty$ to apply Barbalat's lemma. Recall that $g_{3t} \in L_\infty$ according to proposition 1, hence $g_{3t}^2 \in L_\infty$. Differentiating g_{3t}^2 with respect to time, using (88), yields:

$$\frac{dg_{3t}^2}{dt} = \frac{dg_{3t}^2}{d\bar{X}_3}\frac{\bar{X}_3}{dt} \tag{106}$$

$$\frac{d(g_{3t}^2)}{d\bar{X}_3} = \begin{cases} 2(\bar{g}_3 - \bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*b}})\frac{d\bar{g}_3}{d\bar{X}_3} & \text{for } \bar{X}_3 \geq \bar{X}_3^{*b} \\ 0 & \text{for } \bar{X}_3 \in (\bar{X}_3^{*a},\ \bar{X}_3^{*b}) \\ 2(\bar{g}_3 + (-1)\bar{g}_{2b}|_{\bar{X}_2=\bar{X}_2^{*a}})\frac{d\bar{g}_3}{d\bar{X}_3} & \text{for } \bar{X}_3 \leq \bar{X}_3^{*a} \end{cases} \tag{107}$$

$$\frac{d\bar{g}_3}{d\bar{X}_3} = \frac{1}{\tau_{out}} + k_d\frac{K_{ds}}{(K_{ds} + \bar{X}_3 + X_3^{eq})^2}. \tag{108}$$

Therefore,

$$\frac{d(g_{3t}^2)}{d\bar{X}_3} = 0 \text{ for } \bar{X}_3 = \bar{X}_3^{*a}, \text{ and for } \bar{X}_3 = \bar{X}_3^{*b}.$$

Thus, $d(g_{3t}^2)/d\bar{X}_3$ is well-defined and continuous with respect to $\bar{X}_3$. Recall that $\bar{g}_3 \in L_\infty$, $\bar{X}_3 \in L_\infty$ according to Proposition 1. This and Equations (107) and (108) lead to $d(g_{3t}^2)/d\bar{X}_3 \in L_\infty$.

Since $d(g_{3t}^2)/d\bar{X}_3$, $d\bar{X}_3/dt$ are bounded, it follows from (106) that $d(g_{3t}^2)/dt$ is bounded. So far we have proved that $g_{3t}^2 \in L_1$, $g_{3t}^2 \in L_\infty$ and $d(g_{3t}^2)/dt \in L_\infty$. Thus, applying Barbalat's Lemma [27], yields $\lim_{t\to\infty} g_{3t}^2 = 0$. Hence, according to properties (92) and (93), $\bar{X}_3$ converges asymptotically to Ω_3.

5.6. Invariant Properties (Proof of Proposition 3)

The positive invariant nature of the convergence sets of $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ is proved at what follows. A subset of the state space is positively invariant if the system trajectories starting inside it remain inside in the future. In addition, the positive invariant nature of a residual set is guaranteed if $\dot{V} \leq 0$ [37,38]. Consider the compact sets Ω_1 (61), Ω_2 (74) and Ω_3 (94). Let

$$\Omega_{12} = \Omega_1 \cup \Omega_2$$
$$\Omega_{123} = \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

According to Proposition 2,

$$lim_{t\to\infty} g_{1t} = 0$$
$$lim_{t\to\infty} g_{2t} = 0$$
$$lim_{t\to\infty} g_{3t} = 0$$

Therefore, Ω_1, Ω_2, Ω_3, Ω_{12}, Ω_{123} are attractive sets.

The set Ω_1 is positively invariant, what is concluded from:

$$dV_1/dt \leq 0 \text{ for } \bar{X}_1 \in [\bar{X}_1^{*a},\ \bar{X}_1^{*b}],, \tag{109}$$

what follows from Equation (63) and Property (59).

The set Ω_{12} is positively invariant, what is concluded from Equation (109) and:

$$dV_2/dt \leq 0 \text{ for } \bar{X}_1 \in [\bar{X}_1^{*a},\ \bar{X}_1^{*b}] \text{ and } \bar{X}_2 \in [\bar{X}_2^{*a},\ \bar{X}_2^{*b}], \tag{110}$$

which follows from Equation (77), and property (59).

The set Ω_{123} is positively invariant, what is concluded from Equations (109) and (110) jointly with

$$dV_3/dt \leq 0 \text{ for } \bar{X}_2 \in [\bar{X}_2^{*a},\ \bar{X}_2^{*b}] \text{ and } \bar{X}_3 \in [\bar{X}_3^{*a},\ \bar{X}_3^{*b}],$$

which follows from Equation (96) and properties (92) and (72).

6. Example

We consider the model (1) to (3), with functions (4) to (5), subject to Characteristics 1–4, and with the following parameter values, based on [33]:

$$\begin{aligned}
&V = 1680\text{L},\ Q_{in} = 113.4\text{L/day},\ Q_{out} = 31.5\text{L/day},\ k_a = 0.27\text{day}^{-1},\ k_n = 50.4\text{day}^{-1}\text{mg/L},\\
&k_{AN} = 40\text{mg/L},\ k_d = 134\text{day}^{-1}\text{mg/L},\ K_{ds} = 0.25\text{mg/L},\ k_p = 0.01\text{day}^{-1}\text{mg/L},\\
&X^p_{1,in} = 14\text{mg/L},\ X_{2,in} = 55.1\text{mg/L},\ X_{3,in} = 0.007\text{mg/L}.
\end{aligned}$$

Therefore, $\tau_{in} = 14.815$ day, $\tau = 53.33$ day.

We consider $\delta_0 = \pm 1.4 sin\left((2\pi/T_\delta)t\right)$ mg/L, $T_\delta = 2$ days. Therefore, $\max\{|\delta_1|\} = 0.0945$.

Using Equations (6)–(8), we obtain the following equilibrium points: $X_1^{eq} = 3.273$ mg/L, $X_2^{eq} = 3.94$ mg/L, $X_3^{eq} = 0.009$ mg/L. From Equations (55) and (56) it follows that $\bar{X}_1^{*a} = -0.327$, $\bar{X}_1^{*b} = 0.327$. From Equations (69) and (70) it follows that $\bar{X}_2^{*a} = -0.083$, $\bar{X}_2^{*b} = 0.083$. From Equations (89) and (90) it follows that $\bar{X}_3^{*a} = -1.733 \times 10^{-4}$, $\bar{X}_3^{*b} = 1.736 \times 10^{-4}$.

Figure 1 presents the time course of $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$. The lower and upper bounds of the convergence regions, that is, $\bar{X}_1^{*a}$, $\bar{X}_1^{*b}$, $\bar{X}_2^{*a}$, $\bar{X}_2^{*b}$, $\bar{X}_3^{*a}$, $\bar{X}_3^{*b}$ are shown as horizontal dashed-lines. It can be noticed that once the trajectories enter the compact set $\bar{\Omega}_{123}$, they remain inside it.

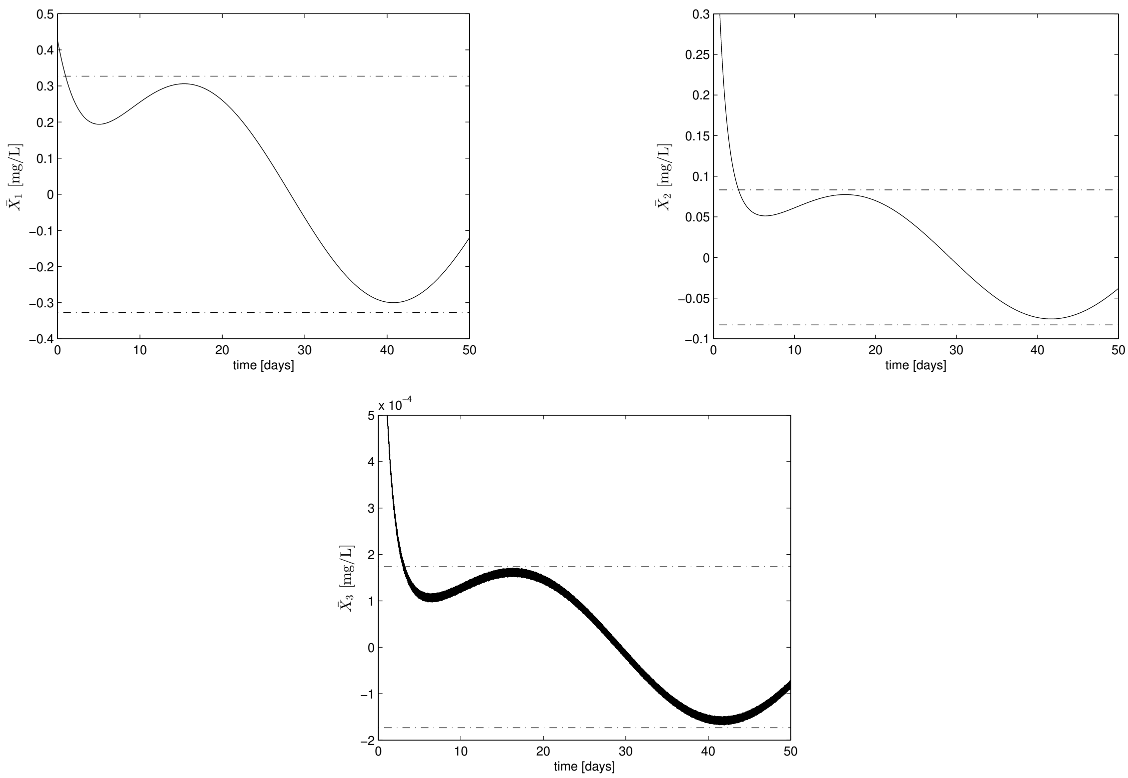

Figure 1. Time course of $\bar{X}_1$ (upper left), $\bar{X}_2$ (upper right), $\bar{X}_3$ (lower left). The lower and upper bounds of the convergence regions, that is, $\bar{X}_1^{*a}$, $\bar{X}_1^{*b}$, $\bar{X}_2^{*a}$, $\bar{X}_2^{*b}$, $\bar{X}_3^{*a}$, $\bar{X}_3^{*b}$ are shown as horizontal dashed-lines.

7. Discussion

It was shown that the asymptotic stability of the bioreaction process considered can be proved by using functions with vertex truncation. The size of the convergence region of the state variables depend on the bounds of the external disturbance. This size can be large, and far from the equilibrium point. A simple and systematic procedure was provided to determine and prove asymptotic convergence of the state variables towards a compact set of asymmetrical bounds, what includes definition of the truncated V_i functions and the truncated forms appearing in its time derivative. Both of these truncated functions exhibit a vanishment for values of the state variables in the convergence region. The analysis is valid for arbitrarily large positive initial values of the state variables, and arbitrarily large size of the convergence regions. The stability analysis was based on that of classical robust adaptive controller design, but the truncated function was different to the common truncated quadratic function, as it involves the model nonlinearities and an asymmetrical vanishment region.

Although the approach was developed for a specific biological process, it can be adapted to other systems, including systems converging to limit cycles.

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References

1. Zhou, G.; Liao, X.; Xu, B.; Yu, P.; Chen, G. Simple algebraic necessary and sufficient conditions for Lyapunov stability of a Chen system and their applications. *Trans. Inst. Meas. Control.* **2018**, *40*, 2200–2210. [CrossRef]
2. Liao, X.; Fu, Y.; Xie, S. On the new results of global attractive set and positive invariant set of the Lorenz chaotic system and the applications to chaos control and synchronization. *Sci. China Ser. F Inf. Sci.* **2005**, *48*, 304–321. [CrossRef]
3. Yu, P.; Liao, X. Globally attractive and positive invariant set of the Lorenz system. *Int. J. Bifurc. Chaos* **2006**, *16*, 757–764. [CrossRef]
4. Li, D.; Lu, J.; Wu, X.; Chen, G. Estimating the bounds for the Lorenz family of chaotic systems. *Chaos Solitons Fractals* **2005**, *23*, 529–534. [CrossRef]
5. Ratschan, S.; She, Z. Providing a basin of attraction to a target region of polynomial systems by computation of Lypunov-like functions. *SIAM J. Control Optim.* **2010**, *48*, 4377–4394. [CrossRef]
6. Papachristodoulou, A.; Prajna, S. On the construction of lyapunov functions using the sum of squares decomposition. In Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, USA, 10–13 December 2002; pp. 3482–3487.
7. Liu, Z.; Wang, F.; Zhang, Y.; Philip Chen, CL. Fuzzy adaptive quantized control for a class of stochastic nonlinear uncertain systems. *IEEE Trans. Cybern.* **2016**, *46*, 524–534. [CrossRef]
8. Boulkroune, A.; Msaad, M.; Farza, M. Adaptive fuzzy system-based variable-structure controller for multivariable nonaffine nonlinear uncertain systems subject to actuator nonlinearities. *Neural Comput. Appl.* **2017**, *28*, 3371–3384. [CrossRef]
9. Cai, M.; Xiang, Z. Adaptive practical finite-time stabilization for uncertain nonstrict feedback nonlinear systems with input nonlinearity. *IEEE Trans. Syst. Man Cybern. Syst.* **2017**, *47*, 1668–1678. [CrossRef]
10. Zhu, Z.; Xia, Y.; Fu, M. Attitude stabilization of rigid spacecraft with finite-time convergence. *Int. J. Robust Nonlinear Control* **2011**, *21*, 686–702. [CrossRef]
11. Rincon, A.; Piarpuzán, D.; Angulo, F. A new adaptive controller for bio-reactors with unknown kinetics and biomass concentration: Guarantees for the boundedness and convergence properties. *Math. Comput. Simul.* **2015**, *112*, 1–13. [CrossRef]

12. Meadows, T.; Weedermann, M.; Wolkowicz, G.S.K. Global analysis of a simplified model of anaerobic digestion and a new result for the chemostat. *SIAM J. Appl. Math.* **2019**, *79*, 668–689. [CrossRef]
13. Zhang, F.; Liao, X.; Zhang, G.; Mu, C. Dynamical analysis of the generalized Lorenz systems. *J. Dyn. Control Syst.* **2017**, *23*, 349–362. [CrossRef]
14. Zhang, F.; Liao, X.; Zhang, G. On the global boundedness of the Lü system. *Appl. Math. Comput.* **2016**, *284*, 332–339. [CrossRef]
15. Liao, X.; Zhou, G.; Yang, Q.; Fu, Y.; Chen, G. Constructive proof of Lagrange stability and sufficient—Necessary conditions of Lyapunov stability for Yang–Chen chaotic system. *Appl. Math. Comput.* **2017**, *309*, 205–221. [CrossRef]
16. Zhang, F.; Mu, C.; Wang, L.; Zhang, G.; Ahmed, I. On the new results of global exponential attractive set. *Appl. Math. Lett.* **2014**, *28*, 30–37. [CrossRef]
17. Mu, C.; Zhang, F.; Shu, Y. On the boundedness of solutions to the Lorenz-like family of chaotic systems. *Nonlinear Dyn.* **2012**, *67*, 987–996. [CrossRef]
18. Su, C.; Feng, Y.; Hong, H.; Chen, X. Adaptive control of system involving complex hysteretic nonlinearities: A generalised Prandtl–Ishlinskii modelling approach. *Int. J. Control* **2009**, *82*, 1786–1793. [CrossRef]
19. Zhou, J.; Zhang, C.; Wen, C. Robust adaptive output control of uncertain nonlinear plants with unknown backlash nonlinearity. *IEEE Trans. Autom. Control* **2007**, *52*, 503–509. [CrossRef]
20. Zhou, J.; Wen, C.; Zhang, Y. Adaptive output control of nonlinear systems with uncertain dead-zone nonlinearity. *IEEE Trans. Autom. Control* **2006**, *51*, 504–511. [CrossRef]
21. Koo, K. Stable adaptive fuzzy controller with time-varying dead-zone. *Fuzzy Sets Syst.* **2001**, *121*, 161–168. [CrossRef]
22. Bhat, S.; Bernstein, D. Finite-time stability of continuous autonomous systems. *SIAM J. Control Optim.* **2000**, *38*, 751–766. [CrossRef]
23. Jian, J.; Deng, X.; Tu, Z. New results of globally exponentially attractive set and synchronization controlling of the Qi chaotic system. In *Advances in Neural Networks—ISNN 2010*; Zhang, L., Kwok, J., Lu, B.L., Eds.; Springer: Berlin, Germany, 2010; pp. 643–650.
24. Yao, B.; Tomizuka, M. Adaptive robust control of SISO nonlinear systems in a semi-strict feedback form. *Automatica* **1997**, *33*, 893–900. [CrossRef]
25. Gao, S.; Ning, B.; Dong, H. Fuzzy dynamic surface control for uncertain nonlinear systems under input saturation via truncated adaptation approach. *Fuzzy Sets Syst.* **2016**, *290*, 100–117. [CrossRef]
26. Cao, C.; Hovakimyan, N. Design and analysis of a novel L_1 adaptive control architecture with guaranteed transient performance. *IEEE Trans. Autom. Control* **2008**, *53*, 586–591. [CrossRef]
27. Ioannou, P.; Sun, J. *Robust Adaptive Control*; Prentice-Hall PTR: Upper Saddle River, NJ, USA, 1996.
28. Wang, X.; Su, C.; Hong, H. Robust adaptive control of a class of nonlinear systems with unknown dead-zone. *Automatica* **2004**, *40*, 407–413. [CrossRef]
29. Zhou, J.; Wen, C.; Zhang, Y. Adaptive backstepping control of a class of uncertain nonlinear systems with unknown backlash-like hysteresis. *IEEE Trans. Autom. Control* **2004**, *49*, 1751–1757. [CrossRef]
30. Wang, Q.; Su, C. Robust adaptive control of a class of nonlinear systems including actuator hysteresis with Prandtl–Ishlinskii presentations. *Automatica* **2006**, *42*, 859–867. [CrossRef]
31. Ge, S.S.; Hang, C.C.; Zhang, T. Adaptive neural network control of nonlinear systems by state and output feedback. *IEEE Trans. Syst. Man Cybern. -Part B Cybern.* **1999**, *29*, 818–828. [CrossRef]
32. Ioannou, P.; Kokotovic, P. Robust redesign of adaptive control. *IEEE Trans. Autom. Control* **1984**, *AC-29*, 202–211. [CrossRef]
33. Xuan, Z.; Chang, N.; Daranpob, A.; Wanielista, M. Modeling subsurface upflow wetland systems for wastewater effluent treatment. *Environ. Eng. Sci.* **2010**, *27*, 879–888. [CrossRef]
34. Ramírez-León, H.; Barrios-Piña, H.; Cuevas-Otero, A.; Torres-Bejarano, F.; Ponce-Palafox, J. Hydraulic and environmental design of a constructed wetland as a treatment for shrimp aquaculture effluents. In *High Performance Computer Applications*; Gitler, I., Klapp, J., Eds.; Springer: Berlin, Germany, 2015; pp. 508–522.
35. Slotine, J.; Li, W. *Applied Nonlinear Control*; Prentice-Hall Inc.: Upper Saddle River, NJ, USA, 1991.
36. Su, C.; Stepanenko, Y.; Svoboda, J.; Leung, T.P. Robust adaptive control of a class of nonlinear systems with unknown backlash-like hysteresis. *IEEE Trans. Autom. Control* **2000**, *45*, 2427–2432. [CrossRef]

37. Blanchini, F. Set invariance in control. *Automatica* **1999**, *35*, 1747–1767. [CrossRef]
38. Cao, Y.-Y.; Lin, Z.; Ward, D.G. Anti-windup design of output tracking systems subject to actuator saturation and constant disturbances. *Automatica* **2004**, *40*, 1221–1228. [CrossRef]