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Some Identities on Type 2 Degenerate Bernoulli Polynomials of the Second Kind

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Abstract: In recent years, many mathematicians studied various degenerate versions of some special polynomials for which quite a few interesting results were discovered. In this paper, we introduce the type 2 degenerate Bernoulli polynomials of the second kind and their higher-order analogues, and study some identities and expressions for these polynomials. Specifically, we obtain a relation between the type 2 degenerate Bernoulli polynomials of the second and the degenerate Bernoulli polynomials of the second, an identity involving higher-order analogues of those polynomials and the degenerate Stirling numbers of the second kind, and an expression of higher-order analogues of those polynomials in terms of the higher-order type 2 degenerate Bernoulli polynomials and the degenerate Stirling numbers of the first kind.

Keywords: type 2 degenerate Bernoulli polynomials of the second kind; degenerate central factorial numbers of the second kind

1. Introduction

In [1,2], Carlitz initiated study of the degenerate Bernoulli and Euler polynomials and obtained some arithmetic and combinatorial results on them. In recent years, many mathematicians have drawn their attention to various degenerate versions of some old and new polynomials and numbers, namely some degenerate versions of Bernoulli numbers and polynomials of the second kind, Changhee numbers of the second kind, Daehee numbers of the second kind, Bernstein polynomials, central Bell numbers and polynomials, central factorial numbers of the second kind, Cauchy numbers, Eulerian numbers and polynomials, Fubini polynomials, Stirling numbers of the first kind, Stirling polynomials of the second kind, central complete Bell polynomials, Bell numbers and polynomials, type 2 Bernoulli numbers and polynomials, type 2 Bernoulli polynomials of the second kind, poly-Bernoulli numbers and polynomials, poly-Cauchy polynomials, and of Frobenius–Euler polynomials, to name a few [3–10] and the references therein.

They have studied those polynomials and numbers with their interest not only in combinatorial and arithmetic properties but also in differential equations and certain symmetric identities [7,9] and references therein, and found many interesting results related to them [3–6,8,10]. It is remarkable that studying degenerate versions is not only limited to polynomials but also extended to transcendental functions. Indeed, the degenerate gamma functions were introduced in connection with degenerate Laplace transforms [11,12].

The motivation for this research is to introduce the type 2 degenerate Bernoulli polynomials of the second kind defined by

$$\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^*(x)\frac{t^n}{n!},$$

and investigate its arithmetic and combinatorial properties. The facts in Section 1 are some known definitions and results that are needed throughout this paper. However, all of the results in Section 2 are new.

We will spend the rest of this section in recalling some necessary stuffs for the next section.

As is known, the type 2 Bernoulli polynomials are defined by the generating function [5,13]

$$\frac{t}{e^t - e^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^*(x)\frac{t^n}{n!}. \tag{1}$$

From (1), we note that

$$B_n^*(x) = 2^{n-1}B_n\left(\frac{x+1}{2}\right), \quad (n \geq 0), \tag{2}$$

where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Also, the type 2 Euler polynomials are given by [5,13]

$$e^{xt}\mathrm{sech}t = \frac{2}{e^t+e^{-t}}e^{xt} = \sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!}. \tag{3}$$

Note that

$$E_n^*(x) = 2^n E_n\left(\frac{x+1}{2}\right), \quad (n \geq 0), \tag{4}$$

where $E_n(x)$ are the ordinary Euler polynomials given by [14,15]

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The central factorial numbers of the second kind are defined as [5,8]

$$x^n = \sum_{k=0}^{n} T(n,k)x^{[k]}, \tag{5}$$

or equivalently as

$$\frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}, \tag{6}$$

where $x^{[0]} = 1$, $x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\cdots\left(x-\frac{n}{2}+1\right), \ (n \geq 1)$.

It is well known that the Daehee polynomials are defined by [16,17]

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{k=0}^{n} D_n(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

The Bernoulli polynomials of the second kind of order r are defined by [15]

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{k=0}^{n} b_n^{(r)}(x)\frac{t^n}{n!}. \tag{8}$$

Note that $b_n^{(r)}(x) = B_n^{(n-r+1)}(x+1), \ (n \geq 0)$. Here $B_n^{(r)}(x)$ are the ordinary Bernoulli polynomials of order r given by [8,15–18]

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{k=0}^{n} B_n^{(r)}(x)\frac{t^n}{n!}. \tag{9}$$

It is known that the Stirling numbers of the second kind are defined by [8]

$$\frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{10}$$

and the Stirling numbers of the first kind by [8]

$$\frac{1}{k!}\log^k(1+t) = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{11}$$

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by [11,12]

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{12}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1)$.

In particular, we let

$$e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{13}$$

In [1,2], Carlitz introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{14}$$

Also, he considered the degenerate Euler polynomials given by [1,2]

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

Recently, Kim-Kim considered the degenerate central factorial numbers of the second kind given by [8,13]

$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^k = \sum_{n=k}^{\infty} T_\lambda(n,k)\frac{t^n}{n!}. \tag{16}$$

Note that $\lim_{\lambda\to 0} T_\lambda(n,k) = T(n,k)$.

2. Type 2 Degenerate Bernoulli Polynomials of the Second Kind

Let $\log_\lambda t$ be the compositional inverse of $e_\lambda(t)$ in (13). Then we have

$$\log_\lambda t = \frac{1}{\lambda}\left(t^\lambda - 1\right). \tag{17}$$

Note that $\lim_{\lambda\to 0}\log_\lambda t=\log t$. Now, we define the *degenerate Daehee polynomials* by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{18}$$

Note that $\lim_{\lambda\to 0}D_{n,\lambda}(x)=D_n(x),\ (n\geq 0)$. In view of (8), we also consider the degenerate Bernoulli polynomials of the second kind of order α given by

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^{\alpha}(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{19}$$

Note that $\lim_{\lambda\to 0}b_{n,\lambda}^{(\alpha)}(x)=b_n^{(\alpha)}(x),\ (n\geq 0)$. From (19), we have

$$\left(\frac{\lambda t}{(1+t)^{\frac{\lambda}{2}}-(1+t)^{-\frac{\lambda}{2}}}\right)^{\alpha}(1+t)^{x-\frac{\lambda\alpha}{2}}=\sum_{n=0}^{\infty}b_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{20}$$

For $\alpha=r\in\mathbb{N}$, and replacing t by $e^{2t}-1$ in (20), we get

$$\begin{aligned}\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\frac{1}{m!}(e^{2t}-1)^m&=\left(\frac{\lambda t}{e^{t\lambda}-e^{-t\lambda}}\right)^r\frac{1}{t^r}(e^{2t}-1)^r e^{(2x-\lambda r)t}\\&=\sum_{k=0}^{\infty}B_k^*\left(\frac{2x}{\lambda}-r\right)\frac{\lambda^k t^k}{k!}\sum_{m=0}^{\infty}S_2(m+r,r)2^{m+r}\frac{1}{\binom{m+r}{r}}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^*\left(\frac{2x}{\lambda}-r\right)\lambda^{n-m}\frac{S_2(m+r,r)}{\binom{m+r}{r}}2^{m+r}\right)\frac{t^n}{n!}.\end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\frac{1}{m!}(e^{2t}-1)^m&=\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\sum_{n=m}^{\infty}S_2(n,m)2^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}b_{m,\lambda}^{(r)}(x)2^nS_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{22}$$

From (21) and (22), we have

$$\sum_{m=0}^{n}b_{m,\lambda}^{(r)}(x)S_2(n,m)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^*\left(\frac{2x}{\lambda}-r\right)\lambda^{n-m}\frac{S_2(m+r,r)}{\binom{m+r}{r}}2^{m+r-n}. \tag{23}$$

Now, we define the *type* 2 *degenerate Bernoulli polynomials of the second kind* by

$$\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}^*(x)\frac{t^n}{n!}. \tag{24}$$

When $x=0$, $b_{n,\lambda}^*=b_{n,\lambda}^*(0)$ are called the type 2 degenerate Bernoulli numbers of the second kind. Note that $\lim_{\lambda\to 0}b_{n,\lambda}^*(x)=b_n^*(x)$, where $b_n^*(x)$ are the type 2 Bernoulli polynomials of the second kind given by

$$\frac{(1+t)-(1+t)^{-1}}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n^*(x)\frac{t^n}{n!}.$$

From (19) and (24), we note that

$$\begin{aligned}\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}(1+t)^x &= \frac{t}{\log_\lambda(1+t)}(1+t)^x\left(1+\frac{1}{1+t}\right)\\ &= \frac{t}{\log_\lambda(1+t)}(1+t)^x + \frac{t}{\log_\lambda(1+t)}(1+t)^{x-1}\\ &= \sum_{n=0}^{\infty}\left(b_{n,\lambda}^{(1)}(x)+b_{n,\lambda}^{(1)}(x-1)\right)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$b_{n,\lambda}^{*}(x) = b_{n,\lambda}^{(1)}(x)+b_{n,\lambda}^{(1)}(x-1).$$

Moreover,

$$\sum_{m=0}^{n} b_{m,\lambda}^{(r)}(x)S_2(n,m) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{*}\left(\frac{2x}{\lambda}-r\right)\lambda^{n-m}\frac{S_2(m+r,r)}{\binom{m+r}{r}}2^{m+r-n},$$

where r is a positive integer.

Now, we observe that

$$\begin{aligned}\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}(1+t)^x &= \sum_{l=0}^{\infty} b_{l,\lambda}^{*}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_m\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}b_{l,\lambda}^{*}(x)_{n-l}\right)\frac{t^n}{n!},\end{aligned} \tag{26}$$

where $(x)_0=1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n\geq 1)$. From (24) and (26), we get

$$b_{n,\lambda}^{*}(x) = \sum_{l=0}^{n}\binom{n}{l}b_{l,\lambda}^{*}(x)_{n-l}, \quad (n\geq 0). \tag{27}$$

For $\alpha \in \mathbb{R}$, let us define the *type* 2 *degenerate Bernoulli polynomials of the second kind of order* α by

$$\left(\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}\right)^{\alpha}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^{*(\alpha)}(x)\frac{t^n}{n!} \tag{28}$$

When $x=0$, $b_{n,\lambda}^{*(\alpha)} = b_{n,\lambda}^{*(\alpha)}(0)$ are called the type 2 degenerate Bernoulli numbers of the second kind of order α.

Let $\alpha = k \in \mathbb{N}$. Then we have

$$\sum_{n=0}^{\infty} b_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!} = \left(\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}\right)^{k}(1+t)^x. \tag{29}$$

By replacing t by $e_\lambda(t)-1$ in (29), we get

$$\begin{aligned}\frac{k!}{t^k}\frac{1}{k!}\left(e_\lambda(t)-e_\lambda^{-1}(t)\right)^k e_\lambda^x(t) &= \sum_{l=0}^{\infty} b_{l,\lambda}^{*(k)}(x)\frac{1}{l!}(e_\lambda(t)-1)^l \\ &= \sum_{l=0}^{\infty} b_{l,\lambda}^{*(k)}(x)\sum_{n=l}^{\infty} S_{2,\lambda}(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} b_{l,\lambda}^{*(k)}(x)S_{2,\lambda}(n,l)\right)\frac{t^n}{n!},\end{aligned} \tag{30}$$

where $S_{2,\lambda}(n,l)$ are the degenerate Stirling numbers of the second kind given by [6]

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{31}$$

On the other hand, we also have

$$\begin{aligned}\frac{k!}{t^k}\frac{1}{k!}\left(e_\lambda(t)-e_\lambda^{-1}(t)\right)^k e_\lambda^x(t) =&\frac{k!}{t^k}\frac{1}{k!}\left(e_\lambda^2(t)-1\right)^k e_\lambda^{x-k}(t) \\ =&\frac{k!}{t^k}\frac{1}{k!}\left(e_{\frac{\lambda}{2}}(2t)-1\right)^k e_\lambda^{x-k}(t) \\ =&\sum_{m=0}^{\infty} S_{2,\frac{\lambda}{2}}(m+k,k)\frac{2^{m+k}}{\binom{m+k}{k}}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x-k)_{l,\lambda}\frac{t^l}{l!} \\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}2^{m+k}}{\binom{m+k}{k}}S_{2,\frac{\lambda}{2}}(m+k,k)(x-k)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{32}$$

Therefore, by (30) and (32), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} b_{l,\lambda}^{*(k)}(x)S_{2,\lambda}(n,l) = \sum_{l=0}^{n}\frac{\binom{n}{l}2^{l+k}}{\binom{l+k}{k}}S_{2,\frac{\lambda}{2}}(l+k,k)(x-k)_{n-l,\lambda}.$$

In particular,

$$2^{n+k}S_{2,\frac{\lambda}{2}}(n+k,k) = \binom{n+k}{k}\sum_{l=0}^{n} b_{l,\lambda}^{*(k)}(k)S_{2,\lambda}(n,l).$$

For $\alpha \in \mathbb{R}$, we recall that the type 2 degenerate Bernoulli polynomials of order α are defined by [5,13]

$$\left(\frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{*(\alpha)}(x)\frac{t^n}{n!}. \tag{33}$$

For $k \in \mathbb{N}$, let us take $\alpha = -k$ and replace t by $\log_\lambda(1+t)$ in (33). Then we have

$$\begin{aligned}\left(\frac{(1+t)-(1+t)^{-1}}{\log_\lambda(1+t)}\right)^k(1+t)^x &= \sum_{l=0}^{\infty}\beta_{l,\lambda}^{*(-k)}(x)\frac{1}{l\,!}\left(\log_\lambda(1+t)\right)^l \\ &= \sum_{l=0}^{\infty}\beta_{l,\lambda}^{*(-k)}(x)\sum_{n=l}^{\infty}S_{1,\lambda}(n.l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\beta_{l,\lambda}^{*(-k)}S_{1,\lambda}(n.l)\right)\frac{t^n}{n!},\end{aligned} \tag{34}$$

where $S_{1,\lambda}(n,l)$ are the degenerate Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log_{\lambda}(1+t))^{k} = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^{n}}{n!}. \tag{35}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(n,l) = S_{1}(n,l)$. Therefore, by (26) and (34), we obtain the following theorem.

Theorem 3. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$b_{n,\lambda}^{*(k)}(x) = \sum_{l=0}^{n} \beta_{l,\lambda}^{*(-k)}(x) S_{1,\lambda}(n,l).$$

We observe that

$$\begin{aligned}
\frac{1}{k!}t^{k} &= \frac{1}{k!}\left((1+t)^{\frac{1}{2}} - (1+t)^{-\frac{1}{2}}\right)^{k}(1+t)^{\frac{k}{2}} \\
&= \frac{1}{k!}\left(e_{\lambda}^{\frac{1}{2}}(\log_{\lambda}(1+t)) - e_{\lambda}^{-\frac{1}{2}}(\log_{\lambda}(1+t))\right)^{k}(1+t)^{\frac{k}{2}} \\
&= \sum_{l=k}^{\infty} T_{\lambda}(l,k)\frac{1}{l!}(\log_{\lambda}(1+t))^{l} \sum_{r=0}^{\infty}\left(\frac{k}{2}\right)_{r}\frac{t^{r}}{r!} \\
&= \sum_{l=k}^{\infty} T_{\lambda}(l,k) \sum_{m=l}^{\infty} S_{1,\lambda}(m,l)\frac{t^{m}}{m!} \sum_{r=0}^{\infty}\left(\frac{k}{2}\right)_{r}\frac{t^{r}}{r!} \\
&= \sum_{m=k}^{\infty}\sum_{l=k}^{m} T_{\lambda}(l,k) S_{1,\lambda}(m,l)\frac{t^{m}}{m!} \sum_{r=0}^{\infty}\left(\frac{k}{2}\right)_{r}\frac{t^{r}}{r!} \\
&= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\sum_{l=k}^{m} T_{\lambda}(l,k) S_{1,\lambda}(m,l)\binom{n}{m}\left(\frac{k}{2}\right)_{n-m}\right)\frac{t^{n}}{n!}.
\end{aligned} \tag{36}$$

On the other hand,

$$\begin{aligned}
\frac{1}{k!}t^{k} &= \left(\frac{t}{\log_{\lambda}(1+t)}\right)^{k}\frac{1}{k!}(\log_{\lambda}(1+t))^{k} \\
&= \sum_{l=0}^{\infty} b_{l,\lambda}^{(k)}\frac{t^{l}}{l!} \sum_{m=k}^{\infty} S_{1,\lambda}(m,k)\frac{t^{m}}{m!} \\
&= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n} S_{1,\lambda}(m,k) b_{n-m,\lambda}^{(k)}\binom{n}{m}\right)\frac{t^{n}}{n!}.
\end{aligned} \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 4. *For $n,k \geq 0$, we have*

$$\sum_{m=k}^{n}\sum_{l=k}^{m} T_{\lambda}(l,k) S_{1,\lambda}(m,l)\binom{n}{m}\left(\frac{k}{2}\right)_{n-m} = \sum_{m=k}^{n} S_{1,\lambda}(m,k) b_{n-m,\lambda}^{(k)}\binom{n}{m}.$$

3. Conclusions

In this paper, we introduced the type 2 degenerate Bernoulli polynomials of the second kind and their higher-order analogues, and studied some identities and expressions for these polynomials. Specifically, we obtained a relation between the type 2 degenerate Bernoulli polynomials of the second and the degenerate Bernoulli polynomials of the second, an identity involving higher-order analogues of those polynomials and the degenerate Stirling numbers of second kind, and an expression of higher-order analogues of those polynomials in terms of the higher-order type 2 degenerate Bernoulli polynomials and the degenerate Stirling numbers of the first kind.

In addition, we obtained an identity involving the higher-order degenerate Bernoulli polynomials of the second kind, the type 2 Bernoulli polynomials and Stirling numbers of the second kind, and an identity involving the degenerate central factorial numbers of the second kind, the degenerate Stirling numbers of the first kind and the higher-order degenerate Bernoulli polynomials of the second kind.

Next, we would like to mention three possible applications of our results. The first one is their applications to identities of symmetry. For instance, in [7] by using the p-adic fermionic integrals it was possible for us to find many symmetric identities in three variables related to degenerate Euler polynomials and alternating generalized falling factorial sums.

The second one is their applications to differential equations. Indeed, in [9] we derived an infinite family of nonlinear differential equations having the generating function of the degenerate Changhee numbers of the second kind as a solution. As a result, from those differential equations we obtained an interesting identity involving the degenerate Changhee and higher-order degenerate Changhee numbers of the second kind.

The third one is their applications to probability. For example, in [19,20] we showed that both the degenerate λ-Stirling polynomials of the second and the r-truncated degenerate λ-Stirling polynomials of the second kind appear in certain expressions of the probability distributions of appropriate random variables.

These possible applications of our results require a considerable amount of work and they should appear as separate papers. We have witnessed in recent years that studying various degenerate versions of some special polynomials and numbers are very fruitful and promising [21]. It is our plan to continue to do this line of research, as one of our near future projects.

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