Inertial Extra-Gradient Method for Solving a Family of Strongly Pseudomonotone Equilibrium Problems in Real Hilbert Spaces with Application in Variational Inequality Problem

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Abstract: In this paper, we propose a new method, which is set up by incorporating an inertial step with the extragradient method for solving a strongly pseudomonotone equilibrium problems. This method had to comply with a strongly pseudomonotone property and a certain Lipschitz-type condition of a bifunction. A strong convergence result is provided under some mild conditions, and an iterative sequence is accomplished without previous knowledge of the Lipschitz-type constants of a cost bifunction. A sufficient explanation is that the method operates with a slow-moving stepsize sequence that converges to zero and non-summable. For numerical explanations, we analyze a well-known equilibrium model to support our well-established convergence result, and we can see that the proposed method seems to have a significant consistent improvement over the performance of the existing methods.

1. Introduction

Let C to be a nonempty closed, convex subset of $\mathbb{E}$ and $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is a bifunction such that $f(u, u) = 0$ for all $u \in C$. The equilibrium problem [1] for the bifunction f on C is defined as follows:

$$\text{Find} \quad u^* \in C \quad \text{such that} \quad f(u^*, v) \geq 0, \text{ for all } v \in C. \tag{1}$$

The equilibrium problem (shortly, *EP*) was originally introduced in the unifying nature by Blum and Oettli [1] in 1994, and provides a comprehensive study on their theoretical properties. This unique formulation of a problem has an absolutely sensational way to deal with a wide range of topics that have emerged from the social sciences, economics, finance, restoration of image, ecology, transport, networking, elasticity and optimization problems (for more details see, [2–4]). The equilibrium problem covers several mathematical problems as a special case, namely minimization problems, the fixed point problems, variational inequality problems (shortly, *VIP*), Nash equilibrium of non-cooperation games, complementarity problems, problem of vector minimization and saddle point problems (see, e.g., [1,5–7]). On the other hand, iterative methods are basic and powerful tools for studying the numerical solution of an equilibrium problem. In this direction, two well-established approaches are used, i.e., the proximal point method [8] and the auxiliary problem principle [9]. The strategy of the proximal point method was originally developed by Martinet [10] for the problems of a monotone variational inequality, and later Rockafellar [11] developed this idea for monotone operators. Moudafi [8] proposed the proximal point method for monotone equilibrium problems. Furthermore, Konnov [12] also provides a different variant of the proximal point method with weaker assumptions in the case of equilibrium problems. Several numerical methods based on these techniques have been developed to solve different classes of equilibrium problems in finite and infinite-dimensional abstract spaces (for more details see, [12–29]). More specifically, Hieu et al. in [30] developed an iterative sequence sequence $\{u_n\}$ recursively as

$$
\begin{cases}
u_0 \in C, \\
v_n = \underset{y \in C}{\arg\min}\{\xi_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\
u_{n+1} = \underset{y \in C}{\arg\min}\{\xi_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2\},
\end{cases}
$$

where $\{\xi_n\}$ is a sequence of positive real numbers satisfying the following conditions:

$$
T_1: \quad \lim_{n \to \infty} \xi_n = 0 \qquad \text{and} \qquad T_2: \quad \sum_{n=0}^{\infty} \xi_n = +\infty. \tag{2}
$$

In addition, the inertial-like methods depend on the approach of the heavy-ball methods of the second-order time dynamic system. Polyak began by considering inertial extrapolation as a speed-up method to solve smooth convex minimization problems. Inertial-like methods are two-step iterative schemes and the next iteration is achieved by making use of the previous two iterations (see [31,32]). An inertial extrapolation term is required to boost the iterative sequence in order to achieve the desired solution. These inertial methods are basically used to accelerate the iterative sequence towards the required solution. Numerical reviews suggest that inertial effects often improve the performance of the algorithm in terms of the number of iterations and time of execution in this context. These two impressive advantages enhance the researcher's interest in developing new inertial methods. There are many methods are already have been established for the different classes of variational inequality problem (for more details see [33–37]).

In this article, we focus on the second direction which consists of projection methods that are well recognized and practically easy to carry out based on their convenient mathematical computation. By relying on the research work of Hieu et al. [30] and Vinh et al. [38], we introduce an inertial extragradient method for solving a specific class of equilibrium problems, where *f* can be a strongly pseudomonotone bifunction. Our method is working without any knowledge of the Lipschitz-type and strongly pseudomonotone constants of a bifunction. The advantage of our method is based on the use of a stepsize sequence that gently converges to zero and non-summable. Due to this aspect and the strong pseudomonotonicity of the bifunction, the strong convergence of our method has been achieved. Nonetheless, we do not need to know such constants beforehand i.e., the input parameters

of the method should not be such constants. In the end, the numerical experiments indicate that the proposed method seems to be more effective than the family of existing ones [30,39,40].

The rest of this paper organized subsequently: In Section 2 we provide some preliminaries and basic results that will be used throughout the paper. Section 3 includes our proposed method and the corresponding convergence result. Section 4 contains some application of our results in the variational inequality problems. Section 5 sets out the numerical experiments to explain the algorithmic efficiency of our proposed algorithm.

2. Background

Now, we provide important lemmas, definitions and other concepts that are useful throughout the convergence analysis. We continue to make use of C as a closed, convex subset of the Hilbert space $\mathbb{E}$. By $\langle .,. \rangle$ and $\|.\|$ we denote the inner product and norm on the Hilbert space respectively, etc. Let $G : \mathbb{E} \to \mathbb{E}$ is a well-defined operator and $VI(G,C)$ is the solution set of a variational inequality problem corresponding operator G over the set C. Moreover, $EP(f,C)$ stands for the solution set of an equilibrium problem upon the set C and u^* is any arbitrary element of $EP(f,C)$ or $VI(G,C)$.

In addition, Let $h : C \to \mathbb{R}$ is a convex function and *subdifferential of h* at $u \in C$ is defined by:

$$\partial h(u) = \{z \in \mathbb{E} : h(v) - h(u) \geq \langle z, v - u \rangle, \ \forall \, v \in C\}.$$

The *normal cone of C* at $u \in C$ is given as

$$N_C(u) = \{z \in \mathbb{E} : \langle z, v - u \rangle \leq 0, \ \forall \, v \in C\}.$$

Definition 1 ([41]). *The metric projection $P_C(u)$ of u onto a closed, convex subset C of $\mathbb{E}$ is define as follows:*

$$P_C(u) = \arg\min_{v \in C}\{\|v - u\|\}.$$

Next, we have various notions of the bifunction monotonicity (see [1,42] for more details).

Definition 2. *The bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on C for $\gamma > 0$ is said to be:*

(i). *Strongly monotone if $f(u,v) + f(v,u) \leq -\gamma\|u - v\|^2$, $\forall \, u,v \in C$;*
(ii). *Monotone if $f(u,v) + f(v,u) \leq 0$, $\forall \, u,v \in C$;*
(iii). *Strongly pseudomonotone if $f(u,v) \geq 0 \implies f(v,u) \leq -\gamma\|u - v\|^2$, $\forall \, u,v \in C$;*
(iv). *Pseudomonotone if $f(u,v) \geq 0 \implies f(v,u) \leq 0$, $\forall \, u,v \in C$;*
(v). *Satisfying the Lipschitz-type condition on C if there are two positive real numbers c_1, c_2 such that*

$$f(u,w) \leq f(u,v) + f(v,w) + c_1\|u - v\|^2 + c_2\|v - w\|^2, \ \forall \, u,v,w \in C.$$

Remark 1. *We obtain the following results from the above definitions.*

$$strongly \ monotone \implies monotone \implies pseudomonotone$$

$$strongly \ monotone \implies strongly \ pseudomonotone \implies pseudomonotone$$

This section concludes with a few specific lemmas that are useful in studying the convergence analysis of our proposed method.

Lemma 1 ([43]). *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{E}$ and $h : C \to \mathbb{R}$ be a convex, subdifferentiable and lower semicontinuous function on C. Moreover, $u \in C$ is a minimizer of a function h if and only if $0 \in \partial h(u) + N_C(u)$, where $\partial h(u)$ and $N_C(u)$ denotes the subdifferential of h at u and the normal cone of C at u, respectively.*

Lemma 2 ([44]). *For every $a, b \in \mathbb{E}$ and $\mu \in \mathbb{R}$ then the subsequent item is true:*

$$\|\mu a + (1 - \mu)b\|^2 = \mu\|a\|^2 + (1 - \mu)\|b\|^2 - \mu(1 - \mu)\|a - b\|^2.$$

Lemma 3 ([45]). *Suppose $\{a_n\}$ and $\{t_n\}$ are two sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq a_n + t_n$ for all $n \in \mathbb{N}$. If $\sum t_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.*

Lemma 4 ([46]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$, then $\liminf_{n \to \infty} b_n = 0$.*

3. Convergence Analysis for an Algorithm

We develop an algorithmic procedure that consists of two strong convex minimization problems with such an inertial term that is used to improve the convergence speed of the iterative sequence, so that it is classified as an inertial extragradient method for strongly pseudomonotone equilibrium problems. We have the following hypothesis on a bifunction that are compulsory to achieve the strong convergence of the iterative sequence generated by Algorithm 1.

Assumption 1. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ such that*

f_1. $f(u, u) = 0$, for all $u \in C$ and f is strongly pseudomonotone on C.

f_2. f satisfy the Lipschitz-type condition through two positive constants c_1 and c_2.

f_3. $f(u, .)$ is convex and subdifferentiable on C for each fixed $u \in C$.

Algorithm 1 (Inertial extragradient algorithm for strongly pseudomonotone equilibrium problems).

Initialization: Choose $u_{-1}, u_0 \in C$, $\theta \in [0, 1)$ with a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty \tag{3}$$

holds. In addition, let $\{\xi_n\}$ be the sequence of positive real numbers that satisfy the following conditions:

$$T_1 : \lim_{n \to \infty} \xi_n = 0 \quad \text{and} \quad T_2 : \sum_{n=0}^{\infty} \xi_n = +\infty. \tag{4}$$

Iterative steps: Choose ϑ_n such that $0 \leq \vartheta_n \leq \beta_n$, where

$$\beta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{5}$$

Step 1: Compute

$$v_n = \arg\min_{y \in C}\{\xi_n f(w_n, y) + \frac{1}{2}\|w_n - y\|^2\},$$

where $w_n = u_n + \vartheta_n(u_n - u_{n-1})$. If $w_n = v_n$; STOP. Otherwise,

Step 2: Compute

$$u_{n+1} = \arg\min_{y \in C}\{\xi_n f(v_n, y) + \frac{1}{2}\|w_n - y\|^2\}.$$

Set $n := n + 1$ and go back to **Iterative steps**.

Remark 2.

(i). *Notice that if $\theta = 0$, in the above method then it is equivalent to the default extragradient method in, e.g., [30].*

(ii). *Evidently, from the expression (3) and (5) we have*

$$\sum_{n=1}^{\infty} \vartheta_n \|u_n - u_{n-1}\| \leq \sum_{n=1}^{\infty} \beta_n \|u_n - u_{n-1}\| < \infty, \tag{6}$$

which implies that

$$\lim_{n \to \infty} \beta_n \|u_n - u_{n-1}\| = 0. \tag{7}$$

Next, we are proving the validity of the stopping criterion for the Algorithm 1.

Lemma 5. *If $v_n = w_n$ in Algorithm 1 then w_n is the solution of a problem (1) over C.*

Proof. By definition of v_n with Lemma 1, we can write as

$$0 \in \partial_2 \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}(v_n) + N_C(v_n).$$

Thus, there is a $\eta \in \partial f(w_n, v_n)$ and $\overline{\eta} \in N_C(v_n)$ such that

$$\xi_n \eta + v_n - w_n + \overline{\eta} = 0.$$

Thus, we have

$$\xi_n \langle \eta, y - v_n \rangle + \langle \overline{\eta}, y - v_n \rangle = 0, \ \forall \, y \in C.$$

Since $\overline{\eta} \in N_C(v_n)$ then $\langle \overline{\eta}, y - v_n \rangle \leq 0$, and with above expression implies that

$$\xi_n \langle \eta, y - v_n \rangle \geq 0, \ \forall \, y \in C. \tag{8}$$

Furthermore, by $\eta \in f(w_n, v_n)$ and the subdifferential definition, we have

$$f(w_n, y) - f(w_n, v_n) \geq \langle \eta, y - v_n \rangle, \ \forall \, y \in C. \tag{9}$$

From the expression (8) and (9) with $\xi_n \in (0, +\infty)$ implies that

$$f(w_n, y) - f(w_n, v_n) \geq 0, \ \forall \, y \in C. \tag{10}$$

By $v_n = w_n$ and under the assumption f_1 given that $f(w_n, y) \geq 0$, for all $y \in C$. □

Lemma 6. *We have the following important inequality from the Algorithm 1.*

$$\xi_n f(v_n, y) - \xi_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, y - u_{n+1} \rangle, \ \forall \, y \in C.$$

Proof. By definition of u_{n+1}, we have

$$0 \in \partial_2 \left\{ \xi_n f(v_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}(u_{n+1}) + N_C(u_{n+1}).$$

Thus, for $\eta \in \partial f(v_n, u_{n+1})$ there exists an $\overline{\eta} \in N_C(u_{n+1})$ such that

$$\xi_n \eta + u_{n+1} - w_n + \overline{\eta} = 0.$$

The above expression can be written as

$$\langle w_n - u_{n+1}, y - u_{n+1} \rangle = \xi_n \langle \eta, y - u_{n+1} \rangle + \langle \overline{\eta}, y - u_{n+1} \rangle, \ \forall \, y \in C.$$

Since $\overline{\eta} \in N_C(u_{n+1})$ then $\langle \overline{\eta}, y - u_{n+1} \rangle \leq 0$ for all $y \in C$.
Thus, we obtain

$$\langle w_n - u_{n+1}, y - u_{n+1} \rangle \leq \xi_n \langle \eta, y - u_{n+1} \rangle, \ \forall \, y \in C. \tag{11}$$

By $\eta \in \partial f(v_n, u_{n+1})$ and by subdifferential definition, we obtain

$$f(v_n, y) - f(v_n, u_{n+1}) \geq \langle \eta, y - u_{n+1} \rangle, \ \forall \, y \in C. \tag{12}$$

Combining the expression (11) and (12) we obtain

$$\xi_n f(v_n, y) - \xi_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, y - u_{n+1} \rangle, \ \forall \, y \in C. \tag{13}$$

$\square$

Lemma 7. *We also have the following important inequality from the Algorithm 1.*

$$\xi_n f(w_n, y) - \xi_n f(w_n, v_n) \geq \langle w_n - v_n, y - v_n \rangle, \ \forall \, y \in C.$$

Proof. We may obtain the proof by following the same step as in the proof of Lemma 6. $\square$

Next, we give a crucial inequality which is useful to prove the boundedness of the iterative sequence generated by Algorithm 1.

Lemma 8. *Suppose that the assumptions f_1-f_4 as in Assumption 1 hold and the $EP(f, C) \neq \emptyset$. Thus, for each $u^* \in EP(f, C)$, we can obtain*

$$\|u_{n+1} - u^*\|^2 \leq \|w_n - u^*\|^2 - (1 - 2c_1 \xi_n)\|w_n - v_n\|^2 - (1 - 2c_2 \xi_n)\|v_n - u_{n+1}\|^2$$
$$- 2\gamma \xi_n \|v_n - u^*\|^2.$$

Proof. It is follow from the Lemma 7 and substituting $y = u_{n+1}$, we obtain

$$\xi_n f(w_n, u_{n+1}) - \xi_n f(w_n, v_n) \geq \langle w_n - v_n, u_{n+1} - v_n \rangle. \tag{14}$$

Next, substituting $y = u^*$ in Lemma 6, we have

$$\xi_n f(v_n, u^*) - \xi_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, u^* - u_{n+1} \rangle. \tag{15}$$

Since $f(u^*, v_n) \geq 0$ then from the assumption (f_1), we have $f(v_n, u^*) \leq -\gamma \|v_n - u^*\|^2$ such that

$$\langle w_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \xi_n f(v_n, u_{n+1}) + \gamma \xi_n \|v_n - u^*\|^2. \tag{16}$$

By using the Lipschitz-type continuity of a bifunction f, we have

$$f(w_n, u_{n+1}) \leq f(w_n, v_n) + f(v_n, u_{n+1}) + c_1 \|w_n - v_n\|^2 + c_2 \|v_n - u_{n+1}\|^2. \tag{17}$$

From the expression (16) and (17), we have

$$\langle w_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \xi_n \{ f(w_n, u_{n+1}) - f(w_n, v_n) \}$$
$$- c_1 \xi_n \|w_n - v_n\|^2 - c_2 \xi_n \|v_n - u_{n+1}\|^2 + \gamma \xi_n \|v_n - u^*\|^2. \tag{18}$$

By the expression (14) and (18), we get

$$
\begin{aligned}
\langle w_n - u_{n+1}, u_{n+1} - u^* \rangle \geq \ & \langle w_n - v_n, u_{n+1} - v_n \rangle \\
& - c_1 \xi_n \| w_n - v_n \|^2 - c_2 \xi_n \| v_n - u_{n+1} \|^2 + \gamma \xi_n \| v_n - u^* \|^2.
\end{aligned}
\tag{19}
$$

Furthermore, we have the following facts:

$$
2 \langle v_n - w_n, v_n - u_{n+1} \rangle = \| w_n - v_n \|^2 + \| u_{n+1} - v_n \|^2 - \| w_n - u_{n+1} \|^2.
$$

$$
2 \langle w_n - u_{n+1}, u_{n+1} - u^* \rangle = \| w_n - u^* \|^2 - \| u_{n+1} - w_n \|^2 - \| u_{n+1} - u^* \|^2.
$$

From above two facts and the expression (19), we obtain

$$
\begin{aligned}
\| u_{n+1} - u^* \|^2 \leq \ & \| w_n - u^* \|^2 - (1 - 2c_1 \xi_n) \| w_n - v_n \|^2 - (1 - 2c_2 \xi_n) \| v_n - u_{n+1} \|^2 \\
& - 2 \gamma \xi_n \| v_n - u^* \|^2.
\end{aligned}
$$

$\square$

Theorem 1. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying the Assumptions 1. Then, for some $u^* \in EP(f, C) \neq \varnothing$, the sequence $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ set up by Algorithm 1 strongly converges to $u^* \in EP(f, C)$.*

Proof. Since $\xi_n \to 0$ then there exist an $n_0 \in \mathbb{N}$ such that

$$
\xi_n \leq \min \left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\} \text{ for all } n \geq n_0.
\tag{20}
$$

By using the above condition on Lemma 8, we have

$$
\| u_{n+1} - u^* \|^2 \leq \| w_n - u^* \|^2, \ \forall \, n \geq n_0.
\tag{21}
$$

Furthermore, the above expression for all $n \geq n_0$ can be written as

$$
\begin{aligned}
\| u_{n+1} - u^* \| & \leq \| u_n + \vartheta_n (u_n - u_{n-1}) - u^* \| \\
& \leq \| u_n - u^* \| + \vartheta_n \| u_n - u_{n-1} \|.
\end{aligned}
\tag{22}
$$

By Lemma 3 with the expression (6) and (22) implies that

$$
\lim_{n \to \infty} \| u_n - u^* \| = l, \text{ for some finite } l \geq 0.
\tag{23}
$$

By the definition of w_n in Algorithm 1, we have

$$
\begin{aligned}
\| w_n - u^* \|^2 & = \| u_n + \vartheta_n (u_n - u_{n-1}) - u^* \|^2 \\
& = \| (1 + \vartheta_n)(u_n - u^*) - \vartheta_n (u_{n-1} - u^*) \|^2 \\
& = (1 + \vartheta_n) \| u_n - u^* \|^2 - \vartheta_n \| u_{n-1} - u^* \|^2 + \vartheta_n (1 + \vartheta_n) \| u_n - u_{n-1} \|^2 \\
& \leq (1 + \vartheta_n) \| u_n - u^* \|^2 - \vartheta_n \| u_{n-1} - u^* \|^2 + 2 \vartheta_n \| u_n - u_{n-1} \|^2.
\end{aligned}
\tag{24}
$$

The above expression with (23) and (7) implies that

$$
\lim_{n \to \infty} \| w_n - u^* \| = l.
\tag{25}
$$

From Lemma 8 and the expression (24), we have

$$
\begin{aligned}
\|u_{n+1} - u^*\|^2 \\
&\leq (1 + \vartheta_n)\|u_n - u^*\|^2 - \vartheta_n\|u_{n-1} - u^*\|^2 + 2\vartheta_n\|u_n - u_{n-1}\|^2 \\
&\quad - (1 - 2c_1\xi_n)\|w_n - v_n\|^2 - (1 - 2c_2\xi_n)\|v_n - u_{n+1}\|^2 - 2\gamma\xi_n\|v_n - u^*\|^2,
\end{aligned} \tag{26}
$$

which further implies that

$$
\begin{aligned}
(1 - 2c_1\xi_n)\|w_n - v_n\|^2 + (1 - 2c_2\xi_n)\|v_n - u_{n+1}\|^2 \\
&\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + \vartheta_n\left(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2\right) + 2\vartheta_n\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{27}
$$

By taking the limit as $n \to \infty$ in the expression (27), we get

$$
\lim_{n\to\infty} \|w_n - v_n\| = \lim_{n\to\infty} \|v_n - u_{n+1}\| = 0. \tag{28}
$$

Thus, the expression (25) and (28) gives that

$$
\lim_{n\to\infty} \|v_n - u^*\| = l. \tag{29}
$$

From the expression (23), (25) and (29) implies that the sequences $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$ are bounded, and for each $u^* \in EP(f, C)$, the $\lim_{n\to\infty} \|u_n - u^*\|^2$, $\lim_{n\to\infty} \|v_n - u^*\|^2$, $\lim_{n\to\infty} \|w_n - u^*\|^2$ exists. Next, we are going to prove that the sequence $\{u_n\}$ strongly converges to u^*. It follows from the expression (26) for each $n \geq n_0$ such that

$$
\begin{aligned}
2\gamma\xi_n\|v_n - u^*\|^2 \\
&\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + \vartheta_n\left(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2\right) + 2\vartheta_n\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{30}
$$

For some $k > n_0$ using in the expression (30) gives that

$$
\begin{aligned}
\sum_{n=n_0}^{k} 2\gamma\xi_n\|v_n - u^*\|^2 \\
&\leq \left(\|u_{n_0} - u^*\|^2 - \|u_{k+1} - u^*\|^2\right) + 2\theta\sum_{n=n_0}^{k}\|u_n - u_{n-1}\|^2 \\
&\quad + \theta\left(\|u_k - u^*\|^2 - \|u_{n_0-1} - u^*\|^2\right) \\
&\leq \|u_{n_0} - u^*\|^2 + \theta\|u_k - u^*\|^2 + 2\theta\sum_{n=n_0}^{k}\|u_n - u_{n-1}\|^2 \\
&\leq M,
\end{aligned} \tag{31}
$$

for some $M \geq 0$ and letting $k \to \infty$ leads to

$$
\sum_{n=1}^{\infty} 2\gamma\xi_n\|v_n - u^*\|^2 < +\infty. \tag{32}
$$

It follows from Lemma 4 and the expression (32) such that

$$
\liminf \|v_n - u^*\| = 0. \tag{33}
$$

Thus, the expression (29) and (33) gives that

$$
\lim_{n\to\infty} \|v_n - u^*\| = 0. \tag{34}
$$

From the expression (7), (28) and using the Cauchy inequality, we have

$$0 \leq \|u_n - v_n\| = \|u_n - u_{n+1}\| + \|u_{n+1} - v_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{35}$$

Finally, we get

$$\lim_{n\to\infty} \|u_n - u^*\| = \lim_{n\to\infty} \|w_n - u^*\| = 0. \tag{36}$$

This completes the proof. □

If we take $\theta = 0$ in the Algorithm 1, we get the result that appeared in the Hieu et al. [30].

Corollary 1. *Let* $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ *be a bifunction satisfying the assumptions* (f_1-f_3). *Thus, for some* $u^* \in EP(f,K) \neq \emptyset$, *the sequence* $\{u_n\}$ *and* $\{v_n\}$ *are generated as follows:*

i.　*Let* $u_0 \in C$ *and compute*

$$\begin{cases} v_n = \underset{y\in C}{\arg\min}\{\xi_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ u_{n+1} = \underset{y\in C}{\arg\min}\{\xi_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \end{cases}$$

where $\{\xi_n\}$ *be the sequence of positive real numbers satisfy the following conditions:*

$$T_1 : \quad \lim_{n\to\infty} \xi_n = 0 \qquad and \qquad T_2 : \quad \sum_{n=0}^{\infty} \xi_n = +\infty. \tag{37}$$

The sequence $\{u_n\}$ *and* $\{v_n\}$ *strongly converges to the solution* $u^* \in EP(f,C)$.

4. Application to Variational Inequality Problems

Now we discuss the application of our results to solve variational inequality problems involving strongly pseudomonotone with Lipschitz continuous operator. An operator $G : \mathbb{E} \to \mathbb{E}$ is called to be

- *strongly pseudomonotone* upon C for $\gamma > 0$ if

$$\langle G(u), v - u \rangle \geq 0 \implies \langle G(v), u - v \rangle \leq -\gamma\|u - v\|^2, \ \forall\, u, v \in C;$$

- *L-Lipschitz continuous* upon C if $\|G(u) - G(v)\| \leq L\|u - v\|, \ \forall\, u, v \in C$.

The problem of variational inequality is to

$$\text{find} \quad u^* \in C \quad \text{such that} \quad \langle G(u^*), v - u^* \rangle \geq 0, \ \forall\, v \in C.$$

Note: Suppose that the bifunction $f(u, v) := \langle G(u), v - u \rangle$ for all $u, v \in C$. Thus, the equilibrium problem convert into problem of variational inequality with $L = \frac{1}{2}c_1 = \frac{1}{2}c_2$. It follows the definition of v_n in the Algorithm 1 and the above definition of bifunction f such that

$$\begin{aligned} v_n &= \underset{y\in C}{\arg\min}\{\xi_n f(w_n, y) + \frac{1}{2}\|w_n - y\|^2\} \\ &= \underset{y\in C}{\arg\min}\{\xi_n \langle G(w_n), y - w_n \rangle + \frac{1}{2}\|w_n - y\|^2\} \\ &= \underset{y\in C}{\arg\min}\left\{\frac{1}{2}\|y - (w_n - \xi_n G(w_n))\|^2\right\} - \frac{\xi_n^2}{2}\|G(w_n)\|^2 \\ &= P_C(w_n - \xi_n G(w_n)), \end{aligned} \tag{38}$$

and likewise u_{n+1} in Algorithm 1 can reduce to

$$u_{n+1} = P_C(w_n - \xi_n G(v_n)).$$

Assumption 2. *We assume that G satisfying the following assumptions:*

G_1.   *G is strongly pseudomonotone on C and $VI(G, C) \neq \emptyset$;*
G_2.   *G is L-Lipschitz continuous upon C for some constant $L > 0$.*

Thus, the Algorithm 1 is reduced to the following algorithm to solve a strongly pseudomonotone variational inequality problem.

Corollary 2. *Assume that $G : C \to \mathbb{E}$ satisfies (G_1-G_2) as in Assumption 2. Let $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ be the sequences generated as follows:*

i.   *Choose $u_{-1}, u_0 \in C$, $\theta \in [0, 1)$ and a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that*

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty, \tag{39}$$

holds. In addition, let $\{\xi_n\}$ be the sequence of positive real numbers which meets the following criteria:

$$T_1 : \quad \lim_{n \to \infty} \xi_n = 0 \quad and \quad T_2 : \quad \sum_{n=0}^{\infty} \xi_n = +\infty. \tag{40}$$

ii.   *Choose ϑ_n such that $0 \leq \vartheta_n \leq \beta_n$ where*

$$\beta_n = \begin{cases} \min \left\{ \theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|} \right\} & if \quad u_n \neq u_{n-1}, \\ \theta & otherwise. \end{cases} \tag{41}$$

iii.   *Compute*

$$\begin{cases} w_n = u_n + \vartheta_n(u_n - u_{n-1}), \\ v_n = P_C(w_n - \xi_n G(w_n)), \\ u_{n+1} = P_C(w_n - \xi_n G(v_n)). \end{cases}$$

Thus, the sequence $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ strongly converges to $u^* \in VI(G, C)$.

By using $\theta = 0$ in the Corollary 2, we get the following results.

Corollary 3. *Assume that $G : C \to \mathbb{E}$ satisfies (G_1-G_2) as in Assumption 2. Let $\{u_n\}$ and $\{v_n\}$ be the sequences generated as follows:*

i.   *Choose $u_0 \in C$ and compute*

$$\begin{cases} v_n = P_C(u_n - \xi_n G(u_n)), \\ u_{n+1} = P_C(u_n - \xi_n G(v_n)), \end{cases}$$

where $\{\xi_n\}$ be the sequence of positive real numbers satisfy the following conditions:

$$T_1 : \quad \lim_{n \to \infty} \xi_n = 0 \quad and \quad T_2 : \quad \sum_{n=0}^{\infty} \xi_n = +\infty. \tag{42}$$

Thus, the sequence $\{u_n\}$ and $\{v_n\}$ strongly converges to $u^* \in VI(G, C)$.

5. Computational Experiment

We present some numerical results to explain the efficiency of our proposed methods. The MATLAB codes run in MATLAB version 9.5 (R2018b) on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB. In all of these examples, we use $u_{-1} = u_0 = v_0 = (1, 1, \cdots, 1, 1)^T$, and x-axis points out to the number of iterations or the time elapsed (in seconds), whereas y-axes show for the value of D_n. For each method, the corresponding stopping criterion is used, which helps the iterative sequence to converge the element of a solution set. Moreover, we use the following values for the error terms and some other terms (n: Dimension of a Hilbert space; N: Total number of samples; iter.: Average number of iterations; time: Average execution time).

(i). For Hieu et al. [30] (shortly, Algo1), we use

$$D_n = \|u_n - v_n\|^2.$$

(ii). For Hieu et al. [39] (shortly, Algo2), we use

$$D_n = \max \left\{ \|u_{n+1} - v_n\|^2, \|u_{n+1} - u_n\|^2 \right\}.$$

(iii). For Hieu et al. [40] (shortly, Algo3), we use $\theta_n = 0.50$ and

$$D_n = \max \left\{ \|u_{n+1} - v_n\|^2, \|u_{n+1} - w_n\|^2 \right\}.$$

(iv). For Algorithm 1 (shortly, Algo4) we use $\theta_n = 0.50$, $\epsilon_n = \frac{1}{n^2}$ and

$$D_n = \|w_n - v_n\|^2.$$

5.1. Example 1

Assume that there will be n firms which produces the same product. Let u sets for a vector in which each entry u_i denotes the amount of the product produce by a firm i. Now choose the cost P as a decreasing affine function that depends upon on the value of $S = \sum_{i=1}^m u_i$, i.e., $P_i(S) = \phi_i - \psi_i S$, where $\phi_i > 0$, $\psi_i > 0$. The profit function for each firm i is described by $F_i(u) = P_i(S)u_i - t_i(u_i)$, where $t_i(u_i)$ is the tax value and cost for generating u_i. Assume that $C_i = [u_i^{\min}, u_i^{\max}]$ is the set of actions corresponding to each firm i, and the strategy for the whole model take the form as $C := C_1 \times C_2 \times \cdots \times C_n$. In fact, each firm tries to reach its peak revenue by choosing the comparable stage of production on the hypothesis that other firms production is the input parameter. The technique generally employed to handle this type of model concentrates mainly on the well-known Nash equilibrium theory. We would like to remind that point $u^* \in C = C_1 \times C_2 \times \cdots \times C_n$ is the solution of equilibrium the model if

$$F_i(u^*) \geq F_i(u^*[u_i]), \ \forall \ u_i \in C_i \text{ for all } i = 1, 2, \cdots, n,$$

with the vector $u^*[u_i]$ represent the vector get from u^* by taking u_i^* with u_i. Finally, we take $f(u, v) := \varphi(u, v) - \varphi(u, u)$ with $\varphi(u, v) := -\sum_{i=1}^n F_i(u[v_i])$ and the problem for finding the Nash equilibrium point for the model may be taken as:

$$\text{Find} \quad u^* \in C : f(u^*, v) \geq 0, \ \forall \ v \in C.$$

In addition, we assume that both the tax and the fee for the production of the unit are increasing as the amount of productivity increases. It follows from [19,22], the bifunction f be taken as

$$f(u, v) = \langle Pu + Qv + q, v - u \rangle,$$

where $q \in \mathbb{R}^n$. Also, $Q - P$ is symmetric negative definite and Q is symmetric positive semidefinite of order n with Lipschitz parameters $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ (for more details see, [22]). During this Example in Section 5.1, both P, Q are arbitrary generated (Choosing two diagonal matrices randomly A_1 and A_2 with entries from $[0, 2]$ and $[-2, 0]$ respectively. Two random orthogonal matrices B_1 and B_2 (RandOrthMat(n)) are able to generate a positive semi definite matrix $M_1 = B_1 A_1 B_1^T$ and negative semi definite matrix $M_2 = B_2 A_2 B_2^T$. Finally, set $Q = M_1 + M_1^T$, $S = M_2 + M_2^T$ and $P = Q - S$.) and entries of q randomly belongs to $[-1, 1]$. The constraint set $C \subset \mathbb{R}^n$ is convex and closed as

$$C := \{u \in \mathbb{R}^n : -5 \le u_i \le 5\}.$$

The numerical results regarding Example in Section 5.1 have shown in Figures 1–9 and Table 1.

Table 1. The numerical findings of the Figures 1–9.

n	N	ζ_n	Algo1 [30]		Algo2 [39]		Algo3 [40]		Algo4	
			iter.	time	iter.	time	iter.	time	iter.	time
10	10	$\frac{1}{n+1}$	83	0.8633	56	0.4295	35	0.2929	19	0.1319
10	10	$\frac{\log(n+3)}{n+1}$	52	0.4297	64	0.4862	40	0.3040	23	0.1896
10	10	$\frac{1}{\log(n+3)}$	94	0.8761	400	5.0501	305	3.4549	82	0.6732
50	10	$\frac{1}{n+1}$	136	1.2545	107	0.9765	69	0.7521	54	0.4691
50	10	$\frac{\log(n+3)}{n+1}$	86	0.6913	80	0.7453	55	0.4792	38	0.3128
50	10	$\frac{1}{\log(n+3)}$	100	0.8427	205	2.2437	175	1.7925	86	0.7685
100	10	$\frac{1}{n+1}$	222	3.0913	150	1.8105	105	1.1990	76	0.8656
100	10	$\frac{\log(n+3)}{n+1}$	100	1.1624	92	1.0639	69	0.7964	36	0.4207
100	10	$\frac{1}{\log(n+3)}$	113	1.3110	211	2.7524	188	2.4022	98	1.1311
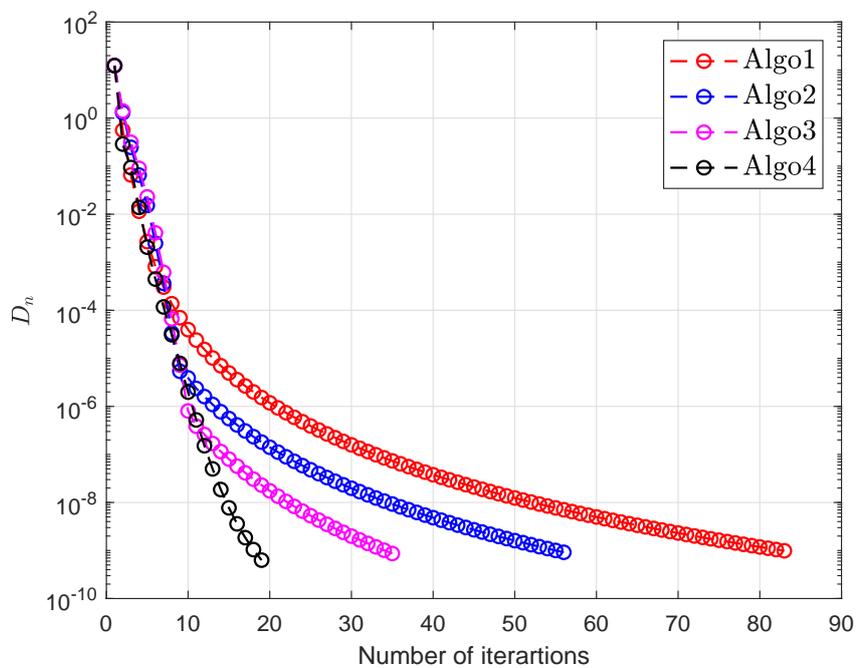
Figure 1. Example 5.1 for $n = 10$ and $\zeta_n = \frac{1}{n+1}$.
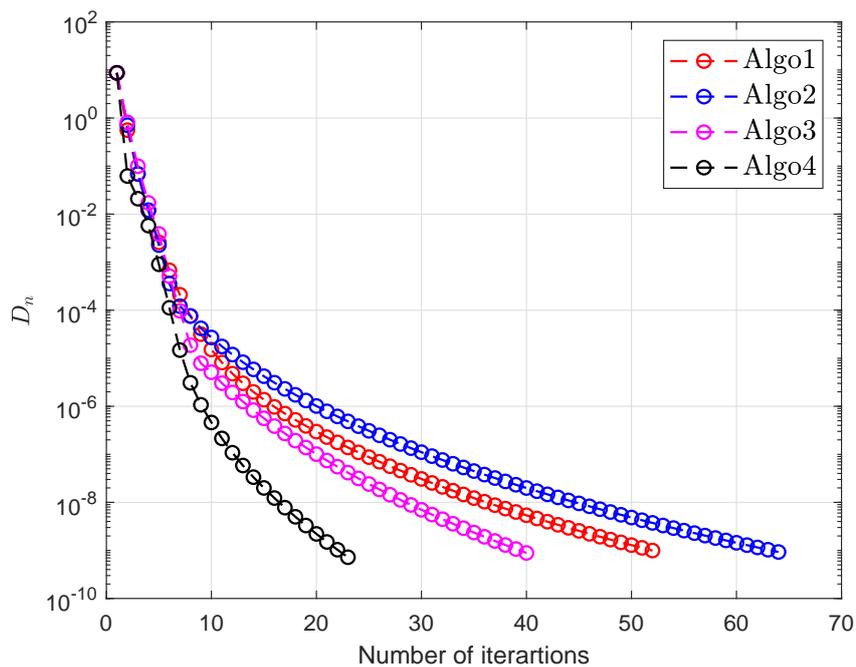
Figure 2. Example 5.1 for $n = 10$ and $\xi_n = \frac{\log(n+3)}{n+1}$.
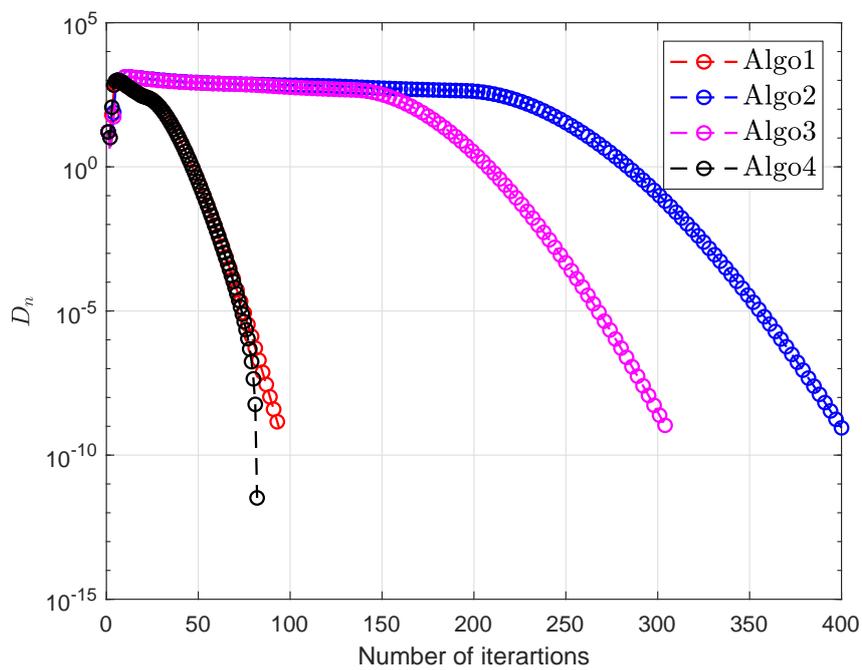
Figure 3. Example 5.1 for $n = 10$ and $\xi_n = \frac{1}{\log(n+3)}$.
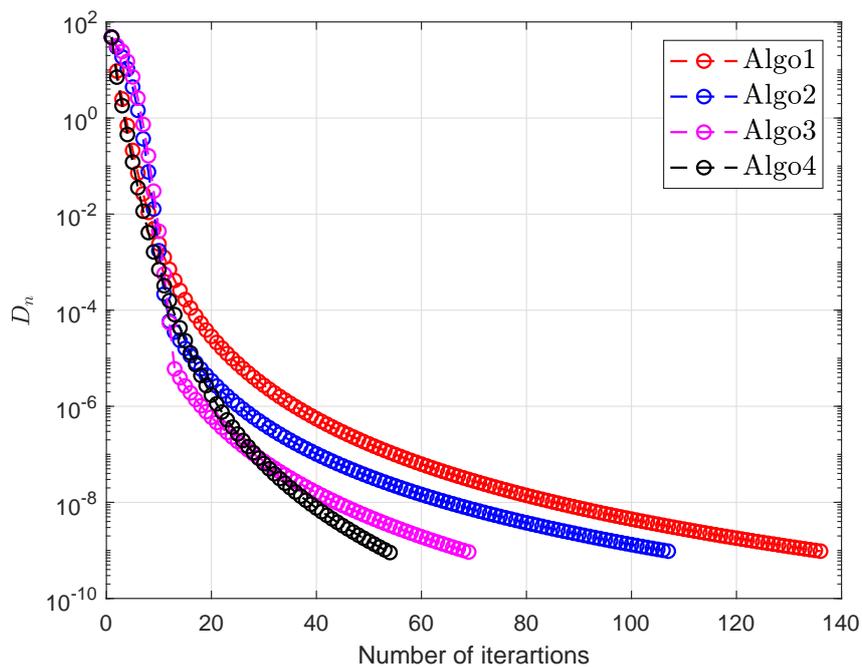
**Figure 4.** Example 5.1 for $n = 50$ and $\xi_n = \frac{1}{n+1}$.
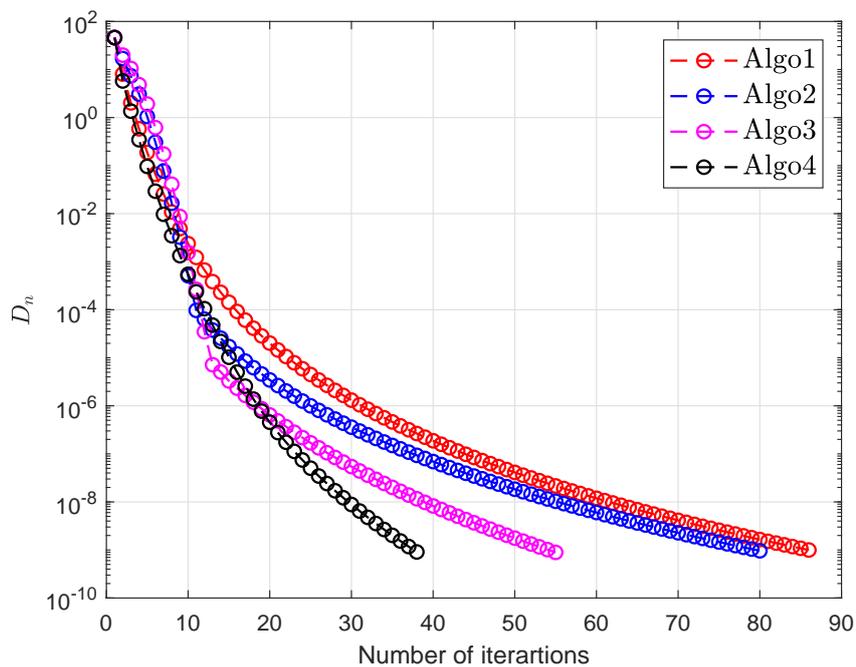
**Figure 5.** Example 5.1 for $n = 50$ and $\xi_n = \frac{\log(n+3)}{n+1}$.
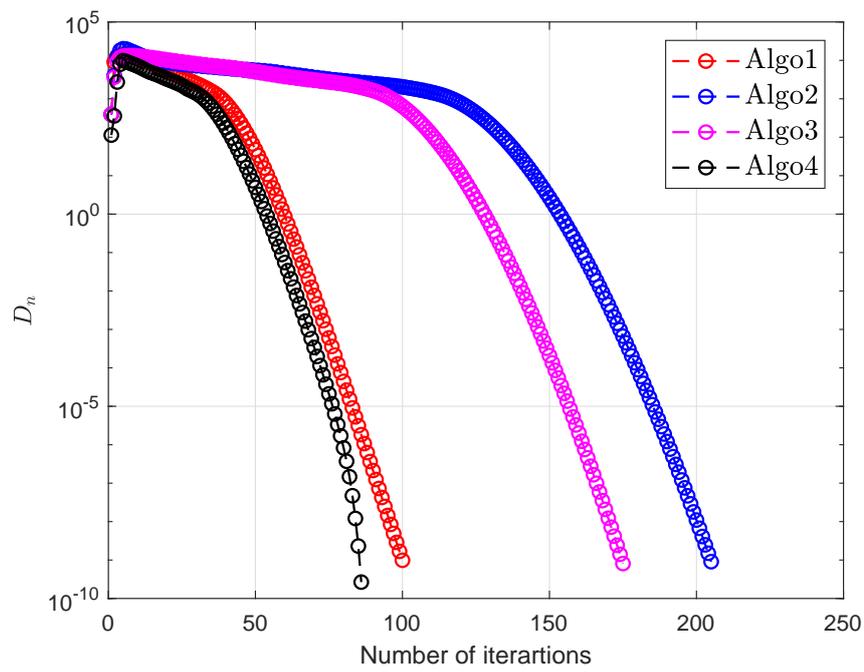
Figure 6. Example 5.1 for $n = 50$ and $\xi_n = \frac{1}{\log(n+3)}$.
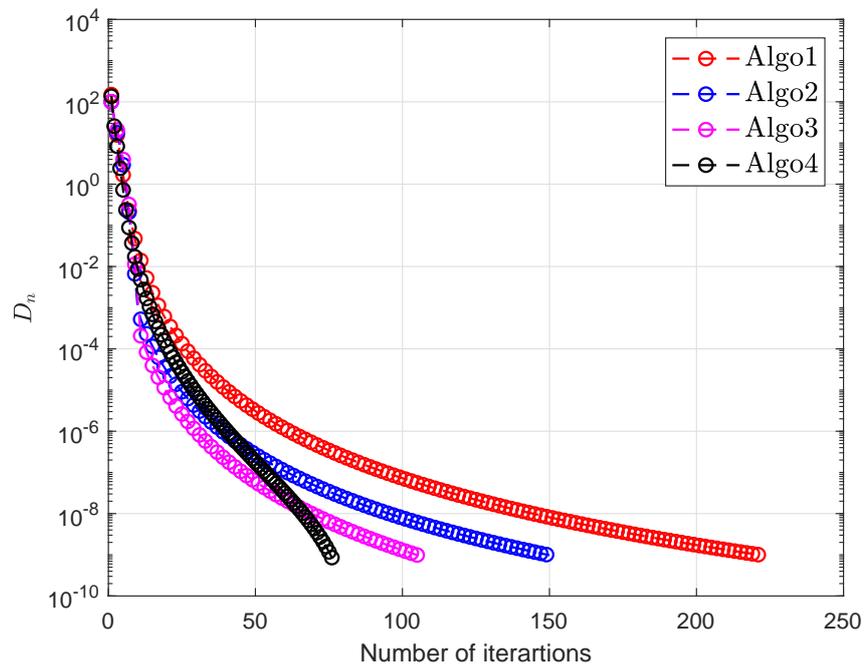
Figure 7. Example 5.1 for $n = 100$ and $\xi_n = \frac{1}{n+1}$.
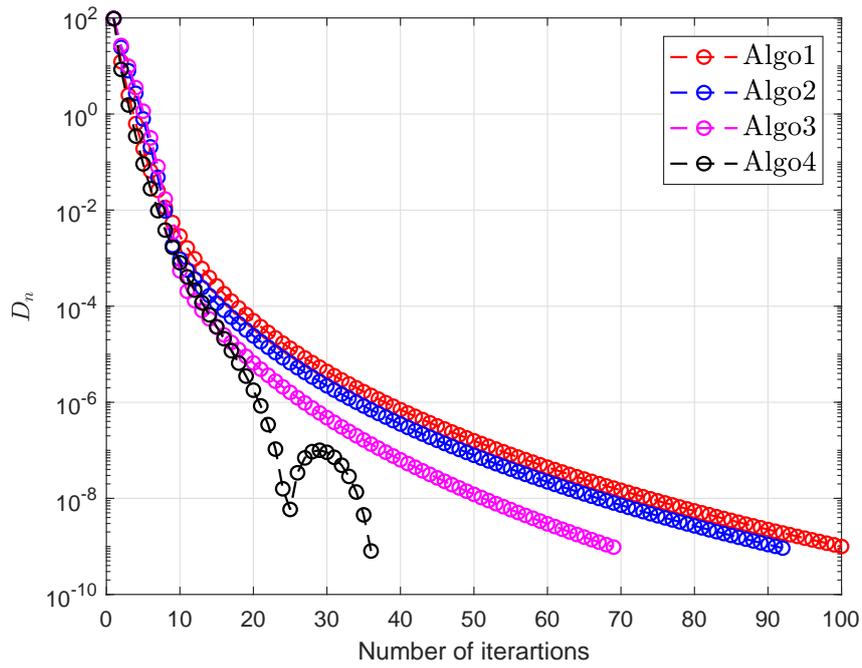
Figure 8. Example 5.1 for $n = 100$ and $\xi_n = \frac{\log(n+3)}{n+1}$.
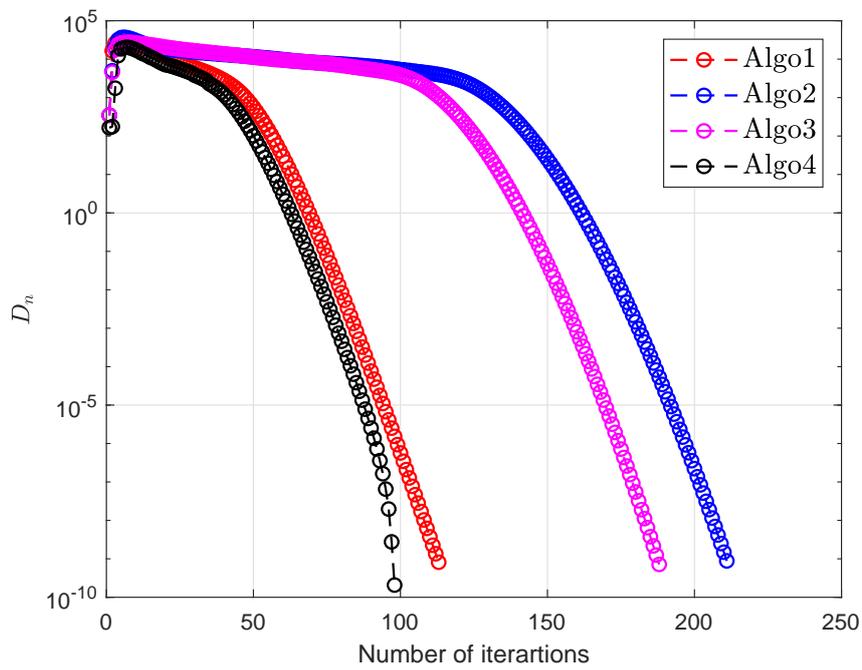
Figure 9. Example in Section 5.1 for $n = 100$ and $\xi_n = \frac{1}{\log(n+3)}$.

5.2. Example 2

Let a bifunction f define on the convex set as

$$f(u,v) = \left\langle \left(BB^T + S + D\right)u, v - u \right\rangle,$$

where D is an $n \times n$ diagonal matrix with nonnegative elements. Moreover, S is an $n \times n$ skew-symmetric matrix and B is an $n \times n$ matrix. The constraint set $C \subset \mathbb{R}^n$ is taken as

$$C = \{u \in \mathbb{R}^n : Au \leq b\}$$

while A is an $100 \times n$ matrix and nonnegative vector b. We can see that bifunction f is γ-monotone through $\gamma = \min\{\text{eig}(BB^T + S + D)\}$ and Lipschitz-like constants are $c_1 = c_2 = \frac{1}{2}\max\{\text{eig}(BB^T + S + D)\}$. In our case we generate the random matrices ($B = rand(n)$, $C = rand(n)$, $S = 0.5C - 0.5C^T$, $D = diag(rand(n, 1))$.) and the numerical results regarding Example in Section 5.2 are shown in Figures 10–19 and Table 2.

Table 2. The experimental results for Figures 10–19.

			Algo1 [30]		Algo2 [39]		Algo3 [40]		Algo4	
n	N	ξ_n	iter.	time	iter.	time	iter.	time	iter.	time
5	10	$\frac{1}{(n+1)\log(n+3)}$	212	2.5360	225	2.6580	179	3.3746	122	1.3161
5	10	$\frac{1}{n+1}$	200	2.1717	254	3.4295	164	1.8637	137	1.7299
5	10	$\frac{\log(n+3)}{n+1}$	181	2.6688	194	2.3646	158	1.8703	106	1.1469
5	10	$\frac{1}{\sqrt{n+1}}$	89	0.9550	132	1.5186	72	0.7889	52	0.5644
5	10	$\frac{1}{\log(n+3)}$	137	1.5127	152	1.8906	89	0.9427	80	0.8514
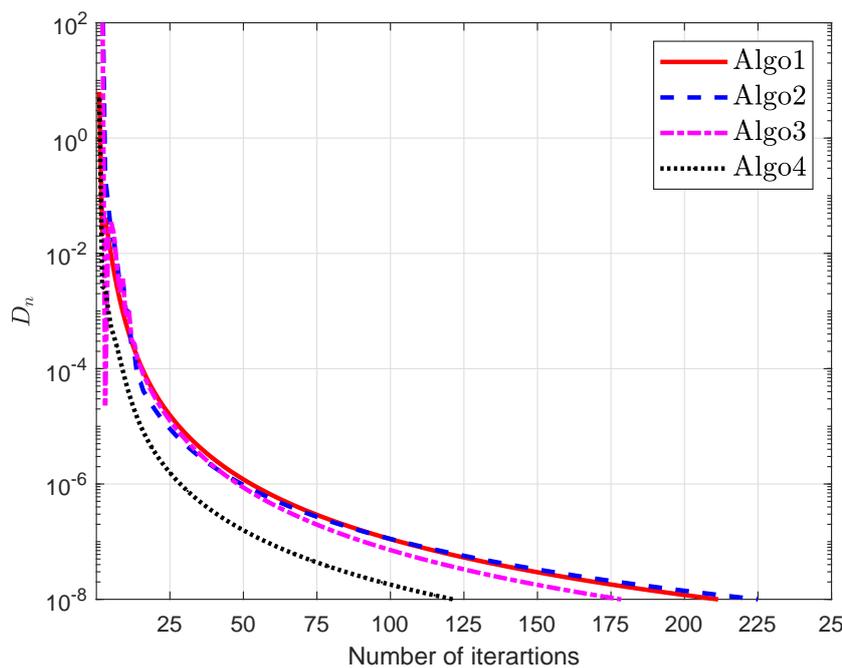
Figure 10. Example in Section 5.2 for $n = 5$ and $\xi_n = \frac{1}{(n+1)\log(n+3)}$.
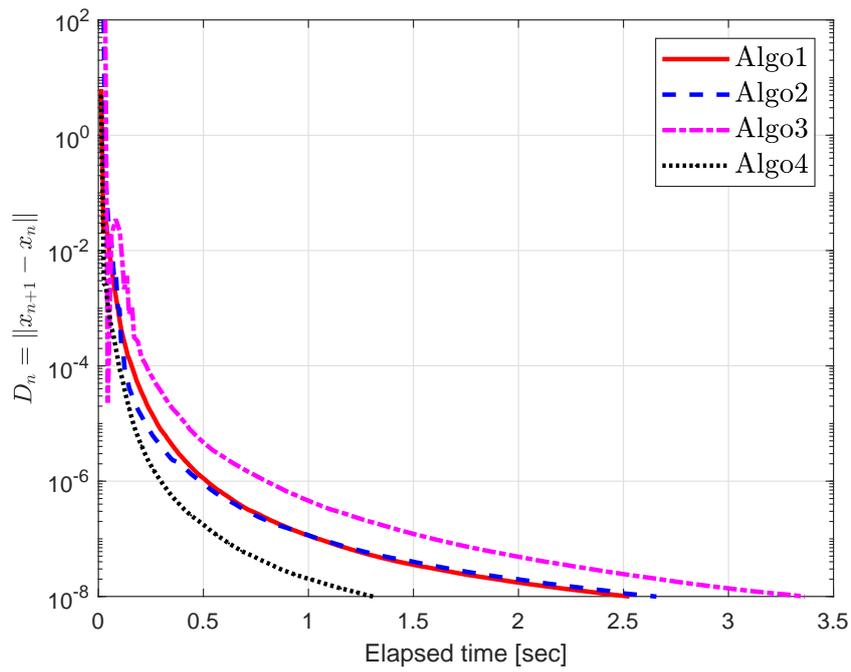
Figure 11. Example in Section 5.2 for $n = 5$ and $\xi_n = \frac{1}{(n+1)\log(n+3)}$.
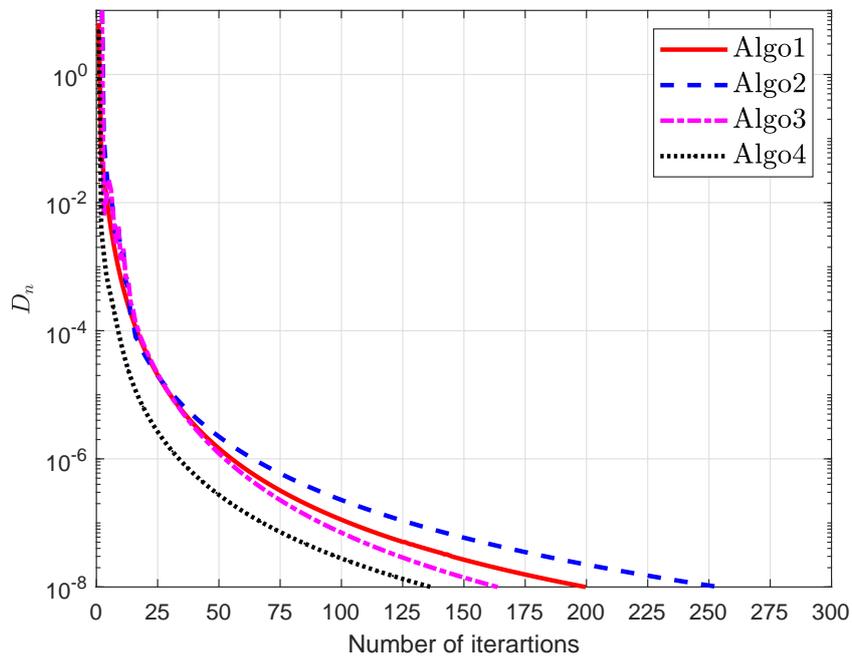

**Figure 12.** Example in Section 5.2 for $n = 5$ and $\xi_n = \frac{1}{n+1}$.
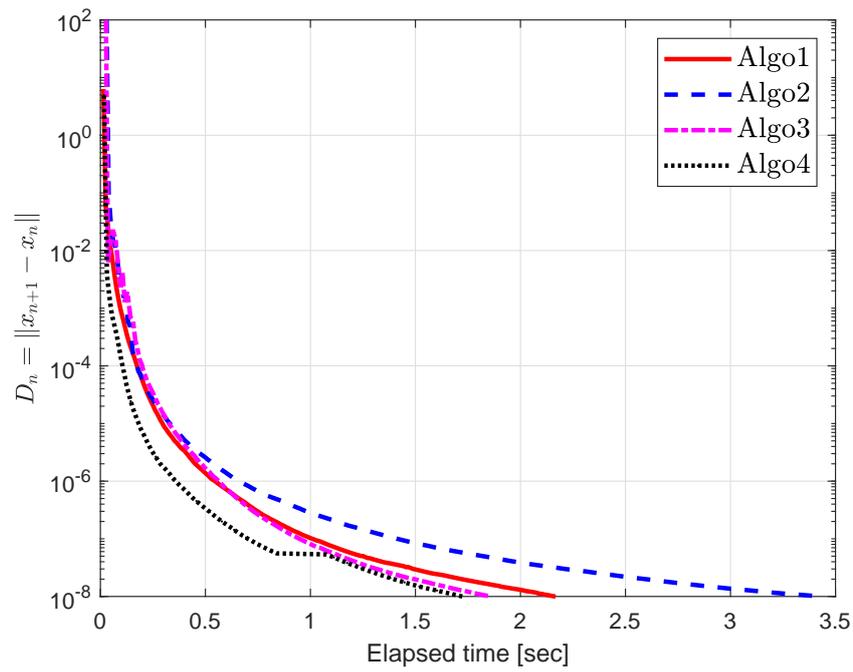
**Figure 13.** Example in Section 5.2 for $n = 5$ and $\xi_n = \frac{1}{n+1}$.
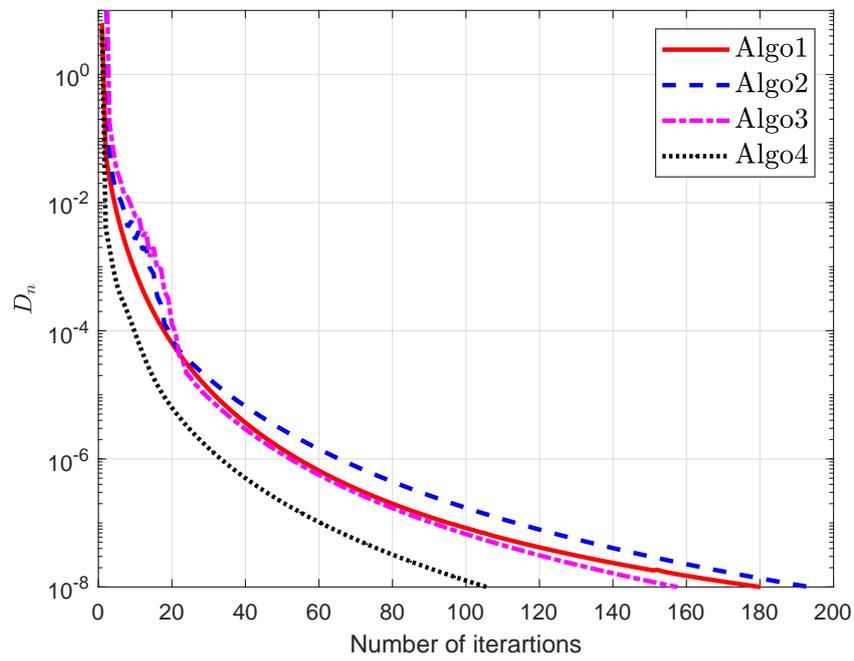
**Figure 14.** Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{\log(n+3)}{n+1}$.
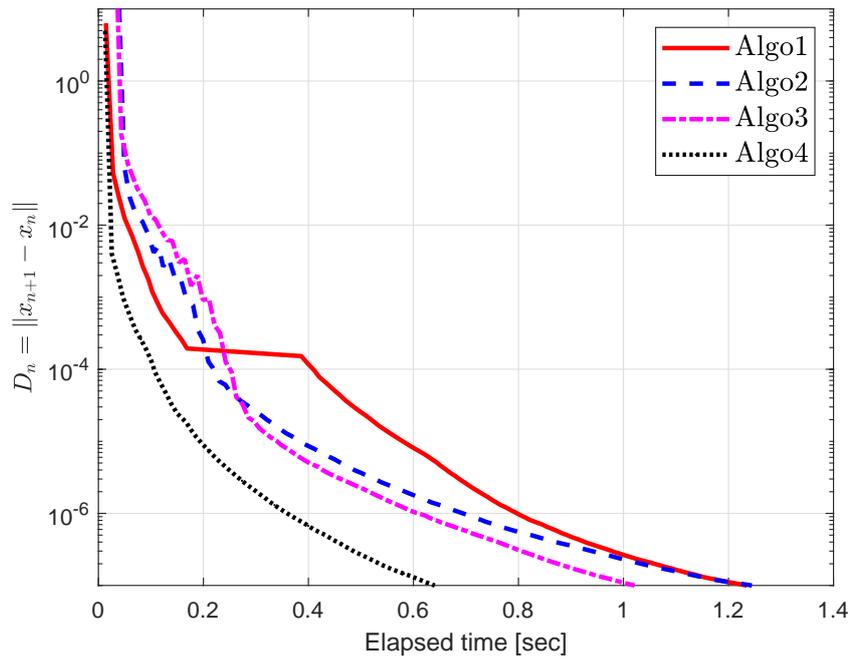
**Figure 15.** Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{\log(n+3)}{n+1}$.
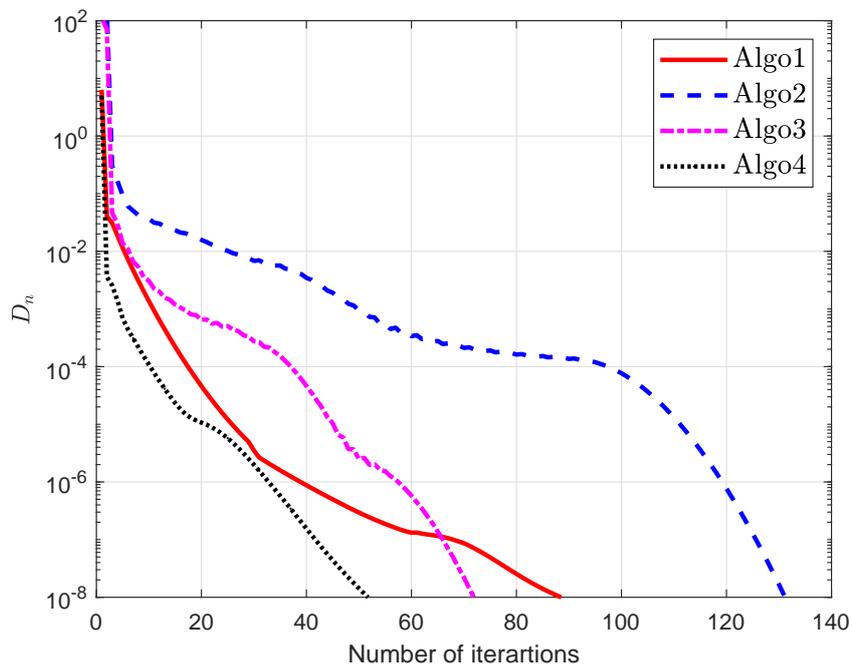
Figure 16. Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{1}{\sqrt{n+1}}$.
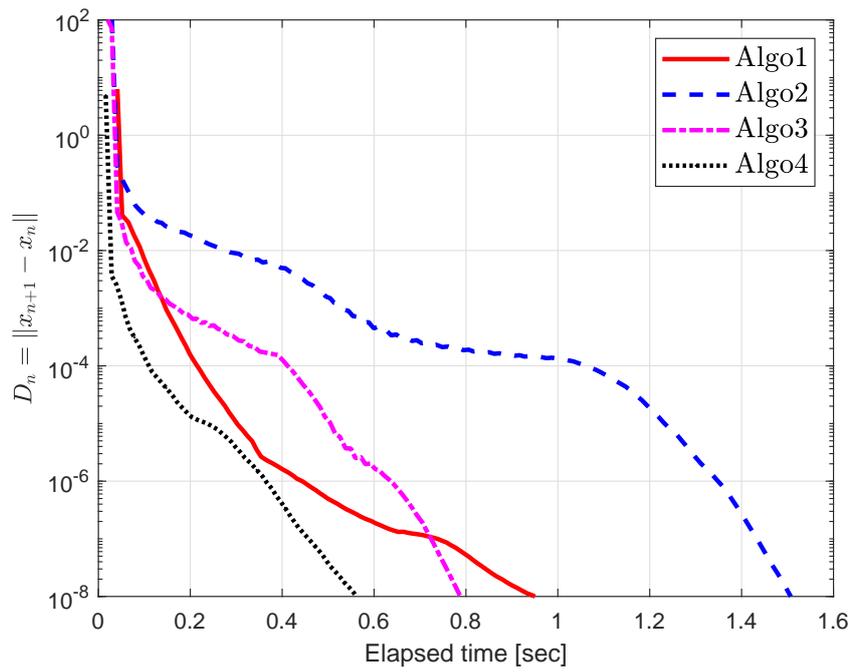
**Figure 17.** Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{1}{\sqrt{n+1}}$.
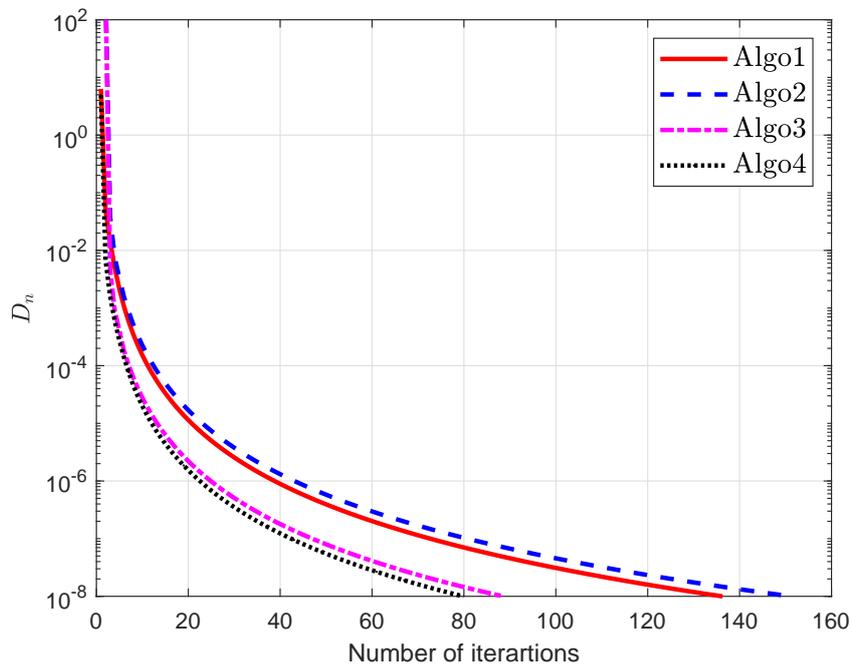
Figure 18. Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{1}{\log(n+3)}$.
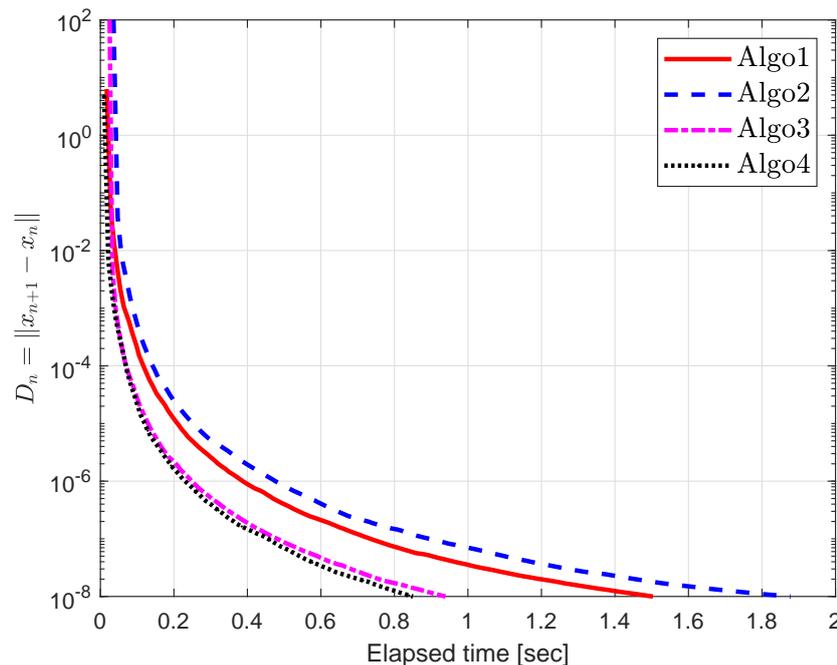
Figure 19. Example in Section 5.1 for $n = 5$ when $\xi_n = \frac{1}{\log(n+3)}$.

Remark 3. *From our numerical experiments we have the following observation:*

1. *There is no need to have prior knowledge of Lipschitz-constant for running algorithms on Matlab.*
2. *The convergence rate of the iterative sequence is based on the convergence rate of the stepsize sequence.*
3. *The convergence rate of the iterative sequence also depends on the nature of the problem and the size of the problem.*
4. *Due to the variable stepsize sequence, a particular value of the stepsize that is not suited to the current iteration of the algorithm often causes disturbance and hump in the behaviour of an iterative sequence.*

6. Conclusions

In this article, we established a new method by associating an inertial term with an extragradient method for dealing with a family of strongly pseudomonotone equilibrium problems. The proposed method requires a sequence of diminishing and non-summable stepsizes and carried out without previous knowledge of the Lipschitz-type constants and the strong pseudo-monotonicity modulus constant. Two numerical experiments have been reported to measure the computational efficiency of our method in comparison to other existing methods. Numerical experiments have pointed out that the method with an inertial scheme performs better than those without inertial scheme.

Conflicts of Interest: The authors declare that they have conflict of interest.

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