Ulam's Type Stability and Generalized Norms

Laura Manolescu

Department of Mathematics, Politehnica University of Timișoara, Piața Victoriei 2, 300006 Timișoara, Romania; laura.manolescu@upt.ro

Received: 20 February 2020; Accepted: 16 March 2020; Published: 1 April 2020

Abstract: A symmetric functional equation is one whose form is the same regardless of the order of the arguments. A remarkable example is the Cauchy functional equation: $f(x + y) = f(x) + f(y)$. Interesting results in the study of the rigidity of quasi-isometries for symmetric spaces were obtained by B. Kleiner and B. Leeb, using the Hyers-Ulam stability of a Cauchy equation. In this paper, some results on the Ulam's type stability of the Cauchy functional equation are provided by extending the traditional norm estimations to their measurements called generalized norm of convex type (v-norm) and generalized norm of subadditive type (s-norm).

Keywords: generalized Hyers-Ulam stability; additive mapping; generalized norm

1. Introduction

Ulam [1] posed the following problem concerning group homomorphism when presenting a talk at the University of Wisconsin:

Let $(G, \circ)$ be a group, $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$ and $\varepsilon > 0$. If there exists a $\delta > 0$ such that $f : G_1 \rightarrow G_2$ satisfies

$$d(f(x \circ y), f(x) * f(y)) \leq \delta, \quad \text{for all } x, y \in G_1$$

then there exists a homomorphism $h : G_1 \rightarrow G_2$ with

$$d(f(x), h(x)) \leq \varepsilon, \quad \text{for all } x \in G_1?$$

The first affirmative answer to this problem was the one provided by Hyers [2], who solved the problem for additive mappings in Banach spaces.

A remarkable generalization of this result was given by Rassias [3]. A further generalization was obtained by Găvruta [4], where he also introduced the concept of generalized Hyers-Ulam-Rassias stability in the spirit of Rassias's approach. Using the idea of [4], Kim [5] proved the modified Hyers-Ulam-Rassias stability for the family of functional equations

$$f(x \circ y) = H(f(x), f(y))(x, y \in S)$$

where H is a homogeneous function and $\circ$ is a square-symmetric operation on the set S . For an instructive treatment of Găvruta's stability theorem, see the paper [6].

For other results and generalizations, see the books [7–11] and their references.

In 1997, Kleiner and Leeb used the Hyers-Ulam stability of a Cauchy equation in the study of the rigidity of quasi-isometries for symmetric spaces (see [12]).

In [13], we proved a fixed-point theorem for a class of operators with suitable properties in very general conditions and some corollaries, which showed that our main result is a useful tool for proving properties of generalized Hyers-Ulam stability for some functional equations in a single variable.

More recently, in the survey [14], the authors presented applications of different fixed point theorems to the theory of stability of functional equations.

In [15], we investigated the approximation of functions by additive and by quadratic mappings, and we discussed the approximation of functions by cubic mappings [16].

In this paper, we extend the main result of [4] in the context of generalized norms.

2. Results

We consider the notion of generalized norm of convex type (v-norm), inspired by [17].

Let $(X, \|\cdot\|)$ be a normed space.

Definition 1. A mapping $\|\cdot\|_v : X \rightarrow \mathbb{R}_+ = [0, \infty)$ is called a generalized norm of convex type or a v-norm if

- (V1) $\|0\|_v = 0$
- (V2) $\left\|\frac{x+y}{2}\right\|_v \leq \frac{\|x\|_v + \|y\|_v}{2}, (\forall)x, y \in X; (\text{property of convexity})$
- (V3) For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, so that if $\|x\|_v < \delta$, then $\|x\| < \varepsilon$;
- (V4) The mapping $x \mapsto \|x\|_v$ from X to $\mathbb{R}_+$ is lower semicontinuous.

Example 1. We provide some examples of v-norms on X .

- (a1) The norm of X is a v-norm.
- (a2) Let $\lambda : X \rightarrow [0, \infty)$ be a convex and continuous function with $\lambda(0) = 0$. Then

$$\|x\|_v := \max\{\|x\|, \lambda(x)\}$$

is a v-norm.

- (a3) Let $h : [0, \infty) \rightarrow [0, \infty)$ be a continuous, nondecreasing mapping so that

$$(\forall)a > 0, \int_0^a h(t)dt > 0.$$

Then

$$\|x\|_v := \int_0^{\|x\|} h(t)dt \text{ is a v-norm.}$$

We prove that the function defined in (a3) is a v-norm.

$$(V1) \|0\|_v = \int_0^0 h(t)dt = 0.$$

(V2) We prove that:

$$\int_0^{\left\|\frac{x+y}{2}\right\|} h(t)dt \leq \frac{1}{2} \left(\int_0^{\|x\|} h(t)dt + \int_0^{\|y\|} h(t)dt \right).$$

We have

$$\left\|\frac{x+y}{2}\right\| \leq \frac{1}{2}(\|x\| + \|y\|)$$

Let $F(u) = \int_0^u h(t)dt$. We prove that:

$$F\left(\frac{u+v}{2}\right) \leq \frac{1}{2}(F(u) + F(v)).$$

For $u \geq v$, we have:

$$\begin{aligned} F(u) + F(v) - 2F\left(\frac{u+v}{2}\right) &= F(u) - F\left(\frac{u+v}{2}\right) + F(v) - F\left(\frac{u+v}{2}\right) \\ &= F'(c_1)\left(u - \frac{u+v}{2}\right) + F'(c_2)\left(v - \frac{u+v}{2}\right) \\ &= \frac{u-v}{2}(h(c_1) - h(c_2)) \geq 0, \end{aligned}$$

since $u \geq c_1 \geq \frac{u+v}{2} \geq c_2 \geq v$ and h is nondecreasing.

(V3) We prove that $(\forall)\varepsilon > 0, (\exists)\delta(\varepsilon) > 0$, so that if x is

$$\int_0^{\|x\|} h(t)dt < \delta(\varepsilon) \Rightarrow \|x\| < \varepsilon.$$

Conversely, there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists x_δ such that:

$$\int_0^{\|x_\delta\|} h(t)dt < \delta \text{ and } \|x_\delta\| \geq \varepsilon_0.$$

It follows that:

$$\int_0^{\varepsilon_0} h(t)dt \leq \int_0^{\|x_\delta\|} h(t)dt < \delta, (\forall) \delta > 0.$$

It follows that $\int_0^{\varepsilon_0} h(t)dt = 0$, in contradiction with the hypothesis on h .

(V4) The mapping $x \mapsto \|x\|_v$ is continuous. If $x_n \rightarrow x$ ($n \rightarrow \infty$), implies that $\|x_n\| \rightarrow \|x\|$ ($n \rightarrow \infty$). We have:

$$\left| \|x_n\|_v - \|x\|_v \right| = \left| \int_{\|x_n\|}^{\|x\|} h(t)dt \right| = \left| \|x_n\| - \|x\| \right| \cdot \left| h(c_n) \right|,$$

where c_n is between $\|x_n\|$ and $\|x\|$, by the integral mean value theorem. It follows:

$$\lim_{n \rightarrow \infty} \left| \|x_n\|_v - \|x\|_v \right| = 0 \cdot h(\|x\|) = 0.$$

So, $\|x_n\|_v \rightarrow \|x\|_v$.

Next, we provide some elementary results on convex norms.

Proposition 1. We have:

- (i) $\|x + y\|_v \leq \frac{1}{2}(\|2x\|_v + \|2y\|_v)$;
- (ii) $\|x\|_v \leq \frac{1}{2^n}\|2^n x\|_v$;
- (iii) $\|x + y\|_v \leq \frac{1}{2^n}(\|2^n x\|_v + \|2^n y\|_v)$;
- (iv) If $\|x_n\|_v \leq a_n$, $n \in \mathbb{N}$, for $(x_n)_{n=1}^\infty \subset X$ and $(a_n)_{n=1}^\infty \subset \mathbb{R}_+$ with $\lim_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} x_n = 0$;
- (v) If $\|x_n - x_m\|_v \leq b_m$, $n > m$ $\lim_{m \rightarrow \infty} b_m = 0$, then (x_n) is a Cauchy sequence;
- (vi) If $\|x - x_n\|_v \leq a_n$ and $\|y - x_n\|_v \leq b_n$, for $a_n \rightarrow 0$ and $b_n \rightarrow 0$, then $x = y$.

Proof. (i) In the condition (V2) of Definition 1, we take $2x$ instead of x , and instead of y , we take $2y$.

(ii) In the condition (V2) of Definition 1, we take $y = 0$ and using the condition (V1), we obtain

$$\left\| \frac{x}{2} \right\|_v \leq \frac{\|x\|_v + \|0\|_v}{2} = \frac{\|x\|_v}{2}$$

If we take $2x$ instead of x in the above inequality, we obtain:

$$\|x\|_v \leq \frac{1}{2} \|2x\|_v.$$

Now, by induction: if $\|x\|_v \leq \frac{1}{2^n} \|2^n x\|_v$, then:

$$\|x\|_v \leq \frac{1}{2^n} \|2^n x\|_v \leq \frac{1}{2^n} \frac{1}{2} \|2^{n+1} x\|_v.$$

(iii) We use induction. For $n = 1$, we have (i). If

$$\|x + y\|_v \leq \frac{1}{2^n} (\|2^n x\|_v + \|2^n y\|_v)$$

then, with (ii), we obtain:

$$\begin{aligned} \|x + y\|_v &\leq \frac{1}{2^n} \left(\frac{1}{2} \|2^{n+1} x\|_v + \frac{1}{2} \|2^{n+1} y\|_v \right) \\ &= \frac{1}{2^{n+1}} (\|2^{n+1} x\|_v + \|2^{n+1} y\|_v). \end{aligned}$$

(iv) We use condition (V3) of Definition 1. Let $\varepsilon > 0$. From (V3), it follows that there exists $\delta = \delta(\varepsilon) > 0$, so that if $\|x\|_v < \delta$, then $\|x\| < \varepsilon$. Since $\lim_{n \rightarrow \infty} a_n = 0$, it follows that there exists $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$, we have $a_n < \delta$. It follows

$$\|x_n\|_v < \delta, \quad n \geq n_0(\varepsilon).$$

Hence, $\|x_n\| < \varepsilon, \quad n \geq n_0(\varepsilon)$.

(v) We use (V3) again. We have $b_n < \delta(\varepsilon), \quad n \geq n_0(\varepsilon) = n_0$. It follows

$$\|x_n - x_{n_0}\|_v < \delta(\varepsilon), \quad n \geq n_0.$$

From (V3), we obtain:

$$\|x_n - x_{n_0}\| < \varepsilon.$$

Let be $m, n \geq n_0$:

$$\|x_n - x_m\| \leq \|x_n - x_{n_0}\| + \|x_{n_0} - x_m\| < 2\varepsilon.$$

(vi) From (iv), we have $x_n \rightarrow x, \quad x_n \rightarrow y$. From the uniqueness of the limit, it follows that $x = y$. $\square$

We prove the stability of additive mappings.

Theorem 1. Let S be an abelian semigroup, X a Banach space, and $\varphi : S \times S \rightarrow [0, \infty)$, such that:

$$\Phi(x, y) := \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{2^{n+1}} < \infty, \quad x, y \in S. \quad (1)$$

Let $\|\cdot\|_v$ be a v -norm on X . If $f : S \rightarrow X$ is so that:

$$\|f(x+y) - f(x) - f(y)\|_v \leq \varphi(x, y), \quad (\forall) x, y \in S, \quad (2)$$

then there exists a unique $T : S \rightarrow X$ additive so that:

$$\|T(x) - f(x)\|_v \leq \Phi(x, x), \quad x \in S. \quad (3)$$

Proof. In (2), we take $x = y$ to obtain:

$$\|f(2x) - 2f(x)\|_v \leq \varphi(x, x), \quad (4)$$

and from (ii):

$$\left\| \frac{f(2x)}{2} - f(x) \right\|_v \leq \frac{1}{2} \|f(2x) - 2f(x)\|_v;$$

hence,

$$\left\| \frac{f(2x)}{2} - f(x) \right\|_v \leq \frac{1}{2} \varphi(x, x). \quad (5)$$

In the above relation, we replace x with $2x$, and we obtain:

$$\left\| \frac{f(2^2x)}{2} - f(2x) \right\|_v \leq \frac{1}{2} \varphi(2x, 2x) \quad (6)$$

and from (i), and relations (4) and (6), we have:

$$\begin{aligned} \left\| \frac{f(2^2x)}{2^2} - f(x) \right\|_v &= \left\| \frac{f(2^2x)}{2^2} - \frac{f(2x)}{2} + \frac{f(2x)}{2} - f(x) \right\|_v \\ &\leq \frac{1}{2} \left[\left\| \frac{f(2^2x)}{2} - f(2x) \right\|_v + \|f(2x) - 2f(x)\|_v \right] \\ &\leq \frac{1}{2} \left[\frac{1}{2} \varphi(2x, 2x) + \varphi(x, x) \right], \end{aligned}$$

hence,

$$\left\| \frac{f(2^2x)}{2^2} - f(x) \right\|_v \leq \frac{1}{2} \varphi(x, x) + \frac{1}{2^2} \varphi(2x, 2x). \quad (7)$$

We prove that :

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\|_v \leq \sum_{k=0}^{n-1} \frac{\varphi(2^k x, 2^k x)}{2^{k+1}}. \quad (8)$$

We use the induction. We suppose that relation (8) holds.

Using (i) and relations (4) and (8), we have:

$$\begin{aligned} \left\| \frac{f(2^{n+1}x)}{2^{n+1}} - f(x) \right\|_v &= \left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2x)}{2} + \frac{f(2x)}{2} - f(x) \right\|_v \\ &\leq \frac{1}{2} \left[\left\| \frac{f(2^{n+1}x)}{2^n} - f(2x) \right\|_v + \|f(2x) - 2f(x)\|_v \right] \\ &\leq \sum_{k=0}^{n-1} \frac{\varphi(2^{k+1}x, 2^{k+1}x)}{2^{k+2}} + \frac{\varphi(x, x)}{2} \end{aligned}$$

hence,

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - f(x) \right\|_v \leq \sum_{i=0}^n \frac{\varphi(2^i x, 2^i x)}{2^{i+1}}.$$

Now, we prove that the sequence $\left\{ \frac{f(2^n x)}{2^n} \right\}$ is a Cauchy sequence. In (8), we take $2^p x$ instead of x and we obtain

$$\left\| \frac{f(2^{n+p}x)}{2^n} - f(2^p x) \right\|_v \leq \sum_{k=0}^{n-1} \frac{\varphi(2^{k+p}x, 2^{k+p}x)}{2^{k+1}}$$

and using (ii):

$$\begin{aligned} \left\| \frac{f(2^{n+p}x)}{2^{n+p}} - \frac{f(2^p x)}{2^p} \right\|_v &\leq \frac{1}{2^p} \left\| \frac{f(2^{n+p}x)}{2^n} - f(2^p x) \right\|_v \\ &\leq \sum_{k=0}^{n-1} \frac{\varphi(2^{k+p}x, 2^{k+p}x)}{2^{k+p+1}} \\ &= \sum_{i=p}^{n+p-1} \frac{\varphi(2^i x, 2^i x)}{2^{i+1}} \\ &\leq \sum_{i=p}^{\infty} \frac{\varphi(2^i x, 2^i x)}{2^{i+1}}. \end{aligned}$$

We use (v). Since $\lim_{p \rightarrow \infty} \sum_{i=p}^{\infty} \frac{\varphi(2^i x, 2^i x)}{2^{i+1}} = 0$, it follows that $\left\{ \frac{f(2^n x)}{2^n} \right\}$ is a Cauchy sequence in Banach space $(X, \|\cdot\|)$.

Since X is a Banach space, there exists

$$T(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}, \text{ for all } x \in X.$$

We prove that T is an additive mapping. From (2), we have:

$$\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\|_v \leq \varphi(2^n x, 2^n y),$$

and from (ii) we have:

$$\begin{aligned} \left\| \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\|_v &\leq \frac{1}{2^n} \|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\|_v \\ &\leq \frac{1}{2^n} \varphi(2^n x, 2^n y). \end{aligned}$$

From (1), it follows that

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n x)}{2^n} = 0,$$

and from (iv), we have

$$\lim_{n \rightarrow \infty} \left\{ \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\} = 0;$$

hence, $T(x + y) - T(x) - T(y) = 0$.

We prove now the relation (3). We have:

$$\frac{f(2^n x)}{2^n} - f(x) \longrightarrow T(x) - f(x), \quad n \rightarrow \infty.$$

Since $x \mapsto \|x\|_v$ is lower semicontinuous, it follows that

$$\liminf \left\| \frac{f(2^n x)}{2^n} - f(x) \right\|_v \geq \|T(x) - f(x)\|_v.$$

From relation (8),

$$\liminf \left\| \frac{f(2^n x)}{2^n} - f(x) \right\|_v \leq \Phi(x, x), x \in S;$$

hence,

$$\|T(x) - f(x)\|_v \leq \Phi(x, x).$$

We prove that T is a unique additive mapping with the property in relation (3). We suppose that there exists two additive mappings T_1 and T_2 , such that

$$\|T_1(x) - f(x)\|_v \leq \Phi(x, x) \text{ and } \|T_2(x) - f(x)\|_v \leq \Phi(x, x), (\forall)x \in S.$$

For any $n \in \mathbb{N}$ and $x \in S$, we have:

$$\|T_1(2^n x) - f(2^n x)\|_v \leq \Phi(2^n x, 2^n x)$$

$$\|T_2(2^n x) - f(2^n x)\|_v \leq \Phi(2^n x, 2^n x)$$

From condition (ii), we get :

$$\left\| T_1(x) - \frac{f(2^n x)}{2^n} \right\|_v \leq \frac{1}{2^n} \Phi(2^n x, 2^n x) \rightarrow 0$$

$$\left\| T_2(x) - \frac{f(2^n x)}{2^n} \right\|_v \leq \frac{1}{2^n} \Phi(2^n x, 2^n x) \rightarrow 0$$

So, from (vi), it follows that $T_1(x) = T_2(x)$. $\square$

Remark 1. It is clear that the result in the above Theorem holds if

$$\sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n x)}{2^{n+1}} < \infty \text{ and } \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0,$$

for any $x, y \in S$.

The following Corollary is a direct extension of a result of Rassias [3].

Corollary 1. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a mapping such that for $0 \leq p < 1$, we have

$$\psi(2t) \leq 2^p \psi(t), t \in \mathbb{R}_+.$$

Let $(E, \|\cdot\|_1)$ be a normed space, X a Banach space, $\varepsilon > 0$, and $\|\cdot\|_v$ be a v -norm on X . If $f : E \rightarrow X$ is so that

$$\|f(x+y) - f(x) - f(y)\|_v \leq \varepsilon \{\psi(\|x\|_1) + \psi(\|y\|_1)\}, (\forall)x, y \in E,$$

then there exists a unique additive mapping T so that

$$\|f(x) - T(x)\|_v \leq \frac{2\varepsilon}{2-2^p} \psi(\|x\|_1), x \in E.$$

Proof. We apply Theorem 1, with $\varphi(x, y) = \varepsilon\{\psi(\|x\|_1) + \psi(\|y\|_1)\}$, $x, y \in E$. We have:

$$\begin{aligned}\Phi(x, y) &\leq \varepsilon \sum_{n=0}^{\infty} \frac{2^{np}}{2^{n+1}} \{\psi(\|x\|_1) + \psi(\|y\|_1)\} \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{1 - 2^{p-1}} \{\psi(\|x\|_1) + \psi(\|y\|_1)\} \\ &= \frac{\varepsilon}{2 - 2^p} \{\psi(\|x\|_1) + \psi(\|y\|_1)\}.\end{aligned}$$

□

An example of a function that verifies the hypothesis of the Corollary 1 is $\psi(t) = t^p$, $t \in \mathbb{R}_+$. We provide an example of application of Theorem 1, which does not follow from the Corollary 1.

Example 2. We consider $(X, \|\cdot\|)$ a Banach space, h a function as in Example 1, a3), $\varepsilon > 0$ and $0 \leq p < 1$. If $f : [0, \infty) \rightarrow X$ is so that

$$\int_0^{\|f(x+y)-f(x)-f(y)\|} h(t)dt \leq \varepsilon(x^p + y^p), \quad (\forall) x, y \in [0, \infty).$$

Then, there exists a unique additive mapping $T : [0, \infty) \rightarrow X$ so that

$$\int_0^{\|f(x)-T(x)\|} h(t)dt \leq \frac{2\varepsilon}{2-2^p} x^p, \quad x \in [0, \infty).$$

Now, we consider the case of generalized norm of subadditive type (or s-norm).

Definition 2. A mapping $\|\cdot\|_s : X \rightarrow \mathbb{R}_+$ is called a generalized norm of subadditive type or a s-norm if

- (S1) $\|x + y\|_s \leq \|x\|_s + \|y\|_s$;
- (S2) For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, so that if $\|x\|_s < \delta$, then $\|x\| < \varepsilon$;
- (S3) The mapping $x \mapsto \|x\|_s$ from X to $\mathbb{R}_+$ is lower semicontinuous.

It is clear that any norm is a generalized norm of the subadditive type. An example of a generalized norm of the subadditive type that is not a norm is the following:

Example 3. Let $(X, \|\cdot\|)$ be a normed space. For all $x \in X$, we define:

$$\|x\|_s := \frac{\|x\|}{\|x\| + 1}.$$

We verify the axioms (S1) – (S3) for this function.

Proposition 2. Elementary properties of s-norms are:

- (i) $\|x\|_s \leq 2 \left\| \frac{x}{2} \right\|_s$;
- (ii) If $\|x_n\|_s \leq a_n$, $n \in \mathbb{N}$, for $(x_n)_{n=1}^{\infty} \subset X$ and $(a_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ with $\lim_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} x_n = 0$;
- (iii) If $\|x_n - x_m\|_s \leq b_m$, $n > m$, $\lim_{m \rightarrow \infty} b_m = 0$, then (x_n) is a Cauchy sequence;
- (iv) If $\|x - x_n\|_s \leq a_n$ and $\|y - x_n\|_c \leq b_n$, for $a_n \rightarrow 0$ and $b_n \rightarrow 0$, then $x = y$.

Following the ideas from Theorem 1, we obtain an analogous result for generalized subadditive norms in the context of 2-divisible abelian semigroup.

We recall that an abelian semigroup S is called 2-divisible if for all $\varepsilon > 0$ there exists a unique x' such that

$$x' + x' = x.$$

We denote such an x' with $\frac{x}{2}$:

Theorem 2. Let S be an abelian semigroup, 2-divisible, X a Banach space and $\varphi_1 : S \times S \rightarrow [0, \infty)$, such that

$$\Phi_1(x, y) := \sum_{n=1}^{\infty} 2^{n-1} \varphi_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty, \quad x, y \in S.$$

Let $\|\cdot\|_s$ be a s -norm on X . If $f : S \rightarrow X$ is so that

$$\|f(x+y) - f(x) - f(y)\|_s \leq \varphi_1(x, y), \quad (\forall) x, y \in S,$$

then there exists a unique $T : S \rightarrow X$ additive so that

$$\|T(x) - f(x)\|_s \leq \Phi_1(x, x), \quad x \in S.$$

The following application of Theorem 2 is a direct extension of a result by Gajda [18].

Corollary 2. Let $p > 1$ and $\psi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a mapping such that

$$\psi_1(2t) \geq 2^p \psi_1(t), \quad t \in \mathbb{R}_+.$$

Let $(E, \|\cdot\|_1)$ be a normed space, $\varepsilon > 0$, $(X, \|\cdot\|)$ be a Banach space, and $\|\cdot\|_s$ be a s -norm on X .

If $f : E \rightarrow X$ is such that

$$\|f(x+y) - f(x) - f(y)\|_s \leq \varepsilon \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\}, \quad (\forall) x, y \in E,$$

then, there exists a unique additive mapping T such that

$$\|f(x) - T(x)\|_s \leq \frac{2\varepsilon}{2^p - 2} \psi_1(\|x\|_1), \quad x \in E.$$

Proof. We apply Theorem 2, with $\varphi_1(x, y) = \varepsilon \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\}$, $x, y \in E$. We have;

$$\begin{aligned} \Phi_1(x, y) &\leq \varepsilon \sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{1}{2^{np}} \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\} \\ &= \frac{\varepsilon}{2} \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \\ &= \frac{\varepsilon}{2} \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\} \cdot \frac{1}{2^{p-1}} \cdot \frac{1}{1 - \frac{1}{2^{p-1}}} \\ &= \frac{\varepsilon}{2^p - 2} \{\psi_1(\|x\|_1) + \psi_1(\|y\|_1)\}. \end{aligned}$$

□

An example of function that verifies the hypothesis of the Corollary 2 is $\psi(t) = t^p$, $t \in \mathbb{R}_+$.

3. Conclusions

In this paper, on a Banach space, two new measurements were introduced called generalized norm of convex type (v-norm) and generalized norm of subadditive type (s-norm). Hence, the area of applications are extended of the general theorem of Găvruta [4] on the Hyers–Ulam stability.

Among the applications of the Ulam's type stability, the paper mentioned applications of Hyers–Ulam–Rassias to approximate testing with error relative to input size [19] and applications

of the theorem of Găvruta [4] to the study of automorphism on C^* -algebras (see pioneering paper of [20]). The present paper has potential applications in these areas.

Funding: This work was supported by a grant of the Romanian Ministry of Research and Innovation, project number 10PFE/16.10.2018, PERFORM-TECH-UPT - “The increasing of the institutional performance of the Polytechnic University of Timișoara by strengthening the research, development and technological transfer capacity in the field of “Energy, Environment and Climate Change”, within Program 1 - Development of the national system of Research and Development, Subprogram 1.2 - Institutional Performance - Institutional Development Projects - Excellence Funding Projects in RDI, PNCDI III”.

Acknowledgments: We want to thank the referees and the academic editor for their careful reading and valuable comments and remarks on the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience Publ.: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Soc. USA* **1941**, *27*, 222–224. [[CrossRef](#)] [[PubMed](#)]
3. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [[CrossRef](#)]
4. Găvruta, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436. [[CrossRef](#)]
5. Kim, G.H. On the stability of functional equations with square-symmetric operation. *Math. Ineq. Appl.* **2001**, *4*, 257–266. [[CrossRef](#)]
6. Gselmann, E.; Szász, A. An instructive treatment of a generalization of Găvruta’s stability theorem. *Sarajevo J. Math.* **2010**, *6*, 3–21.
7. Brzdęk, J.; Popa, D.; Raşa, I.; Xu, B. *Ulam Stability of Operators*; Academic Press: Cambridge, MA, USA, 2018.
8. Cho, Y.-J.; Park, C.; Rassias, T.M.; Saddati, R. *Stability of Functional Equations in Banach Algebras*; Springer International Publishing: Cham, Switzerland, 2015.
9. Czerwik, S. *Stability of Functional Equations of Ulam-Hyers-Rassias Type*; Hadronic Press, Inc.: Palm Harbor, FL, USA, 2003.
10. Hyers, D.H.; Isac, G.; Rassias, T.H. *Stability of Functional Equations in Several Variables*; Birkhäuser: Basel, Switzerland, 1998.
11. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer Science+Business Media: New York, NY, USA, 2011.
12. Kleiner, B.; Leeb, B. Rigidity of quasi-isometries for symmetric spaces Euclidean buildings. *Inst. Hautes Etudes Sci. Publ. Math.* **1997**, *86*, 115–197. [[CrossRef](#)]
13. Cădariu, L.; Găvruta, L.; Găvruta, P. Fixed points and generalized Hyers-Ulam stability. *Abstr. Appl. Anal.* **2012**, *2012*, 712743. [[CrossRef](#)]
14. Brzdęk, J.; Cădariu, L.; Cieplinski, K. Fixed point theory and the Ulam stability. *J. Funct. Spaces* **2014**, *2014*, 829419. [[CrossRef](#)]
15. Găvruta, L.; Găvruta, P. Approximation of functions by additive and by quadratic mappings. In *Mathematical Analysis, Approximation Theory and Their Applications, Book Series: Springer Optimization and Its Application*; Rassias, T.M., Gupta, V., Eds.; Springer International Publishing: Berlin, Germany, 2016; Volume 111, pp. 281–292.
16. Găvruta, P.; Manolescu, L. Approximation by cubic mappings. In *Ulam Type Stability*; Brzdęk, J., Popa, D., Rassias, T.M., Eds.; Springer International Publishing: Berlin, Germany, 22 December 2019; ISBN 978-3-030-28971-3.
17. Kada, O.; Suzuki, T.; Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Math. Japonica* **1996**, *44*, 381–391.
18. Gajda, Z. On stability of additive mappings. *Int. Math. Math. Sci.* **1991**, *14*, 431–434. [[CrossRef](#)]

19. Kiwi, M.; Magniez, F.; Santha, M. Approximate testing with error relative to input size. *J. Comput. Syst. Sci.* **2003**, *66*, 371–392. [[CrossRef](#)]
20. Park, C. On an approximate automorphism on a C^* -algebra. *Proc. Am. Math. Soc.* **2004**, *132*, 1739–1745. [[CrossRef](#)]

© 2020 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).