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Position Dependent Planck's Constant in a Frequency-Conserving Schrödinger Equation

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Abstract: There is controversial evidence that Planck's constant shows unexpected variations with altitude above the earth due to Kentosh and Mohageg, and yearly systematic changes with the orbit of the earth about the sun due to Hutchin. Many others have postulated that the fundamental constants of nature are not constant, either in locally flat reference frames, or on larger scales. This work is a mathematical study examining the impact of a position dependent Planck's constant in the Schrödinger equation. With no modifications to the equation, the Hamiltonian becomes a non-Hermitian radial frequency operator. The frequency operator does not conserve normalization, time evolution is no longer unitary, and frequency eigenvalues can be complex. The wavefunction must continually be normalized at each time in order that operators commuting with the frequency operator produce constants of the motion. To eliminate these problems, the frequency operator is replaced with a symmetrizing anti-commutator so that it is once again Hermitian. It is found that particles statistically avoid regions of higher Planck's constant in the absence of an external potential. Frequency is conserved, and the total frequency equals "kinetic frequency" plus "potential frequency". No straightforward connection to classical mechanics is found, that is, the Ehrenfest's theorems are more complicated, and the usual quantities related by them can be complex or imaginary. Energy is conserved only locally with small gradients in Planck's constant. Two Lagrangian densities are investigated to determine whether they result in a classical field equation of motion resembling the frequency-conserving Schrödinger equation. The first Largrangian is the "energy squared" form, the second is a "frequency squared" form. Neither reproduces the target equation, and it is concluded that the frequency-conserving Schrödinger equation may defy deduction from field theory.

Keywords: Planck's constant; Variable Planck's constant; non-Hermitian operators; Schrödinger equation

1. Introduction

The possibility of the variation of fundamental constants would impact all present physical theory, while all reported variations or interpretations of data concluding a constant has varied are extremely controversial. Examples of work in this area include Dirac's Large Number Hypotheses [1], the Oklo mine from which could be extracted a variation of the fine structure constant [2,3], and the observations of quasars bounding the variation of the latter per year to one part in 10^{17} [4–6]. Recent theoretical work includes the impact of time dependent stochastic fluctuations of Planck's constant [7], and the changes with Planck's constant on mixed quantum states [8]. An authoritative review of the status of the variations of fundamental constants is given in [9].

Publicly available Global Positioning System (GPS) data was used to attempt to confirm the Local Position Invariance (LPI) of Planck's constant under General Relativity [10,11]. LPI is a concept from General Relativity, where all local non-gravitational experimental results in freely falling reference frames should be independent of the location that the experiment is performed in. That foundational rule should hold when the fundamental physical constants are not dependent on the location. If the fundamental constants vary over larger scales than a locally flat frame, but their changes are small locally, then it is the form of the physical laws that should be the same in all locations.

The LPI violation parameter due to variations in Planck's constant is called β_h. The fractional variation of Planck's constant is proportional to the gravitational potential difference and β_h. The value found in [10] for variations in Planck's constant was $|\beta_h| < 0.007$. This parameter is not zero, and is the largest of the violation parameters extracted in the study. The study did not report on the altitude dependence of Planck's constant above the earth. A very recent study involving the Galileo satellites found that GR could explain the frequency shift of the onboard hydrogen maser clocks to within a factor of $(4.5 \pm 3.1) \times 10^{-5}$ [12], improved over Gravity Probe A in 1976 of $\sim 1.4 \times 10^{-4}$, these are the α_{rs} redshift violation values that may be compared to β_h.

Consistent sinusoidal oscillations in the decay rate of a number of radioactive elements with periods of one year taken over a 20-year span has been reported [13–18]. These measurements were taken by six organizations on three continents. As both the strong and weak forces were involved in the decay processes, and might be explainable by oscillations of $\hbar$ influencing the probability of tunneling, an all electromagnetic experiment was conducted, designed specifically to be sensitive to Planck's constant variations [19]. Consistent systematic sinusoidal oscillations of the tunneling voltage of Esaki diodes with periods of one year were monitored for 941 days. The tunnel diode oscillations were attributed to the combined effect of changes in the WKB tunneling exponent going as $\hbar^{-1}$, and changes in the width of the barrier going as $\hbar^2$. The electromagnetic experiment voltage oscillations were correctly predicted to be 180 degrees out of phase with the radioactive decay oscillations. This data can be made available for independent analysis by requesting it from the author of [19].

It is reasonable to suspect that the oscillations of decay rates and tunnel diode voltage are related to the relative position of the sun to the orbiting earth, and that there are resulting oscillations in Planck's constant due to position dependent gravitational effects, or effects with proximity to the sun. It should be mentioned that there have been studies in which it was concluded there was no gravitational dependence to the decay rate oscillations [20,21]. There is also dispute in the literature concerning the reality of the decay rate oscillations [22–24].

Either way, whether by gravitation or by some other mechanism, for the work to be presented, all that matters is that there be a position dependent $\hbar$, and it would be of value to understand the impact on the fundamentals of quantum mechanics and the Schrödinger equation under such a condition, and where conservation of frequency as opposed to energy will be explored as a means to retain Hermitivity.

For the treatment of $\hbar$ in this paper, it is important to emphasize is not as a dynamical field, and leads to energy non-conservation. In another paper by this author, variations in $\hbar$ are treated as a scalar dynamical field, coupling to fields through the derivative terms in the Lagrangian density [25], and the energy is shared between the fields. One of the solutions of [25] suggests that frequency may be a more fundamental dynamical variable than energy, leading to the idea of frequency conservation in this paper, where it arises naturally. This paper concerns issues specific to the Schrödinger equation in a single-particle, non-field theoretic framework, however. In Appendix A of this paper, an attempt will be made to derive a classical field equation of motion (the Schrödinger field) resembling the frequency conserving Schrödinger wavefunction equation developed in the body of the paper, from two Lagrangian densities. The attempt will not be successful.

Variations in $\hbar$ or any fundamental constant may be explainable by treatment as dynamical fields. On the otherhand, they may not be, especially where the spatial dependence is concerned, because there is so little experimental data on the subject. Noone knows with certainty whether they actually are dynamical fields, or not, though much work has been done representing some of them as dynamical fields: Jordan-Brans-Dicke scalar-tensor theory with variable G developed in the late 1950's and early 1960's and note that G is dimensionful; Bekenstein models with variable fine structure constant introduced in 1982 [26,27]; the Cosmon of Wetterich with a field dependent pre-factor to the dynamical terms functioning somewhat like Planck's constant [28,29], falling to a constant value at high fields; the investigations of Albrecht, Magueijo, Moffat, and Barrow on variable c used towards the explanation of the flatness, horizon, homogeneity, and cosmological constant problems [30–33]. For example,

$$\begin{array}{l} S_{GR} = \int \left\{ \begin{array}{c} \frac{(c_o\mathbb{C})^4}{16G_o\pi}\xi R + \\ \frac{(\hbar_o\psi)^2}{2} g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + \frac{(\hbar_o\psi)^2}{2}\frac{w}{\xi} g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\xi + \frac{(\hbar_o\psi)^2}{2} g^{\mu\nu}\nabla_\mu\mathbb{C}\nabla_\nu\mathbb{C} \\ +\lambda\xi\psi\mathbb{C}R + L_m\{\hbar, c, G\} \end{array} \right\} \sqrt{-g}d^4x \\ \frac{\partial}{\partial x_o} = \frac{1}{c_o\mathbb{C}}\frac{\partial}{\partial t} \\ c = c_o\mathbb{C} \\ \hbar = \hbar_o\psi \\ G = G_o/\xi \end{array} \tag{i}$$

Equation (i) shows in a single form an amalgam of possible couplings including a Jordan-Brans-Dicke-like scalar-tensor theory of alternative General Relativity with variable G, an Albrecht-Magueijo-Barrow-Moffat-like field for c, a field for $\hbar$ like that of [26], which is different than the form of Bekenstein's for variable e^2 whose representative field squared divided the derivative terms. There is also the field theory of Modified Gravity (MOG) of Moffat, and the Tensor-Vector-Scalar (TeVeS) gravity of Bekenstein. There are many ways all the constants might be represented as fields, and many ways they might be coupled. Coupling fields together in this way is the accepted approach for the treatment of a constant, but is not the only possible approach, and here, something different will be tried.

What is to follow serves as a starting point for investigating what happens to the most familiar equations in physics, if Planck's constant variations are that of a fixed-background parameter and not a field, and so there is no energy exchange between fields conserving the total. Instead, frequency conservation is explored, and energy is intentionally not conserved. In [34] it will be shown that energy non-conservation leads to a possible explanation of the NASA Flyby Anomaly.

2. Derivation of the Expectation Value Time Derivative

The time derivative of expectation values for a position dependent Planck's constant will be derived. No modification will be made to the form of the Schrödinger equation in this section, and the purpose is to make clear the difficulties that arise, and the special conditions that would have to be imposed on the wavefunction and Planck's constant to maintain the basic framework of quantum mechanics. Then, a modification will be suggested.

Begin with the time-dependent Schrödinger equation in which Planck's constant is allowed to be position dependent, and real,

$$i\hbar(\bar{r})\frac{\partial\psi_u(\bar{r},t)}{\partial t} = -\frac{\hbar^2(\bar{r})}{2m}\nabla^2\psi_u(\bar{r},t) + V(\bar{r})\psi_u(\bar{r},t) \tag{1}$$

The subscript u indicates that the wavefunctions are un-normalized over space at any given time. To separate the time and position variables, divide both sides by $\hbar$,

$$i\frac{\partial\psi_u(\bar{r},t)}{\partial t} = -\frac{\hbar(\bar{r})}{2m}\nabla^2\psi_u(\bar{r},t) + \frac{V(\bar{r})}{\hbar(\bar{r})}\psi_u(\bar{r},t) \tag{2}$$

Let

$$\psi_u(\bar{r},t) = S_u(\bar{r})\varphi(t) \tag{3}$$

Substituting (3) into (2) and dividing both sides by (3) gives,

$$\frac{i}{\varphi(t)}\frac{\partial\varphi(t)}{\partial t} = -\frac{\hbar(\bar{r})}{2m}\frac{1}{S_u(\bar{r})}\nabla^2 S_u(\bar{r}) + \frac{V(\bar{r})}{\hbar(\bar{r})} = \omega \tag{4}$$

where ω is the constant of separation with units of frequency. The left-hand side of (4) has the solution,

$$\varphi(t) = e^{-i\omega t} \tag{5}$$

and the right-hand side of (4) becomes

$$-\frac{\hbar(\bar{r})}{2m}\nabla^2 S_u(\bar{r}) + \frac{V(\bar{r})}{\hbar(\bar{r})} S_u(\bar{r}) = \omega S_u(\bar{r}) \tag{6}$$

Defining the frequency operator F,

$$\hat{F} = -\frac{\hbar(\bar{r})}{2m}\nabla^2 + \frac{V(\bar{r})}{\hbar(\bar{r})} \tag{7}$$

Switching to the Dirac notation, Equation (2) becomes

$$\begin{aligned} i\tfrac{\partial\psi_u(\bar{r},t)}{\partial t} &= \hat{F}\psi_u(\bar{r},t) \rightarrow \\ i\tfrac{\partial|\psi_u(t)\rangle}{\partial t} &= \hat{F}|\psi_u(\bar{r},t)\rangle \end{aligned} \tag{8}$$

Taking the complex conjugate of (8),

$$\begin{aligned} -i\tfrac{\partial\psi_u^*(\bar{r},t)}{\partial t} &= \left(\hat{F}\psi_u(\bar{r},t)\right)^* = \hat{F}^*\psi_u^*(\bar{r},t) = \hat{F}\psi_u^*(\bar{r},t) \rightarrow \\ -i\tfrac{\partial\langle\psi_u(\bar{r},t)|}{\partial t} &= \langle\hat{F}\psi_u(\bar{r},t)| = \langle\psi_u(\bar{r},t)|\hat{F}^\dagger \end{aligned} \tag{9}$$

where the superscript †designates the adjoint operator acting to the right. The frequency operator is not Hermitian, noted from writing out in integral form the problematic part,

$$\begin{aligned} &\int \left(\hbar(\bar{r})\nabla^2\psi\right)^*_{\bullet}\psi d^3r = \int \nabla^2\psi^*_{\bullet}\psi\hbar(\bar{r})d^3r = \int \psi^*\nabla^2(\psi\hbar(\bar{r}))d^3r \\ &\neq \int \psi^*(\hbar(\bar{r})\nabla^2)\psi d^3r \end{aligned} \tag{10}$$

where the lower "." indicates where the operator stops operating. The Hermiticity of the Laplacian has been used in (10), derivable by the use of Green's second identity in the second to the third step, as long as products of $\hbar\psi$, and ψ vanish at the boundary at infinity. The fourth step is what the answer would need to be in order to be Hermitian. Therefore, the frequency operator is non-Hermitian,

$$\hat{F}^\dagger \neq \hat{F} \tag{11}$$

As a result, the normalization will not be conserved, and the frequency eigenvalues may be complex or imaginary. The rate of change of expectation values can now be derived using (8) and (9). The expectation value of an operator is,

$$\langle\hat{A}\rangle = \frac{\langle\psi_u(\bar{r},t)|\hat{A}|\psi_u(\bar{r},t)\rangle}{\langle\psi_u(\bar{r},t)|\psi_u(\bar{r},t)\rangle} = \frac{\langle\hat{A}\rangle_u}{\langle 1\rangle_u} \tag{12}$$

where the denominator is the normalization, and normalization is redone continually for all times. Differentiating (12) with respect to time,

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{\langle 1\rangle_u\partial_t\langle\hat{A}\rangle_u - \partial_t\langle 1\rangle_u\langle\hat{A}\rangle_u}{\langle 1\rangle_u^2} \tag{13}$$

Working out the numerator of (13) and then using (8) and (9),

$$\begin{aligned}
&\partial_t\langle\hat{A}\rangle_u = \partial_t\langle\psi_u(\bar{r},t)|_{\bullet}\hat{A}|\psi_u(\bar{r},t)\rangle + \langle\partial_t\hat{A}\rangle_u + \langle\psi_u(\bar{r},t)|\hat{A}\partial_t|\psi_u(\bar{r},t)\rangle = \\
&\tfrac{1}{i}\langle\psi_u(\bar{r},t)|\hat{A}\hat{F} - \hat{F}^{\dagger}\hat{A}|\psi_u(\bar{r},t)\rangle + \langle\partial_t\hat{A}\rangle_u
\end{aligned} \tag{14}$$

Therefore, from (14), the rate of change of the normalization is

$$\partial_t\langle 1\rangle_u = \frac{1}{i}\langle\psi_u(\bar{r},t)|\hat{F} - \hat{F}^{\dagger}|\psi_u(\bar{r},t)\rangle \tag{15}$$

Substituting (14) and (15) into (13) would give the full-time dependence of the operator A, but this can be written in a cleaner way showing the extra terms that do not show up in normal quantum mechanics. To that end, remembering that F is real,

$$\hat{F}^{*} = \hat{F} \tag{16}$$

so (14) in integral form is

$$\partial_t\langle\hat{A}\rangle_u = -\frac{1}{i}\int \hat{F}\psi_u^{*}{}_{\bullet}\hat{A}\psi_u - \psi_u^{*}\hat{A}F\psi_u d^3r + \langle\partial_t\hat{A}\rangle_u \tag{17}$$

Writing out the first term of (17), and of that, only the part containing the non-Hermitian portion of the frequency operator,

$$\int \hbar(\bar{r})\nabla^2\psi_u^{*}{}_{\bullet}\hat{A}\psi_u d^3r = \int \nabla^2\psi_u^{*}{}_{\bullet}\hbar(\bar{r})\hat{A}\psi_u d^3r = \int \psi_u^{*}\nabla^2\Big(\hbar(\bar{r})\hat{A}\psi_u\Big)d^3r \tag{18}$$

where on going from the second to the third part in (18), Green's second identity was used again with ψ and $\hbar A\psi$ vanishing at the boundary at infinity. Note,

$$\nabla^2\Big(\hbar(\bar{r})\hat{A}\psi_u\Big) = \hbar(\bar{r})\nabla^2\Big(\hat{A}\psi_u\Big) + \hat{A}\psi_{u\bullet}\nabla^2\hbar(\bar{r}) + 2\nabla\hbar(\bar{r})\bullet\nabla(\hat{A}\psi_u) \tag{19}$$

where the large dot between the gradients is the vector dot product. Equation (19) allows (17) to be written as a commutation relationship with extra terms. Defining the functional,

$$I[\hat{A}] = -\frac{i}{2m}\int \psi_u^{*}\Big\{\hat{A}\psi_{u\bullet}\nabla^2\hbar(\bar{r}) + 2\nabla\hbar(\bar{r})\bullet\nabla(\hat{A}\psi_u)\Big\}d^3r \tag{20}$$

$$\partial_t\langle\hat{A}\rangle_u = \langle i[\hat{F},\hat{A}] + \partial_t\hat{A}\rangle_u + I[\hat{A}] \tag{21}$$

and from (20) follows the time dependence of the normalization,

$$\partial_t\langle 1\rangle_u = I[1] = -\frac{i}{2m}\int \psi_u^{*}\Big\{\psi_{u\bullet}\nabla^2\hbar(\bar{r}) + 2\nabla\hbar(\bar{r})\bullet\nabla(\psi_u)\Big\}d^3r \tag{22}$$

Combining (20), (21) and (13), the result is,

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{\langle i[\hat{F},\hat{A}] + \partial_t\hat{A}\rangle_u}{\langle 1\rangle_u} + \frac{\langle 1\rangle_u I[\hat{A}] - \langle\hat{A}\rangle_u I[1]}{\langle 1\rangle_u^2} \tag{23}$$

The second term of (23) appears because F is not Hermitian, and were it not there, (23) would look like the result of normal quantum mechanics.

3. Time Evolution Operator under *F*

Time evolution is no longer unitary. From (5) it is inferred that the time evolution operator is

$$\hat{U} = \exp(-i\hat{F}t) \tag{24}$$

and its adjoint is

$$\hat{U}^{\dagger} = \exp(i\hat{F}^{\dagger}t) \tag{25}$$

Therefore,

$$\langle\hat{A}\rangle_u = \langle\psi_u(\bar{r},t)|\hat{A}|\psi_u(\bar{r},t)\rangle = \langle\psi_u(\bar{r},0)|\hat{U}^{\dagger}\hat{A}\hat{U}|\psi_u(\bar{r},0)\rangle \tag{26}$$

and for the normalization,

$$\langle 1\rangle_u = \langle\psi_u(\bar{r},t)|\psi_u(\bar{r},t)\rangle = \langle\psi_u(\bar{r},0)|\hat{U}^{\dagger}\hat{U}|\psi_u(\bar{r},0)\rangle \tag{27}$$

Since $F \neq F^{\dagger}$, it is seen that $U^{\dagger} \neq U^{-1}$, the normalization is not conserved noting (27), and from (26) for the non-normalized wavefunctions, the expectation values of A are not constants of the motion even if A commutes with F (and therefore U).

4. Result for Expectation Values of Operators Commuting with the Frequency Operator *F*

If A commutes with F then from (20),

$$I[\hat{A}] = aI[1] \tag{28}$$

and from (26) and (27),

$$\langle\hat{A}\rangle_u = a\langle 1\rangle_u \tag{29}$$

and substitution of (28) and (29) into (23) gives that the expectation value time derivative of the operator A is zero. For the non-Hermitian F operator, this result only holds because of the continual normalization procedure at each time.

5. Symmetrized Hermitian Frequency Operator F_h and modified Schrödinger Equation

The basic framework of quantum mechanics is disturbed without modification to the Schrödinger equation for a position dependent $\hbar$, or by imposing special conditions of some sort. Inspecting (20) and (22), one might consider special conditions on the forms of $\hbar$ or ψ so the additional terms are zero, and the operator becomes "effectively Hermitian". It is worth mentioning there is ongoing work on non-Hermitian and complex Hamiltonians being used to describe dissipative and open systems [35,36]. There is also work on complex non-Hermitian Hamiltonians with PT-symmetry that produce real eigenvalues [37,38].

Looking at (14), unusual symmetries or operators such that $AF = F^{\dagger}A$ might also be tried. It was shown in [39] that such a symmetry results in an expectation value that changes with time in inverse proportion to the wavefunction normalization, while the latter is not conserved noting (15).

Instead, to rectify the problems thus far mentioned, without exotic conditions or symmetries, to retain the property that a dynamical variable is a constant of the motion when its operator commutes with the frequency operator, and that normalization be conserved so the wavefunction has a probabilistic interpretation, a modified symmetrical form of F is proposed.

For Hermitian operators P and Q the product operator PQ is not Hermitian unless they commute. However, two symmetrized operators that are Hermitian, are,

$$i[\hat{P},\hat{Q}] \tag{30}$$

$$\hat{P}\hat{Q} + \hat{Q}\hat{P} = \left\{\hat{P},\hat{Q}\right\} \tag{31}$$

Of the two candidates for symmetrizing the non-Hermitian product of Hermitian operators $\hbar(r)$ and ∇^2,

$$\frac{\hbar(\bar{r})\nabla^2 + \nabla^2\hbar(\bar{r})}{2} = \frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\} \tag{32}$$

is the one reducing to the standard Schrödinger equation for constant $\hbar$. Therefore, the symmetrized equation proposed is,

$$i\frac{\partial\psi_u(\bar{r},t)}{\partial t} = -\frac{1}{2m}\frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\}\psi_u(\bar{r},t) + \frac{V(\bar{r})}{\hbar(\bar{r})}\psi_u(\bar{r},t) = \hat{F}_h\psi_u(\bar{r},t) \tag{33}$$

$$\hat{F}_h = -\frac{1}{2m}\frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\} + \frac{V(\bar{r})}{\hbar(\bar{r})} \tag{34}$$

The time dependence of the wavefunction is still given by (5), and the spatial component on separation becomes

$$\hat{F}_h S_u(\bar{r}) = \omega S_u(\bar{r}) \tag{35}$$

The general principles and framework of quantum mechanics is then restored, with the difference being the Hamiltonian is replaced with the symmetrized frequency operator. The previously problematic relations become much more like normal quantum mechanics, namely

$$\partial_t\langle\hat{A}\rangle_u = \langle i[\hat{F}_h, \hat{A}] + \partial_t\hat{A}\rangle_u \tag{36}$$

$$\partial_t\langle 1\rangle_u = 0 \tag{37}$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{\partial_t\langle\hat{A}\rangle_u}{\langle 1\rangle_u} \tag{38}$$

$$\hat{U}_h = \exp(-i\hat{F}_h t) \tag{39}$$

$$\hat{U}_h^\dagger = \exp(i\hat{F}_h t) \tag{40}$$

Since $F_h = F_h{}^\dagger$, it is seen that $U_h{}^\dagger = U_h{}^{-1}$, time evolution is unitary, and the normalization is now again conserved,

$$\langle 1\rangle_u = \langle\psi_u(\bar{r},t)|\psi_u(\bar{r},t)\rangle = \langle\psi_u(\bar{r},0)|\hat{U}_h^\dagger\hat{U}_h|\psi_u(\bar{r},0)\rangle = \langle\psi_u(\bar{r},0)|\psi_u(\bar{r},0)\rangle \tag{41}$$

6. Free Particles under F_h

Since,

$$-\frac{1}{2m}\frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\}S(\bar{r}) = \hat{W}_h S(\bar{r}) = \omega S(\bar{r}) \tag{42}$$

the spatial part of the free-particle wavefunction depends explicitly on the attributes of Planck's constant. The free particle frequency operator W_h is introduced. The wavefunction time dependence is still given by (5); however, the spatial wavefunction of a free particle is not of the usual form,

$$\exp\left(i\bar{k}\cdot\bar{r}\right) \tag{43}$$

A simple but illustrative case will demonstrate the interesting feature that the particle tends to statistically avoid regions of higher $\hbar$. Consider a slight linear gradient in $\hbar$. In one dimension, the free particle wave equation with $V = 0$ becomes,

$$-\frac{1}{2m}\left[\hbar\partial_x^2 + \frac{1}{2}(\partial_x^2\hbar) + (\partial_x\hbar)\partial_x\right]S(x) = \omega S(x) \tag{44}$$

where the parentheses "( )" indicate that the enclosed derivative operations stop on $\hbar$, and do not operate on $S(x)$. For the simplest position dependent Planck's constant,

$$\hbar(x) = \hbar_o + \eta x \tag{45}$$

it is found that,

$$(\hbar_o + \eta x)\partial_x^2 S + \eta \partial_x S + 2m\omega S = 0 \tag{46}$$

The interest is in solutions for $\eta > 0$, and for simplicity in regions where $\eta x/\hbar_o << 1$, so the ηx in the first term of (46) can be dropped. An oscillating solution will be investigated. The result is a second order homogeneous differential equation with solution,

$$S(x) = \exp\left(\frac{-\eta x}{2\hbar_o}\right)(c_1 \exp(ikx) + c_2 \exp(-ikx)) \tag{47}$$

where

$$k = \frac{|\eta^2 - 8m\omega\hbar_o|^{1/2}}{2\hbar_o} \tag{48}$$

and $\eta^2 < 8m\omega\hbar_o$, where the exponential terms can sum to $cos(kx)$ or $sin(kx)$ depending on the boundary conditions, resulting in quantization of frequency in the usual way, by restriction of the allowed values of k.

One sees from (47) that for very small gradients in $\hbar$ the normal free particle solution exp (ikx) is approximated. The wavefunction is concentrated near the region of smaller $\hbar$. A well-defined wavenumber appears, but only as a consequence of the small gradient in $\hbar$. Even though there is no external potential, the particle is not "free" in the usual sense, since the gradient in $\hbar$ plays a role in positioning it. If the particle energy can still be defined as $E = \hbar\omega$, the particle is most likely to be found in regions where its energy is lowest.

The full general solution, retaining the ηx so that the changes in $\hbar$ can become larger is

$$S(x) \propto c_1 \sqrt{\frac{2}{\eta}} I_o\left(i\sqrt{\frac{8m\omega(\hbar_o + \eta x)}{\eta^2}}\right) + c_2 \sqrt{\frac{2}{\eta}} K_o\left(i\sqrt{\frac{8m\omega(\hbar_o + \eta x)}{\eta^2}}\right) \tag{49}$$

where I_o and K_o are the modified Bessel functions of the first and second kind, oscillating functions with a decay envelope. The first term of (49) is the relevant one, as it has no divergences. Noting the square root in the argument containing x, there is not a clearly definable constant wavenumber despite that the particle is "free". Using $I_o\ (iz^{1/2}) = J_o\ (z^{1/2})$ is found the Bessel function of the first kind. For a particle in a box, the infinite sidewall positions must be located such that $L_{1,2} \geq -\hbar_o/\eta$, so that $\hbar$ is positive. The wavefunctions are then concentrated on the low Planck's constant side of the box, decaying to the right of the leftmost sidewall. For quantization, the relation between the frequency and the two of the zeroes of the Bessel function $Z\ [J_o]$ is,

$$\omega_n = \frac{\eta^2 Z_n^2 [J_o]^{1,2}}{8m(\hbar_o + \eta L_{1,2})} \tag{50}$$

which must be solved numerically. The overall form of (49) is shown in Figure 1.

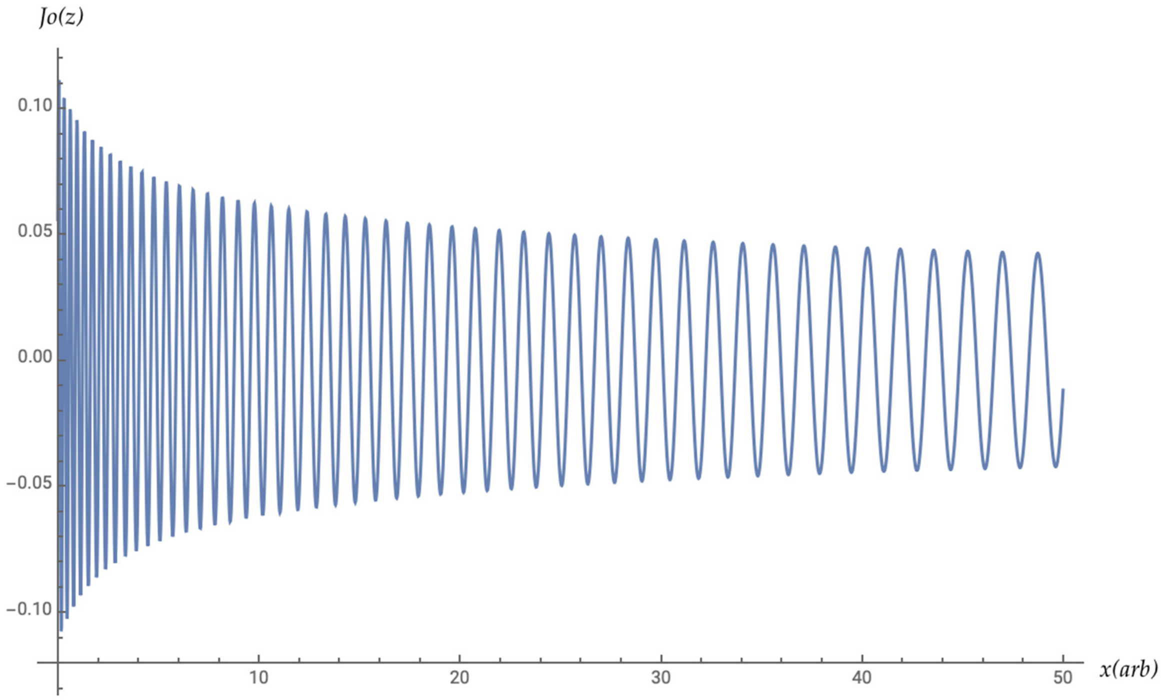

Figure 1. Plot of the overall form Equation (49), demonstrating that the wavefunction amplitude increases when Planck's constant is lower. Planck's constant increases with increasing x position. The argument of the Bessel function is $z = 50(1+x)^{1/2}$.

7. Lack of Conservation of Energy, Momentum, and Ehrenfest's Theorems under F_h

It is not a surprise, given the loss of translational symmetry in the absence of a potential, that momentum should not be conserved, per the results of Noether. In addition, energy is also not conserved, stemming from the lack of a Lagrangian, and action, whose variation could lead to (33) and (34). To continue the analysis, it is easiest to use the most basic methods of quantum mechanics.

Using (36), (38) and writing $V/\hbar = F_h - W_h$ one sees that,

$$\frac{d\langle V(\bar{r})/\hbar(\bar{r})\rangle}{dt} = \langle i[\hat{F}_h, V/\hbar]\rangle = -\frac{d\langle \hat{W}_h(\bar{r})\rangle}{dt} \tag{51a}$$

So that $V/\hbar$ is "potential frequency" and W_h is "kinetic frequency" acting together to conserve total frequency as the particle moves. Energy is not conserved now, and in addition, even if the particle is free, the momentum is also not conserved, both changing value with position in the absence of an external potential. Frequency, however, is conserved. Changes in $V/\hbar$ from a starting to an ending position is the frequency equivalent of work done on or by the system.

On examining the free particle operator W_h, this author is unable to identify a simple operator for momentum. In light of (51a), a possible momentum operator is (51b),

$$\hat{p}_\hbar = \frac{1}{i}\sqrt{\frac{1}{2}\{\hbar(\bar{r}), \nabla^2\}} = \frac{\hat{\tilde{p}}}{\sqrt{\hbar(\bar{r})}} \tag{51b}$$

Although the square root operator is difficult to work with, it could be definable in terms of Fourier transforms. Lacking an operator for the momentum implies there is no relation equivalent to Newton's first and second laws between expectation values as there is in normal quantum mechanics with Ehrenfest's theorems. An attempt at a connection with normal quantum mechanics is made by borrowing its momentum operator, but now with a position dependent $\hbar$,

$$\hat{p}_x = \frac{\hbar(x)}{i}\partial_x \tag{52}$$

from which can be defined a wavenumber operator,

$$\hat{k}_x = \frac{1}{i}\partial_x \tag{53}$$

An infinitesimal displacement operator can be defined as

$$\hat{D}_\varepsilon = 1 + i\varepsilon\hat{k}_x \tag{54}$$

By inspection, the free particle operator W_h is not generally invariant to the infinitesimal displacements owing to $\hbar(x)$, therefore,

$$[\hat{W}_h, \hat{D}_\varepsilon] = i\varepsilon[\hat{W}_h, \hat{k}_x] \neq 0 \tag{55}$$

So neither momentum or wavenumber are conserved by the definitions of normal quantum mechanics by this symmetry argument, for a free particle.

Moreover,

$$\frac{d\langle\hat{x}\rangle}{dt} = \langle i[\hat{F}_h, x]\rangle = \langle i[\hat{W}_h, x]\rangle = \frac{\langle\hat{p}_x\rangle}{m} + \frac{\langle(\partial_x\hbar)\rangle}{2mi} \tag{56}$$

While (56) looks simple enough, the first term is complex, and the second term is always imaginary. It has not been shown whether the imaginary parts of (56) generally exactly cancel for any arbitrary choice of $\hbar$. For the free particle of (47) with its very mild $\hbar$ gradient all is per the norm, as the imaginary terms that result in (56) do exactly cancel, and the righthand side equals $\hbar_o k/m$. For the particle in a box with a slight $\hbar$ gradient of (49) and full solution, it has not been shown that all eigenstates lead to a real result for (56).

For forces,

$$\frac{d\langle\hat{p}_x\rangle}{dt} = \langle i[\hat{F}_h, \hat{p}_x]\rangle = \langle i[\hat{W}_h, \hat{p}_x]\rangle - \langle\hbar(\partial_x(V/\hbar))\rangle = \left.\frac{d\langle\hat{p}_x\rangle}{dt}\right|_{free} - \langle\hbar(\partial_x(V/\hbar))\rangle \tag{57}$$

Equations (56) and (57) reduce to the normal Ehrenfest's theorems for constant $\hbar$, but do not appear like them, otherwise.

So, while particle frequencies are conserved, and local energies, probabilities of particle location, and average values of quantities can all be computed and are real, there seems to be no assured connection with classical dynamics. The position expectation value time derivative being complex or imaginary is difficult to interpret. Consider an analogy in classical mechanics, where a particle sits at the bottom of the harmonic oscillator potential with zero energy and velocity. Integrating the equations of motion, one finds for the velocity $v = (-kx^2/m)^{1/2}$. If the particle is then suddenly found at any position other than $x = 0$ with no source of energy, the particle velocity is imaginary, and the magnitude of the imaginary velocity tells you the extent of the energy non-conservation.

This model may be producing complex position expectation value time derivatives, generally. With a conserved frequency and a position dependent $\hbar$, this suggests $\hbar$ is a minimum at some position in space that serves as the reference of lowest energy, meanwhile the particle wavefunctions may extend to locations where $\hbar$ and energy are larger. Then, the particle has a finite probability to be observed in both high and low energy locations. Complex values of (56) and (57) signify that the particle is forbidden to be there in classical mechanics, and normal quantum mechanics, but is there anyway.

The lack of conservation of energy, while something that is difficult to accept based on the heritage of its use as a guiding law, is not yet a reason to abandon a model. The uncertainty principle, virtual mediating particles, conservation of energy only in locally flat frames in GR, lack of conservation of energy in dynamic spacetimes, and the cosmological constant all attest.

8. Average Value of $\hbar$ under F_h

$$\langle\hbar\rangle = \langle S(\bar{r})|\hbar(\bar{r})|S(\bar{r})\rangle \tag{58}$$

This equation underscores the importance of the position dependence of Planck's constant only over the extent of the substantially non-zero areas of the wavefunction. If Planck's constant does not vary greatly over this region, it may be treated as a constant.

9. Time Dependence of the Expectation Value of $\hbar$ under F_h

As $V/\hbar$ and W_h take up total conserved frequency between them, it is interesting to see if there is a simple quantity taken up by $\hbar$ distinctly. That is, what quantity is stored in $\hbar$? Since F_h and $\hbar$ do not commute,

$$\frac{d}{dt}\langle\hbar\rangle = \langle i[\hat{F}_h, \hbar]\rangle = \langle(-i/4m)[\nabla^2, \hbar^2]\rangle \neq 0 \tag{59}$$

The spatial dependence of Planck's constant would give rise to a temporal dependence as the particle moves through the $\hbar$ field, but there is no simple quantity working in tandem with $\hbar$ to conserve another constant of the motion, generally.

However, in the case where the external potential is constant and non-zero, (51a) shows that $\hbar^{-1}$ becomes the "potential frequency".

10. Indeterminacy of $\hbar$ under F_h

For non-commuting Hermitian operators P and Q, the indeterminacy relationship between them is,

$$\Delta P \Delta Q \geq \frac{1}{2}|\langle -i[\hat{P}, \hat{Q}]\rangle| \tag{60}$$

Since F_h and $\hbar$ do not commute but are Hermitian,

$$\Delta F_h \Delta\hbar \geq \frac{1}{2}|\langle -i[\hat{F}_h, \hbar]\rangle| = \frac{1}{2}|\langle(i/4m)[\nabla^2, \hbar^2]\rangle| \neq 0 \tag{61}$$

Our ability to know the frequency of the particle and the Planck's constant experienced by it simultaneously is mutually limited.

11. Uncertainty under F_h

Using (60), since it can be shown $[x, p_x] = i\hbar(x)$, it is found that $\Delta p_x \Delta x \geq |\langle\hbar(x)\rangle|/2$. Note that there is an integration over the spatial domain in the latter being performed, or the average of $\hbar$. For frequency and time, it can be seen from the same arguments applied in normal quantum mechanics that,

$$\Delta F_h \Delta Q \geq \frac{1}{2}\left|\frac{d\langle\hat{Q}\rangle}{dt}\right| \tag{62}$$

Multiplying by a time increment,

$$\Delta t \Delta F_h \Delta Q \geq \frac{1}{2}\Delta t\left|\frac{d\langle\hat{Q}\rangle}{dt}\right| \geq \frac{1}{2}\Delta Q \tag{63}$$

is found the uncertainty relationship,

$$\Delta F_h \Delta t \geq \frac{1}{2} \tag{64}$$

Multiplying (64) by a position dependent $\hbar$ gives the more familiar relationship in terms of energy and time, and there is no averaging of $\hbar$.

12. Discussion

Field Theory and General Relativity are the cornerstones of modern physics. There seem to be some inherent contradictions in both theories. For example, in field theory, a static field functions much like the fields as envisioned by Faraday. Yet, a static field can be approximated with the tree-level terms of the perturbative expansion to produce an amplitude, with Feynman diagrams showing particle exchange limiting the interaction to the speed of light, equated to the Born approximation amplitude to produce a classical potential. Propagation of a field would therefore appear to be required for the static field to function. For a black hole, the mass, charge, and angular momentum are not censored: they are communicated by non-propagating modes in field theory, the accepted explanation. Changes in the static fields are propagated at the speed of light, but once reestablished, are Faraday-like static fields once again, influencing instantaneously at a distance. Physical constants would be static fields, and like any static field, described by non-propagating modes, Faraday-like, influencing instantaneously at a distance. When matter or charges gravitate into the event horizon of a black hole, the initial non-propagating modes, understood to have been set up long ago, quickly readjust, to produce the new non-propagating mode. Yet, the despite the censorship of the event horizon, somehow, in the particle picture, particles (fields) must propagate from the event horizon to reset the non-propagating modes – thus emission of particles from the horizon would seem to be needed. The descriptions conflict, despite the predictive power. As for General Relativity, it is perfectly acceptable at this time, that energy and momentum are not conserved when there are dynamical changes in spacetime, although *how* the non-conservation evolves is well understood, with the conservation possible only in a locally flat frame. With a cosmological constant, the total energy of the universe increases (explained by negative pressure). The variation of physical constants throughout the universe may also constitute acceptable violations of conservation laws.

Many persons maintain the position that the measurement of a single dimensioned constant in isolation is not physically meaningful, and the only meaningful measurements to be made are those of dimensionless products of the isolated dimensioned constants. The reason given is that the dimensionless constants are free of units that rely on arbitrary standards, and on calibrations of the metrology tools based on them. Both may be influenced by the variation of the constant itself, and also the measurement always involves multiple mechanisms with which other constants are convolved.

The above philosophy is sound when the metrology tools are located in the same place that the physical constants may be varying in, and only one technique is used for the measurement, and that single device-type is changed by the variation itself, and if the standards on which the calibrations are based are in flux. However, it has not yet been experimentally borne out whether multiple techniques used in coordinated concert in the same location could, or could not, attribute the results of all the techniques to a single isolated dimensionful constant changing. It is also possible that a specific experiment could be devised at some point that is sensitive to only one dimensioned constant and designed not to be disturbed by the constant's variations.

An extreme example is to ask what would happen if Planck's constant doubled in the sun, but not on the earth? Would there then be a discernable effect or not, would it be detectable from the earth, could it be determined that it was Planck's constant that was the single constant that had changed, and would it then be worthwhile to attempt to measure the change in isolation?

Suppose that the spatial variation of a physical constant is very gradual, so that locally, it is as if the dimensioned constant were approximately constant, such as the case developed in Section 6 of this work. Then the form of the local physical laws would be the same in the two remote locations X and Y, but the dimensioned physical constants would be different. Experimenters in location Y could make observations on emissions from X with their metrology, exploiting invariants, and communicating results to one another.

Particles emitted from X with local energy E_X traverse the mild $\hbar$-gradient to Y with fixed total frequency ω, where its local energy E_Y can be measured. With no external potential active in the traversal (or the impact subtracted out if there is one), there will be an energy change $\Delta E_{YX} = \Delta\hbar_{YX}\omega$ due to the $\hbar$ gradient. If experimenters in X and Y communicate and both agree on the frequency and report the local energies, the differences measured in $\hbar$ in X and Y could be confirmed. While this may be difficult to arrange, in principle, it can be tested.

According to the model developed here, a particle conserves energy and momentum, and obeys Newton-like laws only locally for a small enough gradient in $\hbar$. This limit is consistent with the tenet that the laws of physics be the same in all locations. Energy conservation and free-particle momentum conservation would become local laws, not be upheld at greater scales. At large scales, energy and momentum are definable artificially in terms of the normal quantum mechanical operators. For a sufficiently mild $\hbar$ gradient, quantum mechanics becomes, locally, per the norm, energy is conserved, frequencies can change, redshifts can occur, position expectation value time derivatives are real, and momentum is an entity.

Though energy would not be conserved over large scales, X cannot benefit by any energy gain at Y, since returning the particle from Y back to X returns it to its original local energy. One may also contemplate manufacturing processes of various sorts, where at Y is required greater work (cost) to execute, relieved by less work (cost) in transporting the items to X. The situation is the opposite for items produced at X then transported to Y. In either direction there is a cancellation effect.

There is the result from this model that free particles have a higher probability to be found in regions of lower $\hbar$. If it were found that $\hbar$ were lower near large masses, then in the absence of an external potential there would be a quantum mechanical reason for mass to tend to locate near other mass. If the opposite, there would be a quantum mechanical reason for mass to avoid mass. One may contemplate whether the seeds of the large-scale structure of the cosmos was due to the variation of a constant, driving matter to collect at the seed origin.

The model requires the definition of a local potential energy V to be put into the frequency operator, and, there are difficulties with Ehrenfest's theorems, as far as identifying a straightforward relationship with classical mechanics. It was rationalized there is some reference point in space in which $\hbar$ is a minimum. The latter is the classical limit, or more precisely, the limit when the classical action $S_c >> \hbar$. In the latter, it is not that $\hbar$ is actually going to zero, rather, masses and kinetic energies are getting very large, and the classical behavior is recovered. In the model of this paper, it is suggested that should $\hbar$ be found to vary spatially anywhere, then somewhere else $\hbar$ is minimum. Recall in the result of Section 6 of this paper, that wavefunctions are concentrated in areas of lower $\hbar$—particles would want to collect in those regions, for reasons beyond gravity, and in collecting, also approach the classical limit.

It would be desirable to find some physical system in which $\hbar$ depended on position to test the model, and this is taken up in [34] in the analysis of the flyby anomaly, and Hulse-Taylor-like binaries. There, the effects of a position-dependent $\hbar$ may be apparent over larger scales.

The first experiments one might consider are those that have been performed already, involving atomic clocks on satellites in orbit about the Earth [10,11]. Increased precision of the instruments may be required to make an undisputed measurement of the variation of a quantity like Planck's constant. The author wonders if variations in Planck's constant, measured with clocks and light, might be somehow be suppressed in some cases, and a scheme is suggested below for how this might happen.

An argument will be offered for how a position-dependent Planck's constant may appear to not violate local position invariance, and how it may appear to be consistent with the Einstein Equivalence Principle, based on experiments, on scales where the observations are not restricted to locally flat frames, involving clocks and light. The argument comes by way of an often-seen pedagogical derivation of the gravitational redshift without full general relativity, and is used here, because at present, there is no higher theory for frequency-conserving Einstein field equations. The prescription leads to the correct formulae given by the higher theory to first order.

Consider a photon falling into a gravitational potential due to its "gravitational mass" $m(r) = \hbar(r)\omega(r)/c^2$, analyzed as if conserving total frequency $\Omega_{TOT} = \omega_\infty$, not total energy. The Newtonian field for a spherical mass of $g = -GM/r^2$ is integrated from ∞ to r to produce the gravitational potential $\varphi = -GM/r$, which is then multiplied by the gravitational photon mass, but without inclusion in the prior integration. This approach produces the result of GR for the gravitational frequency shift to first order. So, with no higher theory of a total frequency-conserving stress tensor, the sum of the kinetic frequency and potential frequency are per (65b), from which (65c) follows. Kinetic frequency is what is measured. Note that $\hbar(r)$ has cancelled in (65c), and is precisely the same expression derived when $\hbar$ is constant. Equation (65a) is the usual expression from GR for a constant $\hbar$, conserving total energy.

If the falling photon is analyzed as conserving total energy $E_{TOT} = \hbar_\infty\omega_\infty$ with a position dependent $\hbar$, then (65d,e) results. A functional form of the LPI violation for $\hbar_\infty/h(r)$ is chosen to resemble (65a), written with the Schwarzschild radius $R_S = 2GM/c^2$. If total frequency is actually conserved and not total energy, the value of the LPI violation parameter β_h returned will be zero, even if $\hbar$ is not constant (at least to first order).

$$\begin{array}{c} \omega_{GR}(r) = \omega_\infty\left(1-\frac{2GM}{rc^2}\right)^{-1/2} \\ \left(\begin{array}{c} \Omega_{TOT} = \omega_\infty = \omega(r) - \frac{GMm}{r}\frac{1}{\hbar(r)} \\ \omega(r) = \omega_\infty\left(1+\frac{GM}{rc^2}\right) \approx \omega_{GR}(r) \end{array}\right)_{\Omega_{TOT}} \\ \left(\begin{array}{c} E_{TOT} = \hbar_\infty\omega_\infty = \hbar(r)\omega(r) - \frac{GMm}{r} \\ \omega(r) = \omega_\infty\left(1+\frac{GM}{rc^2}\right)\frac{\hbar_\infty}{\hbar(r)} \approx \omega_{GR}(r)\frac{\hbar_\infty}{\hbar(r)} \end{array}\right)_{E_{TOT}} \end{array} \tag{65a-e}$$

Equation (66) reduces to the expression in the limit of small deviations seen in [10,40], where ΔU is the gravitational potential. Equation (66) was written to match the expressions of [10,40] in the limit, to allow an analysis on the available data, looking for problems. A systematic dependence of $\hbar$ on altitude was not developed in [10], only that there was a variation with a range per (66) with $|\beta_h| < 0.007$.

$$\frac{\hbar(r)}{\hbar_\infty} = \left(1-\beta_h\frac{R_S}{r}\right)^{1/2} \cong 1+\beta_h\frac{\Delta U}{c^2} \tag{66}$$

Table 1 summarizes the findings of (65a–e), and it is concluded it may be difficult to detect the variation $\hbar(r)$ using falling light or clocks at different altitudes, if total frequency is conserved, even if $\hbar$ truly varies. Tests such as the Pound-Rebka experiment, and observations with clocks on satellites at different altitudes may be completely insensitive to the variation, as such.

Table 1. The case for a photon (or clock) changing position in gravity radially that would register a detectable change in frequency deviating from GR is when total energy is conserved, and Planck's constant is position dependent. It is concluded that a variable Planck's constant may show an *apparent consistency* with the Einstein Equivalence Principle, to first order, for total conserved frequency, in experiments with clocks and light.

Conserved Quantity	$\hbar$ Dependence	$\omega(r)$
$E_{TOT} = \hbar_\infty\omega_\infty$	constant	$\omega_{GR}(r)$
$E_{TOT} = \hbar_\infty\omega_\infty$	position dependent	$\omega_{GR}(r)\,\hbar_\infty/\hbar(r)$
$\Omega_{TOT} = \omega_\infty$	constant	$\omega_{GR}(r)$
$\Omega_{TOT} = \omega_\infty$	position dependent	$\omega_{GR}(r)$

Consider the following thought experiment. Bob is inside a closed elevator in the vicinity of the Earth, held on a rope by an immobilized Alice, above. They both have a clock, which is a perfectly interior-reflecting box of trapped light of frequency ω that they can each measure. If total frequency is conserved, by Table 1, whether $\hbar$ varies or not, or whether Alice slowly lowers Bob, or cuts the rope and allows him to freefall, he will register no change in the frequency of his own clock, and since he cannot see emissions from Alice's clock, he registers no perception of any difference. Thus, the Einstein Equivalence Principle would be apparently consistent, as would the local position invariance of $\hbar$, since when the two clock readings are compared later when Alice and Bob communicate, they will show only the differences predicted by normal GR, whether $\hbar$ varies or not. Now, let Bob kick the box with a known force parallel to the floor of the elevator at several different altitudes in a gradient in $\hbar$. Though the frequency of light in the box to Bob is fixed, the energy of the light in the box is not, hence its gravitational mass changes, as does the result of the kicking experiment as a function of his altitude. Since the result of the kicking experiment varies with his position in spacetime, and the experiment is not gravitational, both the Einstein and Strong Equivalence principles do not actually hold (unless the former is interpreted to hold, if the kicking of the box is interpreted to be a gravitational experiment, since it measures the gravitational mass and inertial mass simultaneously, or if the experiment is considered to be greater in scale than a locally flat frame, as he must kick when he knows he is in a different spacetime position to register a difference). The Weak Equivalence Principle will still hold, despite that the mass of any object becomes position dependent due to the variation of $\hbar$, and despite that different substances will show different ratios of mass change – gravitational and inertial mass are still equal.

Reference [34] shows that objects in non-circular orbits, or elliptical orbits, will enhance the effect of a position dependent Planck's constant, especially a flyby orbit, as a hyperbolic orbit cuts through the isocontours of Planck's constant maximally. The analysis of an entire orbit is a larger scale experiment, from which the variation in Planck's constant can be detected.

The discussion will continue as if energy is conserved. Using $|\beta_h| = 0.007$ and the mass and radius of the Earth, (66) results in very small fractional changes near the surface of the Earth relative to infinity, on the order of one part in 10^{12}. The form (66) does not persist beneath the Earth surface due to volume filling matter. The same order of magnitude for the fractional change is found in the ratio of $\hbar$ at the maximum and minimum radii of the Earth's orbit around the sun. These variations are four orders of magnitude lower than the very best terrestrial laboratory measurement capability, achieving on the order of 10^{-8} relative uncertainty using the superconducting Watt balance [41]. Therefore, the authors of [10] may have used the GPS data to attempt to measure changes four orders of magnitude smaller than the capability of the very best earthbound metrology, if Equation (66) is operative.

The variations taken from (66) are much smaller than the 21 ppm peak-to-peak $\hbar$ variation extracted from the electromagnetic experiment (850 ppm peak-to-peak annual diode voltage variation) in [19], and the 1000–3000 ppm peak-to-peak annual variations of the decay rates in [12–17]. Were Equation (66) actually operative, completely different mechanisms would have to be at work than those in [12–17] in its relation to [19], or β_h would have to be 7 to 9 orders larger to account for the difference. At the surface of the sun using $|\beta_h| = 0.007$ the fractional change in $\hbar$ is 1 part in 10^8, getting closer to the relative uncertainty of the best terrestrial measurement. Thus, (66) may not be the correct description, in light of all the data from the two experiments.

The latter two paragraphs merely underscore that the measurement of variations in a constant such $\hbar$ is in its infancy, its dependence in a gravitational field is unknown, and more experimental work is needed to gain traction. A reanalysis of the GPS data per [10] up to the current date to refine β_h and look specifically for a systematic change in $\hbar$ with altitude may be worthwhile. An independent analysis of the data of the diode experiment in reference [19], along with analyses of the theory of the measurement are both needed. Repeats of all of the experiments by independent investigators with higher precision equipment would be critical.

13. Conclusions

A mathematical study was undertaken concerning how the Schrödinger equation would have to be changed if Planck's constant was position dependent. Notable departures from normal quantum mechanics are described. A frequency operator results, and to make it Hermitian, is augmented with an anti-commutator of the non-Hermitian part, which is the simplest alteration. While total frequency is a constant of the motion, total energy is not, and momentum becomes a non-entity except in regions where Planck's constant has a very small gradient. There are quantities now named "kinetic frequency" and "potential frequency" which together conserve total frequency between them. Wavefunctions are concentrated in regions of lower Planck's constant even in the absence of an external potential. A functional form of Planck's constant near massive bodies is alluded to, based on this authors speculation on [10], and another analysis of the GPS data associated with it might be valuable. Further work might entail finding approximate or exact quantum harmonic oscillator solutions (the author has derived this by two means in 1-D with a linear $\hbar$ gradient, unpublished, and the wavefunctions are those of the normal oscillator multiplied by the same exponential factor as Equations (45) and (47), concentrating the wavefunctions on the lower $\hbar$ side), and working out how to incorporate a position dependent Planck's constant into a canonically quantized field theory (done in [25]). Fuller investigations of the symmetries resulting from those cases could be made. The latter would help determine other dynamical variable operators commuting with the frequency operator, as so far, the only one found is itself. It may also be possible to arrive at the modified Schrödinger equation from a modified Feynman path integral with a position and/or time dependent $\hbar$ (partially examined in [34]). The implications of variations in $\hbar$ before cosmological inflation may play a role in the anisotropy of matter and present large-scale structure, as matter would gather in the minima (presently explained by dark matter and Baryon Acoustic Oscillations). Implications of position-dependent $\hbar$ could also play a role in holding the matter of galaxies together in stable orbits, and the flattening of galaxy rotation curves, since the field in reference [42] (therein Equations (44) and (45)) can represent Planck's Constant.

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Appendix A

Appendix A.1. Classical Field Equation of Motion Compared to Frequency-Conserving Schrödinger Equation

It is known that the Lagrangian density that produces a classical field equation of motion (the Schrödinger field) in the non-relativistic limit, that is the same in functional form as the single-particle Schrödinger wavefunction equation, when that field in the Hamiltonian density is then quantized, it will give the correct description of the non-relativistic single- and multi-particle states. Here, it will be determined if the Lagrangian density of the Planck constant field developed in [25] leads to a classical Schrödinger field equation of motion resembling the frequency-conserving Schrödinger wavefunction equation for which $\hbar$ is not a dynamical field but is position dependent.

The Schrödinger equation referred to features a Hermitian frequency-conserving Hamiltonian when Planck's constant is position dependent only,

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\}\varphi \tag{A1}$$

where the curly bracket signifies an anticommutaor. The φ in (A1) is the single-particle wavefunction with a probability interpretation. The classical fields will not have a probability interpretation.

The goal now is to try to arrive at (A1) as a "supported" field φ, using a "supporting" Planck's constant field $\hbar = \beta\psi$, the latter being real.

Appendix A.2. Energy Squared Lagrangian

The Lagrangian density is usually written in terms of the squares of energies, and the resulting equations are energy conserving. The Lagrangian and Hamiltonian density when Planck's constant is a dynamical field $\hbar = \beta\psi = \beta\chi^{1/2}$ supporting the field φ is from [25],

$$\begin{aligned} L_{\varepsilon^2} &= \tfrac{1}{2}\beta^2\chi\partial_u\varphi\partial^u\varphi - \tfrac{1}{2}m^2c^2\varphi^2 + \tfrac{1}{8}\beta^2\partial_u\chi\partial^u\chi \\ H_{\varepsilon^2} &= \tfrac{1}{2}\beta^2\chi\Big(\dot{\varphi}^2 + (\nabla\varphi)^2\Big) + \tfrac{1}{2}m^2c^2\varphi^2 + \tfrac{1}{8}\beta^2\Big(\dot{\chi}^2 + (\nabla\chi)^2\Big) \end{aligned} \tag{A2a,b}$$

where φ is the supported field. The energy is shared between the two fields per (A2b). The resulting equations of motion for the coupled fields φ and χ are, respectively,

$$\begin{aligned} &\tfrac{1}{4}\Big(\ddot{\chi} - \nabla^2\chi\Big) - \tfrac{1}{2}\Big(\dot{\varphi}^2 - (\nabla\varphi)^2\Big) = 0 \\ &\Big(\dot{\chi}\dot{\varphi} + \chi\ddot{\varphi}\Big) - \Big(\nabla\chi\cdot\nabla\varphi + \chi\nabla^2\varphi\Big) + \tfrac{m^2c^2}{\beta^2}\varphi = 0 \end{aligned} \tag{A3a,b}$$

Replacing $\chi = \psi^2$, then multiplying (A3a) by φ/ψ and adding to (A3b) after division by ψ in the latter, one finds for the equation of motion for the combined fields,

$$\begin{aligned} &\psi\ddot{\varphi} - \tfrac{1}{2}\Big\{\psi, \nabla^2\Big\}\varphi + \tfrac{m^2c^2}{\beta^2}\tfrac{\varphi}{\psi} \\ &+ \\ &2\dot{\psi}\dot{\varphi} + \tfrac{\varphi}{2}\left(\ddot{\psi} + \tfrac{\dot{\psi}^2}{\psi} - \tfrac{(\nabla\psi)^2}{\psi} - \tfrac{\left(\dot{\varphi}^2 - (\nabla\varphi)^2\right)}{\psi}\right) - \nabla\psi\cdot\nabla\varphi = 0 \end{aligned} \tag{A4}$$

The field φ will be decomposed as Equation (A5a,b),

$$\begin{aligned} &\tfrac{\omega_m}{c} = \tfrac{1}{c}\sqrt{\tfrac{m^2c^4}{\beta^2\psi^2}} \\ &\varphi = \widetilde{\varphi}\exp\Big(-i\tfrac{mc}{\beta\psi}x_o\Big) = \widetilde{\varphi}E \end{aligned} \tag{A5a,b}$$

with which the following derivatives are computed,

$$\frac{\dot{\varphi}}{E} = \dot{\widetilde{\varphi}} + \widetilde{\varphi}\underbrace{\left(-\frac{imc}{\beta}\right)\left(\frac{1}{\psi} - \frac{\dot{\psi}}{\psi^2}x_o\right)}_{\dot{E}} \tag{A6}$$

$$\begin{aligned} \tfrac{\ddot{\varphi}}{E} &= \ddot{\widetilde{\varphi}} + 2\dot{\widetilde{\varphi}}\Big(-\tfrac{imc}{\beta}\Big)\Big(\tfrac{1}{\psi} - \tfrac{\dot{\psi}}{\psi^2}x_o\Big) \\ &+\widetilde{\varphi}\left(\underbrace{\left[\left(-\frac{imc}{\beta}\right)\left(\frac{1}{\psi} - \frac{\dot{\psi}}{\psi^2}x_o\right)\right]^2}_{\dot{E}^2} + \Big(-\tfrac{imc}{\beta}\Big)\Big\{-\tfrac{\dot{\psi}}{\psi^2} - \Big(\tfrac{\ddot{\psi}}{\psi^2}x_o + \tfrac{\dot{\psi}}{\psi^2} - 2\tfrac{\dot{\psi}^2}{\psi^3}x_o\Big)\Big\}\right) \end{aligned} \tag{A7}$$

$$\frac{\nabla\varphi}{E} = \nabla\widetilde{\varphi} + \widetilde{\varphi}\underbrace{\left(\frac{imc}{\beta}\frac{\nabla\psi}{\psi^2}x_o\right)}_{\nabla E} \tag{A8}$$

$$\frac{\nabla^2\varphi}{E} = \nabla^2\widetilde{\varphi} + 2\nabla\widetilde{\varphi}\left(\frac{imc}{\beta}\frac{\nabla\psi}{\psi^2}x_o\right) + \widetilde{\varphi}\left(\underbrace{\left[\frac{imc}{\beta}\frac{\nabla\psi}{\psi^2}x_o\right]^2}_{(\nabla E)^2} + \left(\frac{imc}{\beta}\right)\left\{\frac{\nabla^2\psi}{\psi^2}x_o - 2\frac{(\nabla\psi)^2}{\psi^3}x_o\right\}\right) \tag{A9}$$

In Equation (A7), the first term in the underbracket will cancel the mass term of Equation (A4).

If (A6) to (A9) were substituted into (A4), the resulting equation of motion would have a very large number of terms.

Since Equation (A1) was derived with no $\hbar$ time dependence, the time derivative of ψ will be set to zero. Then from (A4), (A6), and (A7) follows,

$$\psi\ddot{\varphi} - \frac{1}{2}\left\{\psi, \nabla^2\right\}\varphi + \frac{m^2c^2}{\beta^2}\frac{\varphi}{\psi} + \frac{\varphi}{2}\left(-\frac{(\nabla\psi)^2}{\psi} - \frac{\left(\dot{\varphi}^2 - (\nabla\varphi)^2\right)}{\psi}\right) - \nabla\psi\cdot\nabla\varphi = 0 \tag{A10}$$

$$\frac{\dot{\varphi}}{E} = \dot{\widetilde{\varphi}} + \widetilde{\varphi}\left(-\frac{imc}{\beta\psi}\right) \tag{A11}$$

$$\frac{\ddot{\varphi}}{E} = \ddot{\widetilde{\varphi}} + 2\dot{\widetilde{\varphi}}\left(-\frac{imc}{\beta\psi}\right) - \frac{m^2c^2}{\beta^2}\frac{\widetilde{\varphi}}{\psi^2} \tag{A12}$$

To approach Equation (A1) some additional conditions must be imposed. The classical limit implies the kinetic energy is much less than the rest energy, therefore

$$\begin{array}{c} \dot{\varphi} \gg \nabla\varphi \\ \dot{\widetilde{\varphi}} \ll m\widetilde{\varphi} \\ \ddot{\widetilde{\varphi}} \ll m\dot{\widetilde{\varphi}} \end{array} \tag{A13a-c}$$

so the first derivative in (A11) and second derivative in (A12) can be dropped, and also $(\nabla\varphi)^2$ in the large bracketed term of (A10). In Equations (A8) and (A9) occurrences of $\nabla\psi$ or $\nabla^2\psi$ that are either second order and/or multiplied by x_o are assumed to be negligible, and only the first terms of (A8) and (A9) remain. This condition implies a combination of an early epoch and/or second order spatial changes. Then, substituting the resulting equations (A11), (A12), (A8) and (A9) into (A10),

$$i\frac{\partial\widetilde{\varphi}}{\partial t} = -\frac{1}{2m}\frac{1}{2}\left\{\beta\psi, \nabla^2\right\}\widetilde{\varphi} \underbrace{- \frac{1}{2m}\nabla\beta\psi\cdot\nabla\widetilde{\varphi}}_{A} \underbrace{+ \frac{mc^2}{4\beta}\left(\frac{\widetilde{\varphi}}{\psi}\right)^3 E(t, \psi(\bar{r}))^2}_{B} \tag{A14}$$

Equation (A14) resembles Equation (A1) with no potential, but with two additional terms that cannot easily be explained away. The extra term B may be argued to vanish if the mass is very large, to the extent that the frequency ω_m is very much larger than the frequency of the non-relativistic field, so that over much less than one cycle of the latter, the term B in (A14) would average to zero. That still leaves the problematic term A.

Appendix A.3. Frequency Squared Lagrangian

Here, it will be determined if a squared-frequency Lagrangian will produce an equation of motion conserving frequency in the non-relativistic limit with the form of (A1). To produce this Lagrangian, Equation (A2a) is divided by $\hbar^2 = (\beta\psi)^2$ and the latter is absorbed into L, but appears in the denominator of the mass term,

$$\begin{aligned} L_{f^2} &= \tfrac{1}{2}\partial_u\varphi\partial^u\varphi - \tfrac{1}{2}\tfrac{m^2c^2\varphi^2}{\beta^2\psi^2} + \tfrac{1}{2}\partial_u\psi\partial^u\psi \\ H_{f^2} &= \tfrac{1}{2}\left(\dot{\varphi}^2 + (\nabla\varphi)^2\right) + \tfrac{1}{2}\tfrac{m^2c^2\varphi^2}{\beta^2\psi^2} + \tfrac{1}{2}\left(\dot{\psi}^2 + (\nabla\psi)^2\right) \end{aligned} \tag{A15a,b}$$

From (A15b) it is seen that the frequency is shared between the two fields. The equations of motion for fields φ and ψ are, respectively,

$$\begin{aligned} \ddot{\varphi} - \nabla^2\varphi + \tfrac{m^2c^2\varphi}{\beta^2\psi^2} &= 0 \\ \ddot{\psi} - \nabla^2\psi - \tfrac{m^2c^2\varphi^2}{\beta^2\psi^3} &= 0 \end{aligned} \tag{A16a,b}$$

Note that the frequency-squared Lagrangian reproduces with (A16a) the Klein-Gordon equation with a variable $\hbar^2$ in the denominator of the mass term, and that the fields are uncoupled if $m = 0$. Multiplying (A16a) by ψ and (A16b) by $\varphi/2$ and adding them produces,

$$\psi\ddot{\varphi} - \frac{1}{2}\left\{\psi, \nabla^2\right\} + \underbrace{\nabla\psi \cdot \nabla\varphi}_{A/\beta} + \frac{\varphi}{2}\ddot{\psi} + \frac{m^2c^2}{\beta^2}\left(\frac{\varphi}{\psi} - \left(\frac{\varphi}{\psi}\right)^3\right) = 0 \tag{A17}$$

Continuing the procedure as before with the field decomposition (A5a,b) also produces a large number of terms, and the approximations eliminate all but the extra term in the underbracket of (A17), the same term as in (A14). The final equation looks like (A14) without the B term.

Appendix A.4. Discussion

Equation (A1) was not derived from field theory, rather, it was found by making the leap that frequency could be a constant of the motion if the Hamiltonian remained Hermitian by the addition of terms to the Schrödinger equation, in face of the fixed background of a position dependent $\hbar$. The expense is that energy and momentum are no longer conserved, even for a free-particle.

The plausibility of Equation (A1) depends on how plausible it is for energy conservation to be an inviolable law. Important quantities and events involved in our present physical understanding seem to violate energy conservation, such as the cosmological constant driving the accelerated expansion of the universe, and the occurrence of the big bang. Also, can the infinite energies of the vacuum state, or states, be said to be conserved? Those observations, in combination with not truly knowing whether constants are inconstant, or if they actually are dynamical fields, or fixed background fields, or neither, and something else entirely, make Equation (A1) viable to contemplate, and there may yet be a specific form of the action that leads to it, other than

$$L = \varphi^\dagger\left(i\frac{\partial\varphi}{\partial t} + \frac{1}{2m}\frac{1}{2}\left\{\hbar(\bar{r}), \nabla^2\right\}\varphi\right) \tag{A18}$$

in which $\hbar \neq \beta\psi$ and is not a supporting dynamical field. The form of Equation (A1) is extremely simple, relative to the equations that result from coupling through the derivative terms in field theory.

Appendix A.5. Conclusions

A form of the action leading to a classical, non-relativistic Schrödinger field equation of motion matching the form of the frequency-conserving Schrödinger wavefunction equation is still sought. Thus far, the latter could not be derived from field theory when there is a supporting dynamical field $\hbar = \beta\psi$.

References

1. Dirac, P.A.M. A New Basis for Cosmology. *Proc. R. Soc. Lond. A* **1938**, *165*, 199–208. [CrossRef]
2. Meshik, A.P. The Workings of an Ancient Nuclear Reactor. *Scientific American*, 26 January 2009.
3. Uzan, J.-P.; Leclercq, B. *The Natural Laws of the Universe: Understanding Fundamental Constants*; Springer: Berlin/Heidelberg, Germany, 2010.

4. Webb, J.K.; Murphy, M.T.; Flambaum, V.V.; Dzuba, V.A.; Barrow, J.D.; Churchill, C.W.; Wolfe, A.M. Further evidence for cosmological evolution of the fine structure constant. *Phys. Rev. Lett.* **2001**, *87*, 091301. [CrossRef] [PubMed]
5. Feng, S.S.; Yan, M.L. Implication of Spatial and Temporal Variations of the Fine-Structure Constant. *Int. J. Theor. Phys.* **2016**, *55*, 1049–1083. [CrossRef]
6. Kraiselburd, L.; Landau, S.J.; Simeone, C. Variation of the fine-structure constant: An update of statistical analyses with recent data. *Astron. Astrophys.* **2013**, *557*, A36. [CrossRef]
7. Mangano, G.; Lizzi, F.; Porzio, A. Inconstant Planck's constant. *Int. J. Mod. Phys. A* **2015**, *30*, 1550209. [CrossRef]
8. De Gosson, M.A. Mixed Quantum States with Variable Planck's Constant. *Phys. Lett. A* **2017**, *381*, 3033–3037. [CrossRef]
9. Uzan, J.P. The fundamental constants and their variation: Observational and theoretical status. *Rev. Mod. Phys.* **2003**, *75*, 403–455. [CrossRef]
10. Kentosh, J.; Mohageg, M. Global positioning system test of the loca lposition invariance of Planck's constant. *Phys. Rev. Lett.* **2012**, *108*, 110801. [CrossRef]
11. Kentosh, J.; Mohageg, M. Testing the local position invariance of Planck's constant in general relativity. *Phys. Essays* **2015**, *28*, 286–289. [CrossRef]
12. Herrmann, S.; Finke, F.; Lülf, M.; Kichakova, O.; Puetzfeld, D.; Knickmann, D.; List, M.; Rievers, B.; Giorgi, G.; Günther, C.; et al. Test of the Gravitational Redshift with Galileo Satellites in an Eccentric Orbit. *Phys. Rev. Lett.* **2018**, *121*, 231102. [CrossRef]
13. Ellis, K.J. The Effective Half-Life of a Broad Beam 238 Pu/Be Total Body Neutron Radiator. *Phys. Med. Biol.* **1990**, *35*, 1079–1088. [CrossRef]
14. Falkenberg, E.D. Radioactive Decay Caused by Neutrinos? *Apeiron* **2001**, *8*, 32–45.
15. Alburger, D.E.; Harbottle, G.; Norton, E.F. Half-Life of 32Si. *Earth Planet. Sci. Lett.* **1986**, *78*, 168–176. [CrossRef]
16. Jenkins, J.H.; Fischbach, E.; Sturrock, P.A.; Mundy, D.W. Analysis of Experiments Exhibiting Time Varying Nuclear Decay Rates: Systematic Effects or New Physics? *arXiv* **2011**, arXiv:1106.1678.
17. Parkhomov, A.G. Researches of Alpha and Beta Radioactivity at Long-Term Observations. *arXiv* **2010**, arXiv:1004.1761.
18. Siegert, H.; Schrader, H.; Schötzig, U. Half-Life Measurements of Europium Radionuclides and theLong-Term Stability of Detectors. *Appl. Radiat. Isot.* **1998**, *49*, 1397. [CrossRef]
19. Hutchin, R.A. Experimental Evidence for Variability in Planck's Constant. *Opt. Photon. J.* **2016**, *6*, 124–137. [CrossRef]
20. Cooper, P.S. Searching for Modifications to the Exponential Radioactive Decay Law with the Cassini Spacecraft. *Astropart. Phys.* **2008**, *31*, 267–269. [CrossRef]
21. Norman, E.B.; Browne, E.; Chan, Y.D.; Goldman, I.D.; Larimer, R.-M.; Lesko, K.T.; Nelson, M.; Wietfeldt, F.E.; Zlimen, I. Half-Life of 44Ti. *Phys. Rev. C* **1998**, *57*, 2010–2016. [CrossRef]
22. Alexeyev, E.N.; Alekseenko, V.V.; Gavriljuk, J.M.; Gangapshev, A.M.; Gezhaev, A.M.; Kazalov, V.V.; Yakimenko, S.P. Experimental test of the time stability of the half-life of alpha-decay Po-214 nuclei. *Astropart. Phys.* **2013**, *46*, 23–28. [CrossRef]
23. Norman, E.B. Additional experimental evidence against a solar influence on nuclear decay rates. *arXiv* **2012**, arXiv:1208.4357.
24. Kossert, K.; Nähle, O.J. Disproof of solar influence on the decay rates of 90Sr/90Y. *Astropart. Phys.* **2015**, *69*, 18–23. [CrossRef]
25. Dannenberg, R. Planck's Constant as a Dynamical Field. *arXiv* **2018**, arXiv:1812.02325.
26. Bekenstein, J.D. Fine-structure constant: Is it really a constant. *Phys. Rev. D* **1982**, *25*, 1527. [CrossRef]
27. Bekenstein, J.D. Fine-structure constant variability, equivalence principle and cosmology. *Phys. Rev. D* **2002**, *66*, 123514. [CrossRef]
28. Wetterich, C. Naturalness of exponential cosmon potentials and the cosmological constant problem. *Phys. Rev. D* **2008**, *77*, 103505. [CrossRef]
29. Wetterich, C. Cosmon inflation. *Phys. Rev. B* **2013**, *726*, 15–22. [CrossRef]
30. Albrecht, A.; Magueijo, J. A time varying speed of light as a solution to cosmological puzzles. *Phys. Rev. D* **1999**, *59*, 043516. [CrossRef]

31. Barrow, J.D. Cosmologies with varying light-speed. *Phys. Rev. D* **1998**, *59*, 043515. [CrossRef]
32. Moffat, J.W. Superluminary universe: A Possible solution to the initial value problem in cosmology. *Int. J. Mod. Phys. D* **1993**, *2*, 351–366. [CrossRef]
33. Moffat, J.W. Variable Speed of Light Cosmology, Primordial Fluctuations and Gravitational Waves. *Eur. Phys. J. C* **2016**, *76*, 13. [CrossRef]
34. Dannenberg, R. Applications of a Feynman Path Integral for a Position Dependent Planck's Constant. *arXiv* **2018**, arXiv:1812.02325.
35. Yan, Y.; Xu, R. Quantum mechanics of dissipative systems. *Annu. Rev. Phys. Chem.* **2005**, *56*, 187–219. [CrossRef] [PubMed]
36. Weiss, U. *Quantum Dissipative Systems*; World Scientific: Singapore, 1999.
37. Bender, C.M.; Boettcher, S. Real spectra in non-Hermitian Hamiltonians having PT symmetry. *Phys. Rev. Lett. V* **1998**, *80*, 5243–5246. [CrossRef]
38. Bender, C.M.; Boettcher, S.; Meisinger, P.N. PT-symmetric quantum mechanics. *J. Math. Phys. V* **1999**, *40*, 2201–2229. [CrossRef]
39. Ángel, M.; Buendía, S.Á.; Muga, J.G. Article Symmetries and Invariants for Non-Hermitian Hamiltonians. *Mathematics* **2018**, *6*, 111. [CrossRef]
40. Berengut, J.C.; Flambaum, V.V.; Ong, A.; Webb, J.K.; Barrow, J.D.; Barstow, M.A.; Holberg, J.B. Limits on the dependence of the fine-structure constant on gravitational, potential from white-dwarf spectra. *Phys. Rev. Lett.* **2013**, *111*, 010801. [CrossRef]
41. Schlamminger, S.; Haddad, D.; Seifert, F.; Chao, L.S.; Newell, D.B.; Liu, R.; Steiner, R.L.; Pratt, J.R. Determination of the Planck constant using a watt balance with a superconducting magnet system at the National Institute of Standards and Technology. *Metrologia* **2014**, *51*, S15. [CrossRef]
42. Rand Dannenberg, Excluded Volume for Flat Galaxy Rotation Curves in Newtonian Gravity and General Relativity. *Symmetry* **2020**, *12*, 398. [CrossRef]