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Chebyshev-Type Integral Inequalities for Continuous Fields of Operators Concerning Khatri–Rao Products and Synchronous Properties

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Received: 30 January 2020; Accepted: 1 March 2020; Published: 5 March 2020

Abstract: We consider bounded continuous fields of self-adjoint operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ . Under certain assumptions on synchronous Khatri–Rao property of the fields of operators, we obtain Chebyshev-type inequalities concerning Khatri–Rao products. We also establish Chebyshev-type inequalities involving Khatri–Rao products and weighted Pythagorean means under certain assumptions of synchronous monotone property of the fields of operators. The Pythagorean means considered here are three classical symmetric means: the geometric mean, the arithmetic mean, and the harmonic mean. Moreover, we derive the Chebyshev–Grüss integral inequality via oscillations when μ is a probability Radon measure. These integral inequalities can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results provide analog results for matrices and integrable functions. Furthermore, our results include the results for tensor products of operators, and Khatri–Rao/Kronecker/Hadamard products of matrices, which have been not investigated in the literature.

Keywords: Chebyshev sum inequality; Khatri–Rao product; Bochner integral; weighted Pythagorean mean

MSC: 47A63; 47A64; 47A80

1. Introduction

In mathematical analysis and applications, analytic inequalities serve as fundamental tools for comparison, approximation, and optimization. The classical Chebyshev sum inequality states that for any real tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ both are increasing or decreasing, we have

$$\frac{1}{n} \sum_{i=1}^n a_i b_i \geq \left(\frac{1}{n} \sum_{i=1}^n a_i \right) \left(\frac{1}{n} \sum_{i=1}^n b_i \right). \quad (1)$$

This inequality was generalized by Matharu and Aujla [1] to the case of positive semidefinite matrices involving the Hadamard (entrywise) product $\circ$: for any matrices $A_1 \geq \dots \geq A_n \geq 0$ and $B_1 \geq \dots \geq B_n \geq 0$, and any positive numbers $\omega_1, \dots, \omega_n$, we have

$$\sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i (A_i \circ B_i) \geq \left(\sum_{i=1}^n \omega_i A_i \right) \circ \left(\sum_{i=1}^n \omega_i B_i \right). \quad (2)$$

To discuss a continuous version of this inequality in a compact form, let us denote for each integrable function $f, g : [a, b] \rightarrow \mathbb{R}$,

$$\mathcal{I}(f) = \frac{1}{b-a} \int_a^b f(x) dx, \quad (3)$$

$$T_{\mathcal{I}}(f, g) = \mathcal{I}(f \cdot g) - \mathcal{I}(f) \cdot \mathcal{I}(g). \quad (4)$$

The latter is called the general Chebyshev functional (see [2]). The Chebyshev functional (4) has many applications in numerical quadrature, probability and statistics, and existence for solutions to certain differential equations. It was obtained in [3] that if such f and g are synchronous on $[a, b]$, that is,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad (5)$$

for all $x, y \in [a, b]$, then

$$T_{\mathcal{I}}(f, g) \geq 0. \quad (6)$$

The opposite inequality of (6) holds if both f and g are opposite-synchronous on $[a, b]$. Operator extensions of this inequality were presented by Moslehian and Bakherad [4]. They generalized the Chebyshev integral inequality (6) to the case of continuous fields of Hilbert space operators with a bounded measurable function involving Hadamard products by using the notion of synchronous Hadamard property. They proved that if two continuous fields $(A_t)_{t \in \Omega}, (B_t)_{t \in \Omega}$ of operators, parametrized by a compact Hausdorff space Ω equipped with a Radon measure μ , have the synchronous Hadamard property, then

$$\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \odot B_t) d\mu(t) \geq \int_{\Omega} \alpha(t) A_t d\mu(t) \odot \int_{\Omega} \alpha(s) B_s d\mu(s), \quad (7)$$

where $\alpha : \Omega \rightarrow [0, \infty)$ is a bounded measurable function. Moreover, they gave some Chebyshev-type inequalities concerning operator means and Hadamard products.

A complement of (6) was introduced by Grüss [5], providing an estimate of the difference between the integral of the product and the product of the integrals for two functions. For any integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$ satisfying the conditions $k \leq f(x) \leq K, l \leq g(x) \leq L$ for all $x \in [a, b]$ and k, K, l, L are real constants, we have

$$|T_{\mathcal{I}}(f, g)| \leq \frac{1}{4} (K - k)(L - l). \quad (8)$$

We can apply Grüss inequalities to estimate error bounds for some integral means and numerical quadrature rules; see e.g., [6,7]. In [8], Gonska, Raşa and Rusu used the terminology Chebyshev–Grüss inequalities referring to Grüss inequalities for (special cases of) generalized Chebyshev functionals $T_{\mathcal{I}}$ which have a general form

$$|T_{\mathcal{I}}(f, g)| \leq E(\mathcal{I}, f, g), \quad (9)$$

where E is an expression in terms of certain properties of $\mathcal{I}$ and some kind of oscillations of f and g . They also established new Chebyshev–Grüss inequalities via discrete oscillations.

On the other hand, in the theory of operator product, the notion of tensor product for Hilbert space operators was extended to the Tracy–Singh product for such operators [9]. Algebraic, order, and analytic properties of the Tracy–Singh product for operator were discussed in [9,10]. The notion of tensor product was also generalized to the Khatri–Rao product for Hilbert space operators in [11].

The work [11] shows that the Khatri–Rao product and the Tracy–Singh product are related via isometric selection operators.

In this paper, we establish new several integral inequalities of Chebyshev-type for continuous fields of self-adjoint operators involving Khatri–Rao products and operator means. In Section 2, we give preliminaries on Khatri–Rao and Tracy–Singh products for operators, and Bochner integrability of continuous field of operators on a locally compact Hausdorff space. In Section 3, we provide Chebyshev-type inequalities involving Khatri–Rao products of operators under the assumption of synchronous Khatri–Rao property. In Section 4, we establish Chebyshev integral inequalities concerning weighted Pythagorean means and Khatri–Rao products under the assumption of synchronous monotone property. We prove Chebyshev–Grüss inequalities via oscillations for continuous fields of operators in Section 5. Our results generalize the matrix result [1] and the result for integrable functions [8]. Moreover, our results include the results for tensor products of operators, and Khatri–Rao/Kronecker/Hadamard products of matrices, which have been not investigated in the literature. Finally, we summarize our work in Section 6.

2. Preliminaries

Throughout this paper, let $\mathbb{H}$, and $\mathbb{K}$ be complex Hilbert spaces. The symbol $\mathcal{B}(\mathbb{H})$ stands to the Banach space of bounded linear operators from $\mathbb{H}$ into itself. The vector space of self-adjoint operators on $\mathbb{H}$ is denoted by $\mathcal{B}(\mathbb{H})^{sa}$. Denote the set of all positive (positive invertible, respectively) operators on $\mathbb{H}$ by $\mathcal{B}(\mathbb{H})^+$ ($\mathcal{B}(\mathbb{H})^{++}$, resp.). For any $A, B \in \mathcal{B}(\mathbb{H})^{sa}$, the situation $A \geq B$ ($A > B$, resp.) means that $A - B \in \mathcal{B}(\mathbb{H})^+$ ($A - B \in \mathcal{B}(\mathbb{H})^{++}$, resp.).

Through this paper, we apply the projection theorem to decompose

$$\mathbb{H} = \bigoplus_{i=1}^n \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{i=1}^n \mathbb{K}_i,$$

where all $\mathbb{H}_i$ and $\mathbb{K}_i$ are Hilbert spaces. Each operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{n,n} \quad \text{and} \quad B = [B_{ij}]_{i,j=1}^{n,n},$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{ij} \in \mathcal{B}(\mathbb{K}_j, \mathbb{K}_i)$ for each i, j .

2.1. Khatri–Rao Product and Tracy–Singh Product for Operators

Recall that the tensor product of $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ is a unique bounded linear operator $A \otimes B \in \mathcal{B}(\mathbb{H} \otimes \mathbb{K})$ such that for all $x \in \mathbb{H}$ and all $y \in \mathbb{K}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Fix a countable orthonormal basis $\mathbb{E}$ of $\mathbb{H}$. Recall that the Hadamard product of $A, B \in \mathcal{B}(\mathbb{H})$ is defined to be the operator $A \odot B \in \mathcal{B}(\mathbb{H})$ such that for all $e \in \mathbb{E}$,

$$\langle (A \odot B)e, e \rangle = \langle Ae, e \rangle \langle Be, e \rangle.$$

It is known that the Hadamard product of $A, B \in \mathcal{B}(\mathbb{H})$ can be expressed as

$$A \odot B = U^*(A \otimes B)U, \tag{10}$$

where $U : \mathbb{H} \rightarrow \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathbb{E}$ (see e.g., [12]).

From the previous setting, we define the Khatri–Rao product of A and B to be the bounded linear operator from $\bigoplus_{i=1}^n \mathbb{H}_i \otimes \mathbb{K}_i$ into itself represented by an operator matrix

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{ij}. \quad (11)$$

We define the Tracy–Singh product of A and B to be the bounded linear operator from $\bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathbb{H}_i \otimes \mathbb{K}_j$ into itself represented by an operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}. \quad (12)$$

The maps $(A, B) \mapsto A \boxdot B$ and $(A, B) \mapsto A \boxtimes B$ are bilinear. Moreover, we have:

Lemma 1 ([9–11]). *Let A, B, C, D be compatible operators. Then*

1. *If $A \geq C \geq 0$ and $B \geq D \geq 0$, then $A \boxdot B \geq C \boxdot D \geq 0$.*
2. *If $A \geq C \geq 0$ and $B \geq D \geq 0$, then $A \boxtimes B \geq C \boxtimes D \geq 0$.*
3. *$(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.*
4. *$(A + C) \boxtimes (B + D) = A \boxtimes B + A \boxtimes D + C \boxtimes B + C \boxtimes D$.*
5. *If A and B are invertible, then $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *If A and B are positive, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \geq 0$.*

Lemma 2 ([11]). *There is an isometry Z such that $A \boxdot B = Z^*(A \boxtimes B)Z$ for any operators $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$.*

2.2. Bochner Integration for Operator-Valued Maps

Throughout this paper, let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ . A continuous map $A : \Omega \rightarrow \mathbb{X} \subseteq \mathcal{B}(\mathbb{H})$ is called a continuous field of operators in $\mathbb{X}$ parametrized by Ω , denoted by $A \in \mathcal{C}(\Omega; \mathbb{X})$. For convenience, for each $t \in \Omega$, we may write A_t instead of $A(t)$. The field A is said to be bounded if there is a constant $M > 0$ such that $\|A_t\| \leq M$ for all $t \in \Omega$. The set of all bounded continuous fields of operators in $\mathbb{X}$ parametrized by Ω is denoted by $\mathfrak{B}\mathcal{C}(\Omega; \mathbb{X})$. If $A \in \mathcal{C}(\Omega; \mathbb{X})$ is such that the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω (e.g., $A \in \mathfrak{B}\mathcal{C}(\Omega; \mathbb{X})$), then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$.

Lemma 3 (see e.g., [13]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, and $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ a bounded linear operator. For any Bochner integrable function $f : \Gamma \rightarrow \mathbb{X}$, the composition $\varphi \circ f$ is also Bochner integrable and*

$$\int_\Gamma (\varphi \circ f) dv = \varphi \int_\Gamma f dv.$$

Proposition 1. *For any $A \in \mathfrak{B}\mathcal{C}(\Omega, \mathcal{B}(\mathbb{H}))$ and $X \in \mathcal{B}(\mathbb{K})$, we have*

$$\int_\Omega A_t d\mu(t) \boxdot X = \int_\Omega (A_t \boxdot X) d\mu(t). \quad (13)$$

Proof. The map $t \mapsto A_t$ is Bochner integrable on Ω because it is continuous. Since the map $T \mapsto T \boxdot X$ is bounded linear operator, we have by Lemma 3 that the map $t \mapsto A_t \boxdot X$ is Bochner integrable on Ω and (13) holds. $\square$

3. Chebyshev-Type Inequalities Involving Khatri–Rao Products of Operators

We introduce the following property, and prove Chebyshev-type inequalities involving Khatri–Rao products of operators.

Definition 1. The fields A and B of operators parametrized by Ω are said to have the synchronous Khatri–Rao property if, for all $s, t \in \Omega$,

$$(A_t - A_s) \boxtimes (B_t - B_s) \geq 0. \quad (14)$$

They are said to have the opposite-synchronous Khatri–Rao property if the reverse of (14) holds for all $s, t \in \Omega$.

The following result is an extension of the Chebyshev integral inequality (6) to the case of operators involving Khatri–Rao products.

Theorem 1. Let $A \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{H})^{sa})$ and $B \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{K})^{sa})$, and let $\alpha : \Omega \rightarrow [0, \infty)$ be a bounded measurable function.

1. If A and B have the synchronous Khatri–Rao property, then

$$\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \boxtimes B_t) d\mu(t) \geq \int_{\Omega} \alpha(t) A_t d\mu(t) \boxtimes \int_{\Omega} \alpha(s) B_s d\mu(s). \quad (15)$$

2. If A and B have the opposite-synchronous Khatri–Rao property, then the reverse of (15) holds.

Proof. By using Lemma 1 and Proposition 1, we have

$$\begin{aligned} & \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \boxtimes B_t) d\mu(t) - \int_{\Omega} \alpha(t) A_t d\mu(t) \boxtimes \int_{\Omega} \alpha(s) B_s d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) (A_t \boxtimes B_t) d\mu(t) d\mu(s) - \iint_{\Omega^2} \alpha(t) \alpha(s) (A_t \boxtimes B_s) d\mu(t) d\mu(s) \\ &= \iint_{\Omega^2} [\alpha(s) \alpha(t) (A_t \boxtimes B_t) - \alpha(t) \alpha(s) (A_t \boxtimes B_s)] d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} [\alpha(s) \alpha(t) (A_t \boxtimes B_t) - \alpha(t) \alpha(s) (A_t \boxtimes B_s)] d\mu(t) d\mu(s) \\ &\quad + \frac{1}{2} \iint_{\Omega^2} [\alpha(t) \alpha(s) (A_s \boxtimes B_s) - \alpha(s) \alpha(t) (A_s \boxtimes B_t)] d\mu(s) d\mu(t) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t - A_s) \boxtimes (B_t - B_s)] d\mu(t) d\mu(s). \end{aligned}$$

Here, we have used Fubini's Theorem [14] to interchange the order of integrals. For the case 1, we have

$$\iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t - A_s) \boxtimes (B_t - B_s)] d\mu(t) d\mu(s) \geq 0 \quad (16)$$

and thus (15) holds. For another case, we get the reverse of (16) and, thus, the reverse of (15) holds. $\square$

For the case $n = 1$, i.e., $\mathbb{H}$ and $\mathbb{K}$ are not decomposed, the synchronous Khatri–Rao property in Definition 1 reduces to the synchronous tensor property:

$$(A_t - A_s) \otimes (B_t - B_s) \geq 0.$$

If two fields A and B of operators parametrized by Ω have the synchronous tensor property, then A and B have the synchronous Hadamard property ([4], Definition 2.1), i.e.,

$$(A_t - A_s) \odot (B_t - B_s) \geq 0$$

for all $s, t \in \Omega$. The following result gives Chebyshev-type inequalities involving tensor products and Hadamard products.

Corollary 1. Let $A, B \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{H})^{sa})$ and let $\alpha : \Omega \rightarrow [0, \infty)$ be a bounded measurable function.

1. If A and B have the synchronous tensor property, then

$$\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \otimes B_t) d\mu(t) \geq \int_{\Omega} \alpha(t) A_t d\mu(t) \otimes \int_{\Omega} \alpha(s) B_s d\mu(s), \quad (17)$$

$$\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \odot B_t) d\mu(t) \geq \int_{\Omega} \alpha(t) A_t d\mu(t) \odot \int_{\Omega} \alpha(s) B_s d\mu(s). \quad (18)$$

2. If A and B have the opposite-synchronous tensor property, then the reverses of (17) and (18) holds.

Proof. For the case $n = 1$, the Khatri–Rao product in Theorem 1 reduces to the tensor product. Assume that A and B have the synchronous tensor property. Using the fact that the Hadamard product is expressed as the deformation of tensor product via the isometry U defined in (10), we obtain

$$\begin{aligned} \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \odot B_t) d\mu(t) &= \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) U^* (A_t \otimes B_t) U d\mu(t) \\ &= U^* \left(\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) (A_t \otimes B_t) d\mu(t) \right) U \\ &\geq U^* \left(\int_{\Omega} \alpha(t) A_t d\mu(t) \otimes \int_{\Omega} \alpha(s) B_s d\mu(s) \right) U \\ &= \int_{\Omega} \alpha(t) A_t d\mu(t) \odot \int_{\Omega} \alpha(s) B_s d\mu(s). \end{aligned}$$

Case 2 for Hadamard products can be similarly treated. $\square$

We can see that the inequality (18) is the same as (7), but they hold under different hypothesis. The next corollary is a discrete version of Theorem 1.

Corollary 2. Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ where $A_i \in \mathcal{B}(\mathbb{H})^{sa}$, $B_i \in \mathcal{B}(\mathbb{K})^{sa}$ and ω_i is a nonnegative number for each $i = 1, \dots, k$.

1. If A and B have the synchronous Khatri–Rao property, then

$$\sum_{i=1}^k \omega_i \sum_{i=1}^k \omega_i (A_i \boxtimes B_i) \geq \left(\sum_{i=1}^k \omega_i A_i \right) \boxtimes \left(\sum_{i=1}^k \omega_i B_i \right). \quad (19)$$

2. If A and B have the opposite-synchronous Khatri–Rao property, then the reverse of (19) holds.

Proof. From the previous theorem, consider the finite space $\Omega = \{1, \dots, k\}$ equipped with the counting measure and $\alpha(i) = \omega_i$ for all $i = 1, \dots, k$. $\square$

This corollary generalizes Chebyshev sum inequalities for the case of real numbers in inequality (1) and for Hadamard product of matrices in [1].

Next, we illustrate Chebyshev-type inequalities for bounded linear operators induced from matrices. Recall that with each $A \in \mathbb{M}_n(\mathbb{C})$ one can naturally associate a bounded linear operator

$$T_A : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad x \mapsto Ax.$$

For any complex matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ partitioned in block-matrix form, we have [11]

$$T_A \boxtimes T_B = T_{A \boxtimes B}. \quad (20)$$

Example 1. Consider $\omega_1 = \omega_2 = \frac{1}{2}$, $A = (T_{A_1}, T_{A_2})$, and $B = (T_{B_1}, T_{B_2})$, where

$$A_1 = \left[\begin{array}{c|cc} -1 & 1 & -1 \\ \hline 1 & 2 & 0 \\ -1 & 0 & -3 \end{array} \right], \quad A_2 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 3 & -1 \\ 0 & -1 & 1 \end{array} \right],$$

$$B_1 = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \\ -1 & 0 \end{array} \middle| \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right], \quad B_2 = \left[\begin{array}{c|c} 2 & 1 \\ \hline 1 & 4 \\ 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ 2 \end{array} \right].$$

First, we check the hypothesis of Corollary 2. Since $A_2 - A_1 \geq 0$, we have $T_{A_2} - T_{A_1} \geq 0$. Similarly, $T_{B_2} - T_{B_1} \geq 0$. By the positivity of the Khatri–Rao product, we get

$$(T_{A_2} - T_{A_1}) \square (T_{B_2} - T_{B_1}) \geq 0,$$

i.e., the fields $A = (T_{A_1}, T_{A_2})$ and $B = (T_{B_1}, T_{B_2})$ have the synchronous Khatri–Rao product property.

Now, we can check that the following matrix is positive semidefinite:

$$\begin{aligned} & A_1 \square B_1 + A_2 \square B_2 - \frac{1}{2}(A_1 + A_2) \square (B_1 + B_2) \\ &= \left[\begin{array}{c|cc} 3 & 1 & -1 & 1 \\ \hline 1 & 9 & 0 & 0 \\ -1 & 0 & 8 & -2 \\ 1 & 0 & -2 & -1 \end{array} \right] - \frac{1}{2} \left[\begin{array}{c|cc} 3 & 2 & -2 & 1 \\ \hline 2 & 3 & 0 & 0 \\ -2 & 0 & 15 & -3 \\ 1 & 0 & -3 & -6 \end{array} \right] \\ &= \frac{1}{2} \left[\begin{array}{c|cc} 3 & 0 & 0 & 1 \\ \hline 0 & 15 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 4 \end{array} \right] \\ &\geq 0. \end{aligned}$$

Thus,

$$A_1 \square B_1 + A_2 \square B_2 \geq \frac{1}{2}(A_1 + A_2) \square (B_1 + B_2).$$

Passing through the induced linear maps, we get

$$T_{A_1 \square B_1 + A_2 \square B_2} \geq T_{\frac{1}{2}(A_1 + A_2) \square (B_1 + B_2)}.$$

Finally, applying the property (20), we have

$$T_{A_1} \square T_{B_1} + T_{A_2} \square T_{B_2} \geq \frac{1}{2}(T_{A_1} + T_{A_2}) \square (T_{B_1} + T_{B_2}),$$

i.e., the inequality (19) in Corollary 2 holds.

4. Chebyshev Integral Inequalities Concerning Weighted Pythagorean Means of Operators

We start this section by introducing order assumptions on continuous fields and supplying preliminaries on weighted arithmetic/geometric/harmonic means of operators. The main part is to establish Chebyshev-type inequalities involving Khatri–Rao products concerning such operator means and order assumptions.

Throughout this section, the space Ω is equipped with a total ordering $\preceq$. Consider the following definitions:

Definition 2. We say that a field A is increasing (decreasing, resp.) whenever $s \preccurlyeq t$ implies $A_s \leq A_t$ ($A_s \geq A_t$, respectively).

Definition 3. Two ordered pairs (X_1, X_2) and (Y_1, Y_2) of self-adjoint operators are said to have the synchronous property if either

$$X_i \leq Y_i \text{ for } i = 1, 2, \text{ or } X_i \geq Y_i \text{ for } i = 1, 2.$$

The pairs (X_1, X_2) and (Y_1, Y_2) are said to have the opposite-synchronous property if either

$$X_1 \leq Y_1 \text{ and } X_2 \geq Y_2, \text{ or } X_1 \geq Y_1 \text{ and } X_2 \leq Y_2.$$

Definition 4. Let A, B, C, D be continuous fields of self-adjoint operators parametrized by Ω . Two ordered pairs (A, B) and (C, D) are said to have the synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the synchronous property for all $t \in \Omega$. The pairs (A, B) and (C, D) are said to have the opposite-synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the opposite-synchronous property for all $t \in \Omega$.

Recall that the three classical Pythagorean means are the following symmetric means: the arithmetic mean, the harmonic mean, and the geometric mean. For each $w \in [0, 1]$, the w -weighted versions of such means are respectively defined for any $A, B \in \mathcal{B}(\mathbb{H})^{++}$ by

$$\begin{aligned} A \nabla_w B &= (1-w)A + wB, \\ A !_w B &= \left[(1-w)A^{-1} + wB^{-1} \right]^{-1}, \\ A \sharp_w B &= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^w A^{\frac{1}{2}}. \end{aligned}$$

These means can be defined for arbitrary positive operators by the following continuity argument with respect to the strong-operator topology:

$$A \sharp_w B = \lim_{\varepsilon \rightarrow 0^+} (A + \varepsilon I) \sharp_w (B + \varepsilon I).$$

For brevity, we write $A \sharp B$ for $A \sharp_{1/2} B$. The Pythagorean means have the following remarkable property: for any $w \in [0, 1]$, $T \in \mathcal{B}(\mathbb{H})$, and $A, B \in \mathcal{B}(\mathbb{H})^+$, we have

$$T^*(A \sigma_w B)T \leq (T^*AT) \sigma_w (T^*BT), \quad (21)$$

here σ is anyone of $\nabla, !, \sharp$.

Lemma 4 (see e.g., [15]). The weighted geometric means, weighted arithmetic means and weighted harmonic means for operators are (jointly) monotone in the sense that if $X_1 \leq X_2$ and $Y_1 \leq Y_2$, then $X_1 \sigma Y_1 \leq X_2 \sigma Y_2$ where σ is any of $\nabla, !, \sharp$.

4.1. Inequalities on Weighted Geometric Means

Recall that a linear map between two operator algebras is said to be positive if it maps positive operators to positive operators.

Lemma 5 ([16]). Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ and $w \in [0, 1]$. Then

$$(A \boxdot B) \sharp_w (C \boxdot D) \geq (A \sharp_w C) \boxdot (B \sharp_w D).$$

Theorem 2. Let $A, B, C, D \in \mathfrak{B}\mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H})^+)$ and let $\alpha : \Omega \rightarrow [0, \infty)$ be a bounded measurable function. If A, B, C, D are either all increasing, or all decreasing, then

$$\int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxminus B_t) \sharp_w (C_t \boxminus D_t)] d\mu(t) \geq \int_{\Omega} \alpha(t) (A_t \sharp_w C_t) d\mu(t) \boxminus \int_{\Omega} \alpha(s) (B_s \sharp_w D_s) d\mu(s). \quad (22)$$

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preccurlyeq t$. By applying Lemmas 1 and 5, Proposition 1, and Fubini's Theorem [14], we have

$$\begin{aligned} & \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxminus B_t) \sharp_w (C_t \boxminus D_t)] d\mu(t) - \int_{\Omega} \alpha(t) (A_t \sharp_w C_t) d\mu(t) \boxminus \int_{\Omega} \alpha(s) (B_s \sharp_w D_s) d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \boxminus B_t) \sharp_w (C_t \boxminus D_t)] d\mu(t) d\mu(s) - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \sharp_w C_t) \boxminus (B_s \sharp_w D_s)] d\mu(t) d\mu(s) \\ &\geq \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \sharp_w C_t) \boxminus (B_t \sharp_w D_t)] d\mu(t) d\mu(s) - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \sharp_w C_t) \boxminus (B_s \sharp_w D_s)] d\mu(t) d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \sharp_w C_t) \boxminus (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxminus (B_s \sharp_w D_s)] d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \sharp_w C_t) \boxminus (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxminus (B_s \sharp_w D_s)] d\mu(t) d\mu(s) \\ &\quad + \frac{1}{2} \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_s \sharp_w C_s) \boxminus (B_s \sharp_w D_s) - (A_s \sharp_w C_s) \boxminus (B_t \sharp_w D_t)] d\mu(s) d\mu(t) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [A_t \sharp_w C_t - A_s \sharp_w C_s] \boxminus [B_t \sharp_w D_t - B_s \sharp_w D_s] d\mu(t) d\mu(s). \end{aligned}$$

If A, B, C, D are all increasing, we have by Lemma 4 that $A_t \sharp_w C_t \geq A_s \sharp_w C_s$ and $B_t \sharp_w D_t \geq B_s \sharp_w D_s$. If A, B, C, D are all decreasing, we have $A_t \sharp_w C_t \leq A_s \sharp_w C_s$ and $B_t \sharp_w D_t \leq B_s \sharp_w D_s$. By Lemma 1, both cases lead to the same conclusion that $(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxminus (B_t \sharp_w D_t - B_s \sharp_w D_s) \geq 0$, and hence (22) holds. $\square$

The next corollary is a discrete version of Theorem 1.

Corollary 3. Let $A = (A_1, \dots, A_k), B = (B_1, \dots, B_k), C = (C_1, \dots, C_k)$ and $D = (D_1, \dots, D_k)$ where $A_i, B_i, C_i, D_i \in \mathcal{B}(\mathbb{H})^+$ for each $i = 1, \dots, k$. If A, B, C, D are either all increasing, or all decreasing, then

$$\frac{1}{k} \sum_{i=1}^k [(A_i \boxminus B_i) \sharp_w (C_i \boxminus D_i)] \geq \left(\frac{1}{k} \sum_{i=1}^k (A_i \sharp_w C_i) \right) \boxminus \left(\frac{1}{k} \sum_{i=1}^k (B_i \sharp_w D_i) \right). \quad (23)$$

Proof. Setting $\Omega = \{1, \dots, k\}$ equipped with the counting measure and $\alpha(i) = \frac{1}{k}$ for all $i = 1, \dots, k$ in Theorem 2, we get the result. $\square$

Operator inequality (23) can be regarded as a generalization of the Chebyshev sum inequality (1). The next goal is to establish a reverse version of Theorem 2.

Lemma 6. Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ be such that $0 < m_1 I \leq A \boxtimes B \leq M_1 I$ and $0 < m_2 I \leq C \boxtimes D \leq M_2 I$. Denote $m = \frac{m_2}{M_1}, M = \frac{M_2}{m_1}$ and

$$\lambda = \frac{Mm^w - mM^w}{(1-w)(M-m)} \cdot \left(\frac{1-w}{w} \cdot \frac{M^w - m^w}{Mm^w - mM^w} \right)^w. \quad (24)$$

Then for any $w \in (0, 1)$, we have

$$\lambda(A \boxminus B) \sharp_w (C \boxminus D) \leq (A \sharp_w C) \boxminus (B \sharp_w D).$$

Proof. Consider a map $\Phi : T \mapsto Z^*TZ$, where Z is the isometry in Lemma 2. Since Φ is a unital positive linear map, we have by ([17], Corollary 3.5) that

$$\lambda[\Phi(A \boxtimes B) \#_w \Phi(C \boxtimes D)] \leq \Phi((A \boxtimes B) \#_w (C \boxtimes D)).$$

From ([16], Theorem 1), we get

$$(A \boxtimes B) \#_w (C \boxtimes D) = (A \#_w C) \boxtimes (B \#_w D).$$

Hence,

$$\begin{aligned} \lambda[(A \boxtimes B) \#_w (C \boxtimes D)] &= \lambda[\Phi(A \boxtimes B) \#_w \Phi(C \boxtimes D)] \\ &\leq \Phi((A \boxtimes B) \#_w (C \boxtimes D)) \\ &= \Phi((A \#_w C) \boxtimes (B \#_w D)) \\ &= (A \#_w C) \boxtimes (B \#_w D). \quad \square \end{aligned}$$

Theorem 3. Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ with $0 < m_1 I \leq A_t \boxtimes B_t \leq M_1 I$ and $0 < m_2 I \leq C_t \boxtimes D_t \leq M_2 I$ for all $t \in \Omega$, and $\alpha : \Omega \rightarrow [0, \infty)$ a bounded measurable function. Let m, M, w, λ as in Lemma 6. If either

1. A, C are increasing and B, D are decreasing, or
2. A, C are decreasing and B, D are increasing,

then

$$\begin{aligned} \lambda \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxtimes B_t) \#_w (C_t \boxtimes D_t)] d\mu(t) \\ \leq \int_{\Omega} \alpha(t) (A_t \#_w C_t) d\mu(t) \boxtimes \int_{\Omega} \alpha(s) (B_s \#_w D_s) d\mu(s). \end{aligned} \quad (25)$$

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preceq t$. By applying Lemmas 1 and 6, Proposition 1, and Fubini's Theorem [14], we have

$$\begin{aligned} &\lambda \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxtimes B_t) \#_w (C_t \boxtimes D_t)] d\mu(t) - \int_{\Omega} \alpha(t) (A_t \#_w C_t) d\mu(t) \boxtimes \int_{\Omega} \alpha(s) (B_s \#_w D_s) d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) [\lambda(A_t \boxtimes B_t) \#_w (C_t \boxtimes D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \#_w C_t) \boxtimes (B_s \#_w D_s)] d\mu(t) d\mu(s) \\ &\leq \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \#_w C_t) \boxtimes (B_t \#_w D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \#_w C_t) \boxtimes (B_s \#_w D_s)] d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \#_w C_t) \boxtimes (B_t \#_w D_t) - (A_t \#_w C_t) \boxtimes (B_s \#_w D_s)] d\mu(t) d\mu(s) \\ &\quad + \frac{1}{2} \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_s \#_w C_s) \boxtimes (B_s \#_w D_s) - (A_s \#_w C_s) \boxtimes (B_t \#_w D_t)] d\mu(s) d\mu(t) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [A_t \#_w C_t - A_s \#_w C_s] \boxtimes [B_t \#_w D_t - B_s \#_w D_s] d\mu(t) d\mu(s). \end{aligned}$$

We have by Lemmas 1 and 4 that $(A_t \#_w C_t - A_s \#_w C_s) \boxtimes (B_t \#_w D_t - B_s \#_w D_s) \leq 0$, and hence (25) holds. $\square$

4.2. Inequalities on Weighted Arithmetic Means

Lemma 7 ([18]). Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^{sa}$ and $w \in [0, 1]$.

1. If (A, B) and (C, D) have the synchronous property, then

$$(A \boxdot B) \nabla_w (C \boxdot D) \geq (A \nabla_w C) \boxdot (B \nabla_w D). \quad (26)$$

2. If (A, B) and (C, D) have the opposite-synchronous property, then the reverse of (26) holds.

Theorem 4. Let $A, B, C, D \in \mathfrak{B}\mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H})^+)$ and let $\alpha : \Omega \rightarrow [0, \infty)$ be a bounded measurable function.

1. If $(A, B), (C, D)$ have the synchronous monotone property and all of A, B, C, D are either increasing or decreasing, then

$$\begin{aligned} \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxdot B_t) \nabla_w (C_t \boxdot D_t)] d\mu(t) \\ \geq \int_{\Omega} \alpha(t) (A_t \nabla_w C_t) d\mu(t) \boxdot \int_{\Omega} \alpha(s) (B_s \nabla_w D_s) d\mu(s). \end{aligned} \quad (27)$$

2. If $(A, B), (C, D)$ have the opposite-synchronous monotone property and if either

- (a) A, C are increasing and B, D are decreasing, or
- (b) A, C are decreasing and B, D are increasing,

then the reverse of (27) holds.

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preceq t$. First, we consider the case 1. We have by using Lemmas 1 and 7, proposition 1, and Fubini's Theorem that

$$\begin{aligned} \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxdot B_t) \nabla_w (C_t \boxdot D_t)] d\mu(t) - \int_{\Omega} \alpha(t) (A_t \nabla_w C_t) d\mu(t) \boxdot \int_{\Omega} \alpha(s) (B_s \nabla_w D_s) d\mu(s) \\ = \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \boxdot B_t) \nabla_w (C_t \boxdot D_t)] d\mu(t) d\mu(s) \\ - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \nabla_w C_t) \boxdot (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\ \geq \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \nabla_w C_t) \boxdot (B_t \nabla_w D_t)] d\mu(t) d\mu(s) \\ - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t \nabla_w C_t) \boxdot (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\ = \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \nabla_w C_t) \boxdot (B_t \nabla_w D_t) - (A_t \nabla_w C_t) \boxdot (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\ = \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [A_t \nabla_w C_t - A_s \nabla_w C_s] \boxdot [B_t \nabla_w D_t - B_s \nabla_w D_s] d\mu(t) d\mu(s). \end{aligned}$$

By Lemmas 1 and 4, we have $(A_t \nabla_w C_t - A_s \nabla_w C_s) \boxdot (B_t \nabla_w D_t - B_s \nabla_w D_s) \geq 0$ and hence (27) holds. The case 2 can be similarly treated. $\square$

We can illustrate Theorem 4 for the case of operators induced from matrices as follows.

Example 2. Consider the following pairs of induced bounded linear operators: $A = (T_{A_1}, T_{A_2})$, $B = (T_{B_1}, T_{B_2})$, $C = (T_{C_1}, T_{C_2})$ and $D = (T_{D_1}, T_{D_2})$, where

$$\begin{aligned} A_1 &= \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right], & A_2 &= \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 4 \end{array} \right], & B_1 &= \left[\begin{array}{c|cc} 3 & 2 & 0 \\ \hline 2 & 2 & 0 \\ 0 & 0 & 1 \end{array} \right], \\ B_2 &= \left[\begin{array}{c|cc} 5 & 3 & 0 \\ \hline 3 & 6 & 2 \\ 0 & 2 & 4 \end{array} \right], & C_1 &= \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], & C_2 &= \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 1 \end{array} \right], \\ D_1 &= \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 1 \end{array} \right], & D_2 &= \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 3 & 2 \\ 0 & 2 & 2 \end{array} \right]. \end{aligned}$$

We can check the hypothesis of Theorem 4: (i) all of A, B, C, D are increasing, and (ii) $(A, B), (C, D)$ have the synchronous monotone property. Set $\Omega = \{1, 2\}$ equipped with the counting measure and $\alpha(i) = \frac{1}{2}$ for all $i = 1, 2$. Let us denote $\nabla = \nabla_{1/2}$. Now, a direct computation reveals that

$$\begin{aligned} & (A_1 \boxminus B_1) \nabla (C_1 \boxminus D_1) + (A_2 \boxminus B_2) \nabla (C_2 \boxminus D_2) - \frac{1}{2} (A_1 \nabla C_1 + A_2 \nabla C_2) \boxminus (B_1 \nabla D_1 + B_2 \nabla D_2) \\ &= \frac{1}{4} \left[\begin{array}{c|cccc} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 6 & 2 & 5 & 2 \\ 0 & 2 & 4 & 2 & 4 \\ 0 & 5 & 2 & 14 & 8 \\ 0 & 2 & 4 & 8 & 10 \end{array} \right] \geq 0, \end{aligned}$$

or equivalently,

$$(A_1 \boxminus B_1) \nabla (C_1 \boxminus D_1) + (A_2 \boxminus B_2) \nabla (C_2 \boxminus D_2) \geq \frac{1}{2} (A_1 \nabla C_1 + A_2 \nabla C_2) \boxminus (B_1 \nabla D_1 + B_2 \nabla D_2).$$

Passing through the induced linear maps and applying the properties $T_{A \nabla B} = T_A \nabla T_B$ and (20), we obtain

$$\begin{aligned} & (T_{A_1} \boxminus T_{B_1}) \nabla (T_{C_1} \boxminus T_{D_1}) + (T_{A_2} \boxminus T_{B_2}) \nabla (T_{C_2} \boxminus T_{D_2}) \\ & \geq \frac{1}{2} (T_{A_1} \nabla T_{C_1} + T_{A_2} \nabla T_{C_2}) \boxminus (T_{B_1} \nabla T_{D_1} + T_{B_2} \nabla T_{D_2}). \end{aligned}$$

Thus, the equality (27) holds in this case.

4.3. Inequalities on Weighted Harmonic Means

Lemma 8 ([18]). Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ and $w \in [0, 1]$. If (A, B) and (C, D) have the opposite-synchronous property, then

$$(A \boxminus B) !_w (C \boxminus D) \geq (A !_w C) \boxminus (B !_w D).$$

Theorem 5. Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ and $\alpha : \Omega \rightarrow [0, \infty)$ a bounded measurable function. If (A, B) and (C, D) have the opposite-synchronous monotone property and all of A, B, C, D are either increasing or decreasing, then

$$\begin{aligned} & \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxminus B_t) !_w (C_t \boxminus D_t)] d\mu(t) \\ & \geq \int_{\Omega} \alpha(t) (A_t !_w C_t) d\mu(t) \boxminus \int_{\Omega} \alpha(s) (B_s !_w D_s) d\mu(s). \end{aligned} \quad (28)$$

Proof. Let $s, t \in \Omega$ with $s \preceq t$. By applying Lemmas 1 and 8, Proposition 1, and Fubini's theorem, we get

$$\begin{aligned} & \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxminus B_t) !_w (C_t \boxminus D_t)] d\mu(t) - \int_{\Omega} \alpha(t) (A_t !_w C_t) d\mu(t) \boxminus \int_{\Omega} \alpha(s) (B_s !_w D_s) d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t \boxminus B_t) !_w (C_t \boxminus D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t !_w C_t) \boxminus (B_s !_w D_s)] d\mu(t) d\mu(s) \\ &\geq \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t !_w C_t) \boxminus (B_t !_w D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t !_w C_t) \boxminus (B_s !_w D_s)] d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [A_t !_w C_t - A_s !_w C_s] \boxminus [B_t !_w D_t - B_s !_w D_s] d\mu(t) d\mu(s). \end{aligned}$$

By Lemmas 1 and 4, $(A_t !_w C_t - A_s !_w C_s) \boxminus (B_t !_w D_t - B_s !_w D_s) \geq 0$ and hence (28) holds. $\square$

Lemma 9 ([18]). Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ with $0 < m_1 I \leq A \boxtimes B \leq M_1 I$ and $0 < m_2 I \leq C \boxtimes D \leq M_2 I$. Denote

$$\lambda = \left[\frac{(m_1 \sharp M_1) \nabla_{1-w} (m_2 \sharp M_2)}{(m_1 \nabla_{1-w} M_2) \sharp (M_1 \nabla_{1-w} m_2)} \right]^2. \quad (29)$$

If (A, B) and (C, D) have the synchronous property, then for any $w \in [0, 1]$,

$$\lambda (A \boxtimes B) !_w (C \boxtimes D) \leq (A !_w C) \boxtimes (B !_w D).$$

Theorem 6. Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ be such that for all $t \in \Omega$, $0 < m_1 I \leq A_t \boxtimes B_t \leq M_1 I$ and $0 < m_2 I \leq C_t \boxtimes D_t \leq M_2 I$. Let $\alpha : \Omega \rightarrow [0, \infty)$ be a bounded measurable function. If (A, B) and (C, D) have the synchronous monotone property and if either

1. A, C are increasing and B, D are decreasing, or
2. A, C are decreasing and B, D are increasing,

then with the constant λ in (29) we have

$$\begin{aligned} & \lambda \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxtimes B_t) !_w (C_t \boxtimes D_t)] d\mu(t) \\ & \leq \int_{\Omega} \alpha(t) (A_t !_w C_t) d\mu(t) \boxtimes \int_{\Omega} \alpha(s) (B_s !_w D_s) d\mu(s). \end{aligned} \quad (30)$$

Proof. Let $s, t \in \Omega$ with $s \preceq t$. We have by using Lemmas 1 and 9, Proposition 1, and Fubini's Theorem that

$$\begin{aligned} & \lambda \int_{\Omega} \alpha(s) d\mu(s) \int_{\Omega} \alpha(t) [(A_t \boxtimes B_t) !_w (C_t \boxtimes D_t)] d\mu(t) - \int_{\Omega} \alpha(t) (A_t !_w C_t) d\mu(t) \boxtimes \int_{\Omega} \alpha(s) (B_s !_w D_s) d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s) \alpha(t) [\lambda (A_t \boxtimes B_t) !_w (C_t \boxtimes D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t !_w C_t) \boxtimes (B_s !_w D_s)] d\mu(t) d\mu(s) \\ &\leq \iint_{\Omega^2} \alpha(s) \alpha(t) [(A_t !_w C_t) \boxtimes (B_t !_w D_t)] d\mu(t) d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t) \alpha(s) [(A_t !_w C_t) \boxtimes (B_s !_w D_s)] d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} \alpha(s) \alpha(t) [A_t !_w C_t - A_s !_w C_s] \boxtimes [B_t !_w D_t - B_s !_w D_s] d\mu(t) d\mu(s). \end{aligned}$$

We have, by Lemmas 1 and 4, $(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxdot (B_t \sharp_w D_t - B_s \sharp_w D_s) \leq 0$ and hence (30) holds. $\square$

Remark 1. When we set $\Omega = \{1, \dots, k\}$ equipped with the counting measure, we get discrete versions of Theorems 3–6. Matrix analogs of our results can be obtained particularly by setting $\mathbb{H} = \mathbb{C}^n$. In this case, our results include Chebyshev-type inequalities for Khatri–Rao products, Kronecker products and Hadamard products of matrices.

5. Chebyshev–Grüss Inequalities via Oscillations

In this section, we prove a Chebyshev–Grüss inequality via oscillations for continuous fields of operators. The basic setup is as follows.

Let Ω be a compact Hausdorff space equipped with a probability Radon measure μ . For any $A \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H}))$ and $B \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{K}))$, we define

$$\begin{aligned} A \boxdot B &= (A_t \boxdot B_t)_{t \in \Omega}, \\ \mathcal{I}(A) &= \int_{\Omega} A_t d\mu(t), \\ \text{osc}(A) &= \max\{\|A_t - A_s\| : (t, s) \in \text{supp}(\mu \times \mu)\}. \end{aligned}$$

Here, we recall that the support of the product measure $\mu \times \mu$ is defined by

$$\text{supp}(\mu \times \mu) = \{(t, s) \in \Omega^2 \mid (\mu \times \mu)(G) > 0 \text{ for all open sets } G \subseteq \Omega^2 \text{ containing } (t, s)\}$$

We call $\text{osc}(A)$ the oscillation of the field A .

In the next theorem, we generalize Grüss inequality (8) to the case of operators concerning Khatri–Rao products.

Theorem 7. For any $A \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H})^{sa})$ and $B \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{K})^{sa})$, we have

$$\mathcal{I}(A \boxdot B) - \mathcal{I}(A) \boxdot \mathcal{I}(B) \leq \frac{1}{2} \text{osc}(A) \text{osc}(B) (\mu \times \mu)(\Omega^2 \setminus \Delta) I, \quad (31)$$

where $\Delta = \{(t, t) : t \in \Omega\}$.

Proof. By using Lemma 1, Proposition 1 and Fubini's theorem, we have

$$\begin{aligned} \mathcal{I}(A \boxdot B) - \mathcal{I}(A) \boxdot \mathcal{I}(B) &= \int_{\Omega} d\mu(s) \int_{\Omega} A_t \boxdot B_t d\mu(t) - \int_{\Omega} A_t d\mu(t) \boxdot \int_{\Omega} B_s d\mu(s) \\ &= \iint_{\Omega^2} A_t \boxdot B_t d\mu(t) d\mu(s) - \iint_{\Omega^2} A_t \boxdot B_s d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} (A_t - A_s) \boxdot (B_t - B_s) d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2 \setminus \Delta} (A_t - A_s) \boxdot (B_t - B_s) d\mu(t) d\mu(s) \\ &\leq \frac{1}{2} \text{osc}(A) \text{osc}(B) (\mu \times \mu)(\Omega^2 \setminus \Delta) I. \quad \square \end{aligned}$$

Theorem 7 can be viewed as an operator version of ([8], Theorem 7).

Corollary 4. For any $A_i \in \mathcal{B}(\mathbb{H})^{sa}$ and $B_i \in \mathcal{B}(\mathbb{K})^{sa}$, $i = 1, \dots, k$, we have

$$\sum_{i=1}^k (A_i \boxdot B_i) - \left(\sum_{i=1}^k A_i \right) \boxdot \left(\sum_{i=1}^k B_i \right) \leq \frac{k(k-1)}{2} \max_{1 \leq i, j \leq k} \|A_i - A_j\| \cdot \max_{1 \leq i, j \leq k} \|B_i - B_j\| I.$$

Proof. Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$. Set Ω to be the finite space $\{1, \dots, k\}$ equipped with the counting measure in Theorem 7, we have

$$(\mu \times \mu)(\Omega^2 \setminus \Delta) = \frac{k(k-1)}{2}, \quad \text{supp}(\mu \times \mu) = \Omega^2$$

and thus $\text{osc}(A) = \max_{1 \leq i, j \leq k} \|A_i - A_j\|$, $\text{osc}(B) = \max_{1 \leq i, j \leq k} \|B_i - B_j\|$. $\square$

6. Conclusions and Future Work

We investigate integral inequalities of Chebyshev-type for bounded continuous fields of self-adjoint operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ . Under certain assumptions on the synchronous Khatri–Rao property of the fields of operators, we obtain Chebyshev-type inequalities concerning Khatri–Rao products. We also establish Chebyshev-type inequalities involving Khatri–Rao products and weighted Pythagorean means under the assumption of synchronous monotone property of the fields of operators. Moreover, we derive Chebyshev–Grüss integral inequality via oscillations when μ is a probability Radon measure. These integral inequalities can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results include Chebyshev-type inequalities for the tensor product of operators and Khatri–Rao/Kronecker/Hadamard products of matrices.

For future work, we may investigate Chebyshev-type inequalities when replacing weighted Pythagorean means by quasi-arithmetic power means, the logarithmic mean, or general operator means.

Author Contributions: All authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: The first author expresses his gratitude towards Thailand Science Research and Innovation (Thailand Research Fund) for providing the Royal Golden Jubilee Ph.D. Scholarship, grant no. PHD60K0225 to support his Ph.D. study.

Acknowledgments: This work was supported by Thailand Science Research and Innovation (Thailand Research Fund).

Conflicts of Interest: The authors declare no conflict of interest.

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