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A Note on Weakly *S*-Noetherian Rings

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Abstract: Let R be a commutative ring with identity and S a (not necessarily saturated) multiplicative subset of R. We call the ring R to be a weakly S-Noetherian ring if every S-finite proper ideal of R is an S-Noetherian R-module. In this article, we study some properties of weakly S-Noetherian rings. In particular, we give some conditions for the Nagata's idealization and the amalgamated algebra to be weakly S-Noetherian rings.

Keywords: S-finite, weakly S-Noetherian ring, S-Noetherian module, Nagata's idealization, amalgamated algebra along an ideal.

1. Introduction

Let R be a commutative ring with identity and M a unitary R-module. Recall that M is a *Noetherian module* if the ascending chain condition on submodules of M holds; and R is a *Noetherian ring* if R is a Noetherian R-module, or equivalently, every ideal of R is finitely generated. In commutative algebra, a Noetherian ring is a relevant topic. Due to its importance, not only Noetherian rings but also several kinds of rings related to Noetherian rings have been studied by many mathematicians. Weakly Noetherian rings and S-Noetherian rings are examples of rings related to Noetherian rings.

Let R be a commutative ring with identity. It is clear that if R is a Noetherian ring, then every proper ideal of R is a Noetherian R-module. In [1], Mahdou and Hassani posed the following question: if every finitely generated proper ideal of R is a Noetherian R-module, can we conclude that R is a Noetherian ring? From this point of view, they introduced the concept of weakly Noetherian rings. They defined R to be a *weakly Noetherian ring* if every finitely generated proper ideal of R is a Noetherian R-module. It is obvious that every Noetherian ring is a weakly Noetherian ring. In [1], the authors found an example of a weakly Noetherian ring which is not a Noetherian ring and studied several properties of weakly Noetherian rings.

Let R be a commutative ring with identity, S a (not necessarily saturated) multiplicative subset of R and M a unitary R-module. In [2], the authors introduced the concept of "almost finitely generated" to study Querre's characterization of divisorial ideals in integrally closed polynomial rings. Later, Anderson and Dumitrescu abstracted this notion to any commutative ring and introduced the concepts of S-Noetherian rings. Recall from [3] (Definition 1) that an ideal I of R is *S-finite* if there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$; and R is an *S-Noetherian ring* if each ideal of R is S-finite. Furthermore, we say that M is *S-finite* if $sM \subseteq F$ for some $s \in S$ and some finitely generated R-submodule F of M; and M is *S-Noetherian* if every R-submodule of M is S-finite. For more on S-Noetherian rings and S-finiteness, the readers can refer to [3–9].

Let R be a commutative ring with identity and S a (not necessarily saturated) multiplicative subset of R. If R is an S-Noetherian ring, then every ideal of R is S-finite; so every S-finite proper ideal of R is an S-Noetherian R-module. Hence it might be natural to ask if R is an S-Noetherian ring when every S-finite proper ideal of R is an S-Noetherian R-module. From this view, we define the notion of weakly

S-Noetherian rings. We say that R is a *weakly S-Noetherian ring* if every S-finite proper ideal of R is an S-Noetherian R-module. Clearly, every S-Noetherian ring is a weakly S-Noetherian ring. It is easy to see that R is a weakly S-Noetherian ring if and only if whenever I and J are proper ideals of R such that $I \subseteq J$ and J is S-finite, I is an S-finite ideal of R. If $\overline{S}$ is the saturation of S in R and I is an ideal of R, then I is S-finite if and only if I is $\overline{S}$-finite [3] (Proposition 2(c)); so R is a weakly S-Noetherian ring if and only if R is a weakly $\overline{S}$-Noetherian ring.

In this article, we study some properties of weakly S-Noetherian rings. In Section 2, we study basic properties of weakly S-Noetherian rings. We show that R is a weakly S-Noetherian ring in which every maximal ideal is S-finite if and only if R is an S-Noetherian ring. We also prove that R is a weakly Noetherian ring if and only if R is a weakly P-Noetherian ring for all prime ideals P of R. In Section 3, we study weakly S-Noetherian rings via the Nagata's idealization and the amalgamated algebra along an ideal. (Relevant definitions and notation will be reviewed in Section 3.) We show that if $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring, then R is a weakly S-Noetherian ring; and if R is a weakly S-Noetherian ring and M is an S-Noetherian R-module, then $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring. We also prove that if $R \bowtie^f J$ is a weakly S'-Noetherian ring and J is an S-finite R-module, then R is a weakly S-Noetherian ring; and if R is a weakly S-Noetherian ring and J is an S-Noetherian R-module contained in $J(T)$, then $R \bowtie^f J$ is a weakly S'-Noetherian ring.

Let R be a commutative ring with identity and S a multiplicative subset of R. If S contains 0, then every ideal of R is S-finite; so R is always an S-Noetherian ring. Hence in this paper, we assume that S does not contain 0 for avoiding the trivial case.

2. Basic properties

We start this section with some relations between a weakly S-Noetherian ring and an S-Noetherian ring.

Proposition 1. *Let R be a commutative ring with identity and S a multiplicative subset of R. Then the following assertions hold.*

(1) If R is a weakly S-Noetherian ring which contains a nonunit regular element, then R is an S-Noetherian ring. In particular, if R is an integral domain, then R is a weakly S-Noetherian ring if and only if R is an S-Noetherian ring.
(2) R is a weakly S-Noetherian ring in which every maximal ideal is S-finite if and only if R is an S-Noetherian ring.

Proof. (1) Let I be a proper ideal of R and take any nonunit regular element a of R. Then $aI \subseteq (a) \subsetneq R$. Since R is a weakly S-Noetherian ring and (a) is an S-finite proper ideal of R, we have that aI is also an S-finite ideal of R; so there exist an element $s \in S$ and a finitely generated ideal F of R such that $saI \subseteq F \subseteq aI$. Clearly, $F = aF_1$ for some finitely generated ideal F_1 of R; so $saI \subseteq aF_1 \subseteq aI$. Since a is a regular element of R, we have that $sI \subseteq F_1 \subseteq I$. Hence I is an S-finite ideal of R. Thus R is an S-Noetherian ring.

If R is a field, then the second assertion is obvious. If R is an integral domain which is not a field, then the equivalence follows from the first assertion.

(2) ($\Rightarrow$) Let I be a proper ideal of R. Then $I \subseteq M$ for some maximal ideal M of R. Since R is a weakly S-Noetherian ring and M is an S-finite ideal of R, we have that I is also an S-finite ideal of R. Thus R is an S-Noetherian ring.

($\Leftarrow$) This implication is obvious. □

Let $R \subseteq T$ be an extension of commutative rings with identity and S a (not necessarily saturated) multiplicative subset of R. We say that S is an *anti-Archimedean subset* of R if $\bigcap_{n\geq 1} s^n R \cap S \neq \emptyset$ for every $s \in S$. Let $R + XT[X] = \{f \in T[X] \mid f(0) \in R\}$ be a composite polynomial ring. Then it was shown that if S is an anti-Archimedean subset of R, then $R + XT[X]$ is an S-Noetherian ring if and only

if R is an S-Noetherian ring and T is an S-finite R-module [9] (Corollary 3.7(1)) (or [4] (Theorem 3.7)). In particular, if $R = T$, then the Hilbert basis theorem for S-Noetherian rings holds as follows: If S is an anti-Archimedean subset of R, then R is an S-Noetherian ring if and only if $R[X]$ is an S-Noetherian ring [3] (Proposition 9) (or [10] (Corollary 3.8(1))).

Corollary 1. *Let $R \subseteq T$ be an extension of commutative rings with identity and S an anti-Archimedean subset of R. Then the following conditions hold.*

(1) R is an S-Noetherian ring and T is an S-finite R-module if and only if $R + XT[X]$ is a weakly S-Noetherian ring.
(2) R is an S-Noetherian ring if and only if $R[X]$ is a weakly S-Noetherian ring.

Proof. (1) Note that X is a nonunit regular element of $R + XT[X]$; so by Proposition 1(1), we have that $R + XT[X]$ is a weakly S-Noetherian ring if and only if $R + XT[X]$ is an S-Noetherian ring. Thus the result follows directly from [9] (Corollary 3.7(1)) (or [4] (Theorem 3.7)).

(2) This can be obtained by applying $R = T$ to (1). □

Let $R \subseteq T$ be an extension of commutative rings with identity, $R + XT[\![X]\!] = \{f \in T[\![X]\!] \mid f(0) \in R\}$ a composite power series ring and S a multiplicative subset of R. Then it was shown that if S is an anti-Archimedean subset of R consisting of regular elements, then $R + XT[\![X]\!]$ is an S-Noetherian ring if and only if R is an S-Noetherian ring and T is an S-finite R-module (cf. [9] (Theorem 4.4) (or [4]) (Theorem 3.8)). As a special case, it was proved that if S is an anti-Archimedean subset of R consisting of regular elements, then R is an S-Noetherian ring if and only if $R[\![X]\!]$ is an S-Noetherian ring [3] (Proposition 10) (or [10] (Corollary 3.8(2))).

Corollary 2. *Let $R \subseteq T$ be an extension of commutative rings with identity and S an anti-Archimedean subset of R consisting of regular elements. Then the following assertions hold.*

(1) R is an S-Noetherian ring and T is an S-finite R-module if and only if $R + XT[\![X]\!]$ is a weakly S-Noetherian ring.
(2) R is an S-Noetherian ring if and only if $R[\![X]\!]$ is a weakly S-Noetherian ring.

Proof. The results can be shown by similar arguments as in the proof of Corollary 1. □

The next examples show that a weakly T-Noetherian ring need not be a weakly S-Noetherian ring, where $S \subseteq T$ are multiplicative sets.

Example 1. *Let p be a prime integer, $R = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$, S the multiplicative subset of R generated by $(1, p, p, \dots)$ and T the multiplicative subset of R generated by the set $\{(1, p, p, \dots), (1, p, p, 0, 0, \dots)\}$.*

(1) For each $i \geq 2$ and $j \geq 1$, let

$$a_{ij} = \begin{cases} p & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and for each $n \geq 2$, let $a_n = (a_{n1}, a_{n2}, \dots)$. Let I be the ideal of R generated by the set $\{a_n \mid n \geq 2\}$ and J the ideal of R generated by $(p, p, \dots)$. Then J is a finitely generated ideal of R; so J is an S-finite ideal of R. Suppose to the contrary that I is an S-finite ideal of R. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sI \subseteq F \subseteq I$. Note that $s = (1, p, p, \dots)^n$ for some integer $n \geq 1$. Since F is finitely generated, we have that $F \subseteq (a_2, \dots, a_m)$ for some $m \geq 2$. Note that the $(m + n)$th coefficient of $(1, p, p, \dots)^n a_{m+n}$ is p^{n+2}; so $(1, p, p, \dots)^n a_{m+n} \notin F$. This is a contradiction. Hence I is not an S-finite ideal of R. Thus R is not a weakly S-Noetherian ring.
(2) Let I be an ideal of R. Then $(1, p, p, 0, 0, \dots)I$ is a finite ideal of R; so I is a T-finite ideal of R. Hence R is a T-Noetherian ring. Thus R is a weakly T-Noetherian ring.

Example 2. *Let D be a Noetherian ring, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over D, I the ideal of $D[\mathbf{X}]$ generated by the set $\{X_iX_jX_k \mid i, j, k \text{ are pairwise distinct}\}$ and $R = D[\mathbf{X}]/I$. For each $f \in D[\mathbf{X}]$, let $\overline{f}$ denote the homomorphic image of f in R. Let S be the multiplicative subset of R generated by $\overline{X_1}$ and let T be the multiplicative subset of R generated by $\overline{X_1}$ and $\overline{X_2}$.*

(1) Let A_1 be the ideal of R generated by the set $\{\overline{X_1}\,\overline{X_i} \mid i \geq 2\}$ and let A_2 be the ideal of R generated by $\overline{X_1}$. Then A_2 is a principal ideal of R; so A_2 is an S-finite ideal of R. Suppose to the contrary that A_1 is an S-finite ideal of R. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sA_1 \subseteq F \subseteq A_1$. Note that $F \subseteq (\overline{X_1}\,\overline{X_2}, \ldots, \overline{X_1}\,\overline{X_m})$ for some integer $m \geq 2$. However, an easy calculation shows that $s\overline{X_1}\,\overline{X_{m+1}} \notin (\overline{X_1}\,\overline{X_2}, \ldots, \overline{X_1}\,\overline{X_m})$; so $s\overline{X_1}\,\overline{X_{m+1}} \notin F$. This is a contradiction. Hence A_1 is not an S-finite ideal of R. Thus R is not a weakly S-Noetherian ring.

(2) Let A be a proper ideal of R. Then for any $\overline{f} \in A$, there exists an element $g \in D[X_1, X_2]$ such that $\overline{g} = \overline{X_1}\,\overline{X_2}\,\overline{f}$. Let C be the ideal of $D[X_1, X_2]$ generated by the set $\{g \in D[X_1, X_2] \mid \overline{g} \in \overline{X_1}\,\overline{X_2}A\}$. Since $D[X_1, X_2]$ is a Noetherian ring, we have that $C = (g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in D[X_1, X_2]$. Note that $\overline{X_1}\,\overline{X_2}A = (\overline{g_1}, \ldots, \overline{g_n})$; so A is a T-finite ideal of R. Hence R is a T-Noetherian ring. Thus R is a weakly T-Noetherian ring.

Let R and T be commutative rings with identity and $\phi : R \to T$ a ring homomorphism. For an ideal I of R, the *extension* I^e of I is the ideal of T generated by $\phi(I)$; and for an ideal A of T, the *contraction* of A is the ideal $A^c = \{r \in R \mid \phi(r) \in A\}$ of R.

Proposition 2. *Let R and T be commutative rings with identity, $\phi : R \to T$ a ring homomorphism and S a multiplicative subset of R. Suppose that $A^{ce} = A$ for all ideals A of T and B^c is an S-finite ideal of R for all $\phi(S)$-finite ideals B of T. If R is a weakly S-Noetherian ring, then T is a weakly $\phi(S)$-Noetherian ring.*

Proof. Let $A \subseteq B$ be proper ideals of T such that B is $\phi(S)$-finite. Then by the assumption, B^c is an S-finite ideal of R. If $B^c = R$, then by the assumption, we have that $B = B^{ce} = R^e = T^{ce} = T$, which is a contradiction. So B^c is a proper ideal of R. Since R is a weakly S-Noetherian ring, we have that A^c is an S-finite ideal of R; so there exist an element $s \in S$ and a finitely generated ideal F of R such that $sA^c \subseteq F \subseteq A^c$. Therefore we obtain

$$\phi(s)A = \phi(s)A^{ce} = (sA^c)^e \subseteq F^e \subseteq A^{ce} = A.$$

Note that F^e is a finitely generated ideal of T. Hence A is a $\phi(S)$-finite ideal of T. Thus T is a weakly $\phi(S)$-Noetherian ring. □

Corollary 3. *Let $R \subseteq T$ be an extension of commutative rings with identity and S a multiplicative subset of R. Suppose that $(A \cap R)T = A$ for all ideals A of T and $B \cap R$ is an S-finite ideal of R for all S-finite ideals B of T. If R is a weakly S-Noetherian ring, then T is also a weakly S-Noetherian ring.*

Proof. By considering the natural ring monomorphism $\phi : R \to T$, the result follows from Proposition 2. □

Let R be a commutative ring with identity and S a multiplicative subset of R. Let $Q = \{r \in R \mid rS = \{0\}\}$ and $\phi : R \to R/Q$ the canonical ring epimorphism. Then $\phi(S)$ is a regular multiplicative subset of R. We denote by R_S the quotient ring $(R/Q)_{\phi(S)}$. Let $\tau : R/Q \to (R/Q)_{\phi(S)}$ be the natural ring homomorphism and $\psi = \tau \circ \phi$. Then it is well known that $\psi(s)$ is a unit in R_S for all $s \in S$ and $A = A^{ce}$ for all ideals A of R_S. Hence we have the following result.

Corollary 4. *Let R be a commutative ring with identity and S a multiplicative subset of R. Suppose that A^c is an S-finite ideal of R for all finitely generated ideals A of R_S. If R is a weakly S-Noetherian ring, then R_S is a weakly Noetherian ring.*

Proof. By Proposition 2, we have that R_S is a weakly $\psi(S)$-Noetherian ring. Since $\psi(s)$ is a unit in R_S for all $s \in S$, we have that R_S is a weakly Noetherian ring. □

Corollary 5. *Let R be a commutative ring with identity and S a multiplicative subset of R. Suppose that for every finitely generated ideal I of R, we have that $(IR_S)^c = (I : s)$ for some $s \in S$. If R is a weakly S-Noetherian ring, then R_S is a weakly Noetherian ring.*

Proof. Let A be a finitely generated ideal of R_S. Then there exists a finitely generated ideal I of R such that $A = IR_S$. By the assumption, we have that $A^c = (I : s)$ for some $s \in S$. Since $s(I : s) \subseteq I \subseteq (I : s)$, we have that $(I : s)$ is an S-finite ideal of R. Hence A^c is an S-finite ideal of R. Thus by Corollary 4, we have that R_S is a weakly Noetherian ring. □

Proposition 3. *Let R and T be commutative rings with identity, $\phi : R \to T$ a ring homomorphism and S a multiplicative subset of R. Suppose that $I^{ec} = I$ for all ideals I of R. If T is a weakly $\phi(S)$-Noetherian ring, then R is a weakly S-Noetherian ring.*

Proof. Let $I \subseteq J$ be proper ideals of R such that J is S-finite. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sJ \subseteq F \subseteq J$; so $\phi(s)J^e \subseteq F^e \subseteq J^e$. Note that F^e is a finitely generated ideal of T; so J^e is a $\phi(S)$-finite ideal of T. If $J^e = T$, then $J = J^{ec} = T^c = R$, which is a contradiction. So J^e is a proper ideal of T. Since T is a weakly $\phi(S)$-Noetherian ring, we have that I^e is also a $\phi(S)$-finite ideal of T; so there exist an element $u \in S$ and a finitely generated ideal G of T such that $\phi(u)I^e \subseteq G \subseteq I^e$. Since G is finitely generated, we have that $G \subseteq (a_1, \ldots, a_n)^e$ for some $a_1, \ldots, a_n \in I$. Therefore we obtain

$$uI = (uI)^{ec} = (\phi(u)I^e)^c \subseteq G^c \subseteq (a_1, \ldots, a_n)^{ec} = (a_1, \ldots, a_n) \subseteq I.$$

Hence I is an S-finite ideal of R. Thus R is a weakly S-Noetherian ring. □

Corollary 6. *Let $R \subseteq T$ be an extension of commutative rings with identity and S a multiplicative subset of R. Suppose that $IT \cap R = I$ for all ideals I of R. If T is a weakly S-Noetherian ring, then R is also a weakly S-Noetherian ring.*

Proof. Let $\phi : R \to T$ be the natural monomorphism. Then by the assumption, we have that $I^{ec} = I$ for all ideals I of R. Thus this result follows directly from Proposition 3. □

Corollary 7. *Let R be a commutative ring with identity, S a multiplicative subset of R and $\psi : R \to R_S$ the natural ring homomorphism. Suppose that $I^{ec} = I$ for all ideals I of R. Then the following assertions are equivalent.*

(1) R is a weakly S-Noetherian ring.
(2) R_S is a weakly Noetherian ring.
(3) R_S is a weakly $\psi(S)$-Noetherian ring.

Proof. (1) ⇒ (2) Let A be a finitely generated ideal of R_S. Then there exists a finitely generated ideal I of R such that $A = I^e$. Since $I = I^{ec}$, we have that A^c is a finitely generated ideal of R. Thus by Corollary 4, we have that R_S is a weakly Noetherian ring.

(2) ⇔ (3) This equivalence follows from the fact that $\psi(s)$ is a unit in R_S for all $s \in S$.

(3) ⇒ (1) This implication follows from Proposition 3. □

Remark 1. *Let R be a commutative ring with identity and S a multiplicative subset of R. If R is a weakly Noetherian ring, then R is a weakly S-Noetherian ring. To see this, let J be an S-finite proper ideal of R and I a subideal of J. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sJ \subseteq F \subseteq J$. Note*

that $sI \subseteq F$. Since R is a weakly Noetherian ring, we have that sI is a finitely generated ideal of R. Hence I is an S-finite ideal of R, which implies that J is an S-Noetherian R-module. Thus R is a weakly S-Noetherian ring.

We give an example of a weakly S-Noetherian ring which is not a weakly Noetherian ring. This shows that the converse of Remark 1 does not hold in general.

Example 3. *Let p be a prime integer and $R = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$.*

(1) *Let $S = \{(1, p, p, 0, 0, \dots), (1, 0, p^2, 0, 0, \dots), (1, 0, 0, \dots)\}$. Then S is a multiplicative subset of R. Let I be an ideal of R. Then for any $s \in S$, sI is a finite ideal of R; so I is an S-finite ideal of R. Hence R is an S-Noetherian ring, and thus R is a weakly S-Noetherian ring.*
(2) *Let I be the ideal of R generated by $(1, p, p, \dots)$ and J the ideal of R generated by the set $\{(1, p, 0, 0, \dots), (1, 0, p, 0, 0, \dots), (1, 0, 0, p, 0, 0, \dots), \dots\}$. Then J is a subideal of the principal ideal I. However, J is not finitely generated. Thus R is not a weakly Noetherian ring.*

Example 4. *Let D be a Noetherian ring, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over D, I the ideal of $D[\mathbf{X}]$ generated by the set $\{X_i X_j \mid i \neq j\}$ and $R = D[\mathbf{X}]/I$. For each $f \in D[\mathbf{X}]$, let $\overline{f}$ denote the homomorphic image of f in R.*

(1) Note that $(\overline{X_1}) \subseteq (\overline{X_1}, \overline{X_2}) \subseteq \cdots$ is a strictly ascending chain of ideals of R; so R is not a Noetherian ring. Since R contains a nonunit regular element $\overline{1 + X_1}$, we have that R is not a weakly Noetherian ring [1] (Theorem 1(2)).
(2) Let S be the multiplicative subset of R generated by $\overline{X_1}$ and let s be any element of S. Let B be a proper ideal of R. Then for any $\overline{f} \in B$, we can find an element $g \in D[X_1]$ such that $\overline{g} = s\overline{f}$. Let C be the ideal of $D[X_1]$ generated by the set $\{g \in D[X_1] \mid \overline{g} \in sB\}$. Since $D[X_1]$ is a Noetherian ring, we have that $C = (g_1, \dots, g_n)$ for some $g_1, \dots, g_n \in D[X_1]$. Note that $sB = (\overline{g_1}, \dots, \overline{g_n})$; so B is an S-finite ideal of R. Hence R is an S-Noetherian ring. Thus R is a weakly S-Noetherian ring.

Let R be a commutative ring with identity and P a prime ideal of R. Then $R \setminus P$ is a multiplicative subset of R. We define R to be a *weakly P-Noetherian ring* if R is a weakly $(R \setminus P)$-Noetherian ring. The next result is a characterization of weakly Noetherian rings.

Proposition 4. *Let R be a commutative ring with identity. Then the following conditions are equivalent.*

(1) R is a weakly Noetherian ring.
(2) R is a weakly P-Noetherian ring for all prime ideals P of R.
(3) R is a weakly M-Noetherian ring for all maximal ideals M of R.

Proof. (1) $\Rightarrow$ (2) This implication was shown in Remark 1.

(2) $\Rightarrow$ (3) This implication is obvious.

(3) $\Rightarrow$ (1) Suppose that R is a weakly M-Noetherian ring for all maximal ideals M of R. Let $I \subseteq J$ be proper ideals of R such that J is finitely generated. Then for each maximal ideal M of R, there exist an element $s_M \in R \setminus M$ and a finitely generated ideal F_M of R such that $s_M I \subseteq F_M \subseteq I$. Note that $\{s_M \mid M \text{ is a maximal ideal of } R\}$ is not contained in any maximal ideal of R; so we can choose $s_{M_1}, \dots, s_{M_n} \in R$ such that $(s_{M_1}, \dots, s_{M_n}) = R$. Therefore we obtain

$$I = (s_{M_1}, \dots, s_{M_n})I \subseteq F_{M_1} + \cdots + F_{M_n} \subseteq I,$$

which implies that $I = F_{M_1} + \cdots + F_{M_n}$. Hence I is a finitely generated ideal of R. Thus R is a weakly Noetherian ring. □

Let R be a commutative ring with identity. Recall that R is *quasi-local* if R has only one maximal ideal. As an easy consequence of Proposition 4, we obtain

Corollary 8. *Let R be a commutative ring with identity. If R is a quasi-local ring with maximal ideal M, then R is a weakly Noetherian ring if and only if R is a weakly M-Noetherian ring.*

Example 5. *Let F be a field, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over F and $F[\![\mathbf{X}]\!]_1$ the ring of formal power series of type one over F,* i.e., *$F[\![\mathbf{X}]\!]_1$ is the union of the ascending net of rings $F[\![\mathbf{B}]\!]$, where $\mathbf{B}$ runs over all finite subsets of $\mathbf{X}$. Let A be the ideal of $F[\![\mathbf{X}]\!]_1$ generated by the set $\{X_iX_j \mid i \neq j\}$ and let $R = F[\![\mathbf{X}]\!]_1/A$. For each $f \in F[\![\mathbf{X}]\!]_1$, set $\overline{f} = f + A$.*

(1) Let J be a finitely generated proper ideal of R and I a subideal of J. Then $J = (\overline{f_1}, \ldots, \overline{f_m})$ for some $f_1, \ldots, f_m \in F[\![\mathbf{X}]\!]_1$; so $f_1, \ldots, f_m \in F[\![X_1, \ldots, X_n]\!]$ for some integer $n \geq 1$. Therefore $(f_1, \ldots, f_m)F[\![X_1, \ldots, X_n]\!] \subseteq (X_1, \ldots, X_n)F[\![X_1, \ldots, X_n]\!]$, which shows that $(f_1, \ldots, f_m)F[\![\mathbf{X}]\!]_1 \subseteq (X_1, \ldots, X_n)F[\![\mathbf{X}]\!]_1$. Hence $J \subseteq (\overline{X_1}, \ldots, \overline{X_n})R$, which indicates that every element of I is of the form $\overline{g_1}\overline{X_1} + \cdots + \overline{g_n}\overline{X_n}$ for some $g_i \in F[\![X_i]\!]$ for each $i \in \{1, \ldots, n\}$. Let C be the ideal of $F[\![X_1, \ldots, X_n]\!]$ generated by the set $\{g_1X_1 + \cdots + g_nX_n \mid$ for each $i = 1, \ldots, n$, $g_i \in F[\![X_i]\!]$ and $\overline{g_1}\overline{X_1} + \cdots + \overline{g_n}\overline{X_n} \in I\}$. Since $F[\![X_1, \ldots, X_n]\!]$ is a Noetherian ring, we have that $C = (h_1, \ldots, h_p)$ for some $h_1, \ldots, h_p \in F[\![X_1, \ldots, X_n]\!]$; so $I = (\overline{h_1}, \ldots, \overline{h_p})$. Hence R is a weakly Noetherian ring. Thus by Remark 1, we have that R is a weakly S-Noetherian ring for any multiplicative subset S of R.

(2) Let S be the set of units in R. Then S is a multiplicative subset of R. Note that $(\overline{X_1}, \overline{X_2}, \ldots)$ is not a finitely generated ideal of R; so R is not a Noetherian ring. Thus R is not an S-Noetherian ring.

Proposition 5. *Let $n \geq 2$ be an integer. Let $R_1, \ldots, R_n$ be commutative rings with identity and $S_1, \ldots, S_n$ multiplicative subsets of $R_1, \ldots, R_n$, respectively. Let $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} S_i$. Then the following assertions are equivalent.*

(1) R is a weakly S-Noetherian ring.
(2) For all $i = 1, \ldots, n$, R_i is an S_i-Noetherian ring.

Proof. (1) $\Rightarrow$ (2) Let I_1 be a proper ideal of R_1. Then $I_1 \times (0) \times \cdots \times (0) \subseteq R_1 \times (0) \times \cdots \times (0)$ are proper ideals of R. Note that $R_1 \times (0) \times \cdots \times (0)$ is an S-finite ideal of R. Since R is a weakly S-Noetherian ring, we have that $I_1 \times (0) \times \cdots \times (0)$ is also an S-finite ideal of R; so there exist an element $(s_1, \ldots, s_n) \in S$ and a finitely generated ideal $F_1 \times \cdots \times F_n$ of R such that $(s_1, \ldots, s_n)(I_1 \times (0) \times \cdots \times (0)) \subseteq F_1 \times \cdots \times F_n \subseteq I_1 \times (0) \times \cdots \times (0)$. Therefore $s_1I_1 \subseteq F_1 \subseteq I_1$. Note that F_1 is a finitely generated ideal of R_1. Hence I_1 is an S_1-finite ideal of R_1. Thus R_1 is an S_1-Noetherian ring.

A similar argument as above shows that R_i is an S_i-Noetherian ring for all $i = 2, \ldots, n$.

(2) $\Rightarrow$ (1) Suppose that for all $i = 1, \ldots, n$, R_i is an S_i-Noetherian ring. Then R is an S-Noetherian ring [10] (Corollary 2.9). Thus R is a weakly S-Noetherian ring. □

The following example shows that the S_i-Noetherian condition in Proposition 5(2) cannot be replaced by the weakly S_i-Noetherian condition.

Example 6. *Take R_1 a weakly S_1-Noetherian ring which is not an S_1-Noetherian ring as in Example 5. Let $n \geq 2$ be an integer. Let $R_2, \ldots, R_n$ be commutative rings with identity and $S_2, \ldots, S_n$ multiplicative subsets of $R_2, \ldots, R_n$, respectively. Let $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} S_i$. Then there exists an ideal I_1 of R_1 which is not S_1-finite; so $I_1 \times (0) \times \cdots \times (0)$ is not an S-finite ideal of R. Note that $I_1 \times (0) \times \cdots \times (0) \subseteq R_1 \times (0) \times \cdots \times (0)$ and $R_1 \times (0) \times \cdots \times (0)$ is an S-finite ideal of R. Hence, R is never a weakly S-Noetherian ring.*

3. Some Extensions of Weakly S-Noetherian Rings

In this section, we study the weakly S-Noetherian property in the amalgamated algebra along an ideal and the Nagata's idealization. To do this, we require the next lemma.

Let R be a commutative ring with identity. Recall that the *Jacobson radical* of R is the intersection of all maximal ideals of R and is denoted by $J(R)$. Let S be a multiplicative subset of R. For an ideal I of R with $I \cap S = \emptyset$, let $S/I = \{s + I \mid s \in S\}$. Then S/I is a multiplicative subset of R/I.

Lemma 1. *Let R be a commutative ring with identity and S a multiplicative subset of R. If I is an ideal of R with $I \cap S = \emptyset$, then the following assertions hold.*

(1) If R is a weakly S-Noetherian ring and I is an S-finite ideal of R, then R/I is a weakly (S/I)-Noetherian ring.
(2) If R/I is a weakly (S/I)-Noetherian ring and I is an S-Noetherian R-module contained in $J(R)$, then R is a weakly S-Noetherian ring.

Proof. (1) Let $J_1/I \subseteq J_2/I$ be proper ideals of R/I such that J_2/I is (S/I)-finite. Then we can find $s \in S$ and $a_1, \ldots, a_n \in R$ such that $(s + I)(J_2/I) \subseteq (a_1 + I, \ldots, a_n + I) \subseteq J_2/I$; so $sJ_2 \subseteq (a_1, \ldots, a_n) + I \subseteq J_2$. Since I is an S-finite ideal of R, there exist an element $t \in S$ and a finitely generated ideal B of R such that $tI \subseteq B \subseteq I$; so $stJ_2 \subseteq t(a_1, \ldots, a_n) + B \subseteq J_2$. Therefore J_2 is an S-finite proper ideal of R. Since R is a weakly S-Noetherian ring and $J_1 \subseteq J_2$, we have that J_1 is also an S-finite ideal of R; so we can find an element $u \in S$ and a finitely generated ideal C of R such that $uJ_1 \subseteq C \subseteq J_1$. Hence $(u + I)(J_1/I) \subseteq (C + I)/I \subseteq J_1/I$, which means that J_1/I is an (S/I)-finite ideal of R/I. Thus R/I is a weakly (S/I)-Noetherian ring.

(2) Let $I_1 \subseteq I_2$ be proper ideals of R such that I_2 is S-finite. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sI_2 \subseteq F \subseteq I_2$. Set $J_1 = I_1 + I$ and $J_2 = I_2 + I$. Since $I \subseteq J(R)$, we have that J_2 is a proper ideal of R. Also, we have that $(s + I)(J_2/I) \subseteq (F + I)/I \subseteq J_2/I$; so J_2/I is an (S/I)-finite proper ideal of R/I. Since R/I is a weakly (S/I)-Noetherian ring and $J_1/I \subseteq J_2/I$, we have that J_1/I is also an (S/I)-finite ideal of R/I. Note that $(I_1 \cap I) \cap S = \emptyset$. By an easy calculation, we have that $I_1/(I_1 \cap I)$ is an $(S/(I_1 \cap I))$-finite ideal of $R/(I_1 \cap I)$. Since I is an S-Noetherian R-module, we have that $I_1 \cap I$ is an S-finite ideal of R. By a similar argument as in the proof of (1), we have that I_1 is an S-finite ideal of R. Thus R is a weakly S-Noetherian ring. □

Let R be a commutative ring with identity and M a unitary R-module. Then the *Nagata's idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M = \{(r, m) \mid r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. Clearly, $(1, 0)$ is the identity of $R(+)M$. Also, it was shown that if A is a maximal ideal of $R(+)M$, then $A = Q(+)M$ for some maximal ideal Q of R [11] (Theorem 3.2(1)); and if Q is a maximal ideal of R, then $Q(+)M$ is a maximal ideal of $R(+)M$ (cf. [12] (Theorem 25.1(3))). Hence $J(R(+)M) = J(R)(+)M$ [11] (Theorem 3.2(1)). For more on the Nagata's idealization, the readers can refer to [11,12].

The next example shows that the "S-Noetherian" condition in Lemma 1(2) cannot be replaced by the "S-finite" condition.

Example 7. *Let K be a field, V an infinite dimensional K-vector space, $T = K(+)V$ and M the unique maximal ideal of T. Let $R = T(+)T$, $I = \{(0, \mathbf{O})\}(+)T$ and $S = \{((1, \mathbf{O}), (0, \mathbf{O}))\}$, where $\mathbf{O}$ is the zero vector in V.*

(1) R/I is isomorphic to T; so R/I is a weakly Noetherian ring which is not a Noetherian ring (cf. [1] (Example 1)). Hence R/I is a weakly (S/I)-Noetherian ring by Remark 1.
(2) Note that $J(R) = J(T)(+)T$; so $I \subseteq J(R)$.
(3) Note that I is a finitely generated ideal of R; so I is an S-finite R-module.
(4) R is not a weakly S-Noetherian ring because I is an S-finite R-module but $\{(0, \mathbf{O})\}(+)M$ is not an S-finite R-module.

Let R be a commutative ring with identity, M a unitary R-module and S a multiplicative subset of R. Then it is clear that $S(+)M$ is a multiplicative subset of $R(+)M$.

Theorem 1. *Let R be a commutative ring with identity, S a multiplicative subset of R and M a unitary R-module. Then the following statements hold.*

(1) If $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring, then R is a weakly S-Noetherian ring.
(2) If R is a weakly S-Noetherian ring and M is an S-Noetherian R-module, then $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring.

Proof. (1) Let $I_1 \subseteq I_2$ be proper ideals of R such that I_2 is S-finite. Then there exist an element $s \in S$ and a finitely generated ideal F of R such that $sI_2 \subseteq F \subseteq I_2$; so we obtain

$$(s,0)(I_2(+)I_2M) \subseteq F(+)FM \subseteq I_2(+)I_2M.$$

Note that $F(+)FM$ is a finitely generated ideal of $R(+)M$; so $I_2(+)I_2M$ is an $(S(+)M)$-finite ideal of $R(+)M$. Since $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring and $I_1(+)I_1M \subseteq I_2(+)I_2M$, we have that $I_1(+)I_1M$ is an $(S(+)M)$-finite ideal of $R(+)M$; so there exist $(t,m) \in S(+)M$ and $(a_1,b_1),\ldots,(a_n,b_n) \in R(+)M$ such that

$$(t,m)(I_1(+)I_1M) \subseteq ((a_1,b_1),\ldots,(a_n,b_n)) \subseteq I_1(+)I_1M.$$

Hence $tI_1 \subseteq (a_1,\ldots,a_n) \subseteq I_1$, which indicates that I_1 is an S-finite ideal of R. Thus R is a weakly S-Noetherian ring.

(2) Let $(I_1(+)M)/((0)(+)M) \subseteq (I_2(+)M)/((0)(+)M)$ be proper ideals of $(R(+)M)/((0)(+)M)$ such that $(I_2(+)M)/((0)(+)M)$ is $((S(+)M)/((0)(+)M))$-finite. Then there exist $(t,m) \in S(+)M$ and $(a_1,b_1),\ldots,(a_n,b_n) \in R(+)M$ such that

$$\begin{aligned}((t,m)+((0)(+)M))((I_2(+)M)/((0)(+)M)) &\subseteq ((a_1,b_1)+((0)(+)M),\ldots,(a_n,b_n)+((0)(+)M))\\ &\subseteq (I_2(+)M)/((0)(+)M);\end{aligned}$$

so $tI_2 \subseteq (a_1,\ldots,a_n) \subseteq I_2$. Therefore I_2 is an S-finite ideal of R. Since R is a weakly S-Noetherian ring and $I_1 \subseteq I_2$, we have that I_1 is also an S-finite ideal of R; so there exist an element $w \in S$ and a finitely generated ideal F of R such that $wI_1 \subseteq F \subseteq I_1$. Hence we obtain

$$\begin{aligned}((w,0)+((0)(+)M))((I_1(+)M)/((0)(+)M)) &\subseteq (F(+)M)/((0)(+)M)\\ &\subseteq (I_1(+)M)/((0)(+)M).\end{aligned}$$

Note that $(F(+)M)/((0)(+)M)$ is a finitely generated ideal of $(R(+)M)/((0)(+)M)$; so $(I_1(+)M)/((0)(+)M)$ is an $((S(+)M)/((0)(+)M))$-finite ideal of $(R(+)M)/((0)(+)M)$. Thus $(R(+)M)/((0)(+)M)$ is a weakly $((S(+)M)/((0)(+)M))$-Noetherian ring.

Let A be an $(R(+)M)$-submodule of $(0)(+)M$. Then $A = (0)(+)N$ for some R-submodule N of M. Since M is an S-Noetherian R-module, we have that N is S-finite; so there exist an element $s \in S$ and a finitely generated R-submodule L of N such that $sN \subseteq L$. Therefore we obtain

$$(s,0)((0)(+)N) \subseteq (0)(+)L \subseteq (0)(+)N.$$

Note that $(0)(+)L$ is a finitely generated $(R(+)M)$-module. Hence A is an $(S(+)M)$-finite $(R(+)M)$-module. Thus $(0)(+)M$ is an $(S(+)M)$-Noetherian $(R(+)M)$-module.

It is clear that $J(R(+)M)$ contains $(0)(+)M$. Thus by Lemma 1(2), we have that $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring. □

Corollary 9 ((*cf.* [8] (Theorem 3.8))). *Let R be a commutative ring with identity, S a multiplicative subset of R and M a unitary R-module. Then the following assertions are equivalent.*

(1) R is an S-Noetherian ring and M is an S-Noetherian R-module.
(2) $R(+)M$ is an $(S(+)M)$-Noetherian ring.

Proof. (1) $\Rightarrow$ (2) Suppose that R is an S-Noetherian ring. Then by Proposition 1(2), we have that R is a weakly S-Noetherian ring in which every maximal ideal is S-finite. Since M is an S-Noetherian R-module, we have that $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring by Theorem 1(2). Let A be a maximal ideal of $R(+)M$. Then $A = Q(+)M$ for some maximal ideal Q of R. Note that Q is an S-finite ideal of R; so there exist $s \in S$ and $r_1, \ldots, r_n \in R$ such that $sQ \subseteq (r_1, \ldots, r_n) \subseteq Q$. Also, since M is an S-Noetherian R-module, we have that M is an S-finite R-module; so there exist $t \in S$ and $m_1, \ldots, m_k \in M$ such that $tM \subseteq Rm_1 + \cdots + Rm_k$. Hence we obtain

$$(st, 0)(Q(+)M) \subseteq ((tr_1, 0), \ldots, (tr_n, 0), (0, m_1), \ldots, (0, m_k)) \subseteq Q(+)M,$$

which means that $Q(+)M$ is an $(S(+)M)$-finite ideal of $R(+)M$. Thus by Proposition 1(2), we have that $R(+)M$ is an $(S(+)M)$-Noetherian ring.

(2) $\Rightarrow$ (1) Suppose that $R(+)M$ is an $(S(+)M)$-Noetherian ring. Then by Proposition 1(2), we have that $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring in which every maximal ideal is $(S(+)M)$-finite. Hence by Theorem 1(1), we have that R is a weakly S-Noetherian ring. Let Q be a maximal ideal of R. Then $Q(+)M$ is a maximal ideal of $R(+)M$. Since $R(+)M$ is an $(S(+)M)$-Noetherian ring, we have that $Q(+)M$ is an $(S(+)M)$-finite ideal of $R(+)M$. Therefore we can find $(s, m) \in S(+)M$ and $(a_1, m_1), \ldots, (a_n, m_n) \in R(+)M$ such that $(s, m)(Q(+)M) \subseteq ((a_1, m_1), \ldots, (a_n, m_n)) \subseteq Q(+)M$. Hence $sQ \subseteq (a_1, \ldots, a_n) \subseteq Q$, which indicates that Q is an S-finite ideal of R. Thus by Proposition 1(2), we have that R is an S-Noetherian ring.

Let N be an R-submodule of M. Then $(0)(+)N$ is an ideal of $R(+)M$. Since $R(+)M$ is an $(S(+)M)$-Noetherian ring, there exist $(s, m) \in S(+)M$ and $n_1, \ldots, n_k \in N$ such that $(s, m)((0)(+)N) \subseteq ((0, n_1), \ldots, (0, n_k))$. Hence $sN \subseteq Rn_1 + \cdots + Rn_k$, which means that N is an S-finite R-module. Thus M is an S-Noetherian R-module. □

The next example shows that the "S-Noetherian" condition in Theorem 1(2) cannot be replaced by the "S-finite" condition.

Example 8. *Let K be a field, V an infinite dimensional K-vector space, $R = K(+)V$, $S = \{(1, \mathbf{O})\}$ and N the maximal ideal of R, where $\mathbf{O}$ is the zero vector in V.*

(1) R is a weakly Noetherian ring which is not a Noetherian ring [1] (Example 1). Thus by Remark 1, we have that R is a weakly S-Noetherian ring which is not an S-Noetherian ring.
(2) Note that R is a finitely generated R-module; so R is an S-finite R-module. However, by (1), we have that R is not an S-Noetherian R-module.
(3) Note that $\{(0, \mathbf{O})\}(+)R$ is an $(S(+)R)$-finite $(R(+)R)$-module but $\{(0, \mathbf{O})\}(+)N$ is not an $(S(+)R)$-finite $(R(+)R)$-module. Hence $R(+)R$ is not a weakly $(S(+)R)$-Noetherian ring.

Let R be a commutative ring with identity and M a unitary R-module. Then $\mathrm{ann}_R(M) = \{r \in R \mid rM = \{0\}\}$ is an ideal of R and is called the *annihilator* of M in R.

Proposition 6. *Let R be a commutative ring with identity, S a multiplicative subset of R and M a unitary R-module. If M is an S-finite R-module such that $\mathrm{ann}_R(M)$ is a maximal ideal of R, then the following assertions hold.*

(1) M is an S-Noetherian R-module.
(2) R is a weakly S-Noetherian ring if and only if $R(+)M$ is a weakly $(S(+)M)$-Noetherian ring.

Proof. Let $Q = \mathrm{ann}_R(M)$.

(1) Suppose that M is an S-finite R-module. Then we can choose an element $s \in S$ and a finitely generated R-submodule F of M such that $sM \subseteq F$. Note that by the definition of Q, we have that

M can be regarded as an (R/Q)-vector space; so F is an (R/Q)-subspace of M. Hence F is a finite dimensional (R/Q)-vector subspace of M.

Let N be an R-submodule of M. Then $sN \subseteq sM \subseteq F$; so sN is a finite dimensional (R/Q)-vector space. Therefore sN is a finitely generated R-module. Hence N is an S-finite R-module. Thus M is an S-Noetherian R-module.

(2) It follows immediately from Theorem 1 and (1). □

Finally, we study the weakly S-Noetherian property in the amalgamated algebra along an ideal with respect to a ring homomorphism. To do this, we recall the definition of the amalgamated algebra. Let R and T be commutative rings with identity, $f : R \to T$ a ring homomorphism and J an ideal of T. Then the subring $R \bowtie^f J$ of $R \times T$ is defined as follows:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}.$$

We call the ring $R \bowtie^f J$ the *amalgamation of R with T along J with respect to f*. Let $\pi : T \to T/J$ be the canonical epimorphism and $\widehat{f} = \pi \circ f$. Then $R \bowtie^f J$ is the pullback $\widehat{f} \times_{T/J} \pi$ of $\widehat{f}$ and π as follows:

$$\begin{array}{ccc} R \bowtie^f J = \widehat{f} \times_{T/J} \pi & \xrightarrow{p_R} & R \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ T & \xrightarrow{\pi} & T/J, \end{array}$$

where p_R (resp., p_T) is the restriction to $R \bowtie^f J$ of the projection of $R \times T$ onto R (resp., T). Note that J can be regarded as an R-module with the R-module structure naturally induced by f in the following way:

$$r \cdot j = f(r)j$$

for all $r \in R$ and $j \in J$. For a multiplicative subset S of R, let $S' = \{(s, f(s)) \mid s \in S\}$. Then it is easy to see that S' is a multiplicative subset of $R \bowtie^f J$. For more on the amalgamated algebra along an ideal and the relation between the amalgamation and the Nagata's idealization, the readers can refer to [13–16] (Remark 2.8).

Theorem 2. *Let R and T be commutative rings with identity, $f : R \to T$ a ring homomorphism and J an ideal of T. Then the following statements hold.*

(1) *If $R \bowtie^f J$ is a weakly S'-Noetherian ring and J is an S-finite R-module, then R is a weakly S-Noetherian ring.*
(2) *If R is a weakly S-Noetherian ring and J is an S-Noetherian R-module contained in $J(T)$, then $R \bowtie^f J$ is a weakly S'-Noetherian ring.*

Proof. (1) Let $A_1 \subseteq A_2$ be proper ideals of R such that A_2 is S-finite. Then we can find an element $s \in S$ and a finitely generated ideal $F = (x_1, \ldots, x_m)$ of R such that $sA_2 \subseteq F \subseteq A_2$; so $(s, f(s))(A_2 \bowtie^f J) \subseteq F \bowtie^f J \subseteq A_2 \bowtie^f J$. Since J is an S-finite R-module, there exist an element $t \in S$ and a finitely generated R-submodule G of J such that $t \cdot J \subseteq G$. Let $\{y_1, \ldots, y_k\}$ be a set of generators of G and let C be the ideal of $R \bowtie^f J$ generated by the set $\{(x_i, f(x_i)) \mid i = 1, \ldots, m\} \cup \{(0, y_i) \mid i = 1, \ldots, k\}$. Then we have

$$(st, f(st))(A_2 \bowtie^f J) \subseteq tF \bowtie^f G \subseteq C \subseteq A_2 \bowtie^f J.$$

Therefore $A_2 \bowtie^f J$ is an S'-finite ideal of $R \bowtie^f J$. Since $R \bowtie^f J$ is a weakly S'-Noetherian ring and $A_1 \bowtie^f J \subseteq A_2 \bowtie^f J$, we have that $A_1 \bowtie^f J$ is an S'-finite ideal of $R \bowtie^f J$; so there exist $w \in S$, $a_1, \ldots, a_n \in A_1$ and $b_1, \ldots, b_n \in J$ such that

$$(w, f(w))(A_1 \bowtie^f J) \subseteq ((a_1, f(a_1) + b_1), \ldots, (a_n, f(a_n) + b_n)) \subseteq A_1 \bowtie^f J.$$

Hence $wA_1 \subseteq (a_1, \ldots, a_n) \subseteq A_1$, which means that A_1 is an S-finite ideal of R. Thus R is a weakly S-Noetherian ring.

(2) Define the map $\varphi : R \bowtie^f J \to R \times \{0\}$ by $\varphi(r, f(r) + j) = (r, 0)$. Then φ is a ring epimorphism with $\mathrm{Ker}(\varphi) = \{0\} \times J$; so $(R \bowtie^f J)/\mathrm{Ker}(\varphi)$ is isomorphic to R. Also, note that $S' \cap \mathrm{Ker}(\varphi) = \emptyset$ and $\varphi(s, f(s)) = (s, 0)$ for all $s \in S$; so $S'/\mathrm{Ker}(\varphi)$ is isomorphic to S. Since R is a weakly S-Noetherian ring, we have that $(R \bowtie^f J)/\mathrm{Ker}(\varphi)$ is a weakly $(S'/\mathrm{Ker}(\varphi))$-Noetherian ring.

We next show that $\mathrm{Ker}(\varphi)$ is an S'-Noetherian $(R \bowtie^f J)$-module. Let N be an $(R \bowtie^f J)$-submodule of $\mathrm{Ker}(\varphi)$. Then $N = \{0\} \times I$ for some R-submodule I of J. Since J is an S-Noetherian R-module, we have that I is an S-finite R-module; so there exist $s \in S$ and $a_1, \ldots, a_n \in I$ such that $s \cdot I \subseteq R \cdot a_1 + \cdots + R \cdot a_n$. Therefore we obtain

$$(s, f(s))N \subseteq (R \bowtie^f J)(0, a_1) + \cdots + (R \bowtie^f J)(0, a_n) \subseteq N,$$

which means that N is an S'-finite $(R \bowtie^f J)$-module. Hence $\mathrm{Ker}(\varphi)$ is an S'-Noetherian $(R \bowtie^f J)$-module.

Let M be any maximal ideal of $R \bowtie^f J$. Since $J \subseteq J(T)$, we have that $M = P \bowtie^f J$ for some maximal ideal P of R [14](Proposition 2.6). Therefore $\mathrm{Ker}(\varphi) \subseteq M$. Hence $\mathrm{Ker}(\varphi) \subseteq J(R \bowtie^f J)$.

Thus by Lemma 1(2), we have that $R \bowtie^f J$ is a weakly S'-Noetherian ring. □

Let R be a commutative ring with identity, $id_R : R \to R$ the identity function and I an ideal of R. Then we simply write $R \bowtie I$ instead of $R \bowtie^{id_R} I$. By Theorem 2, we obtain

Corollary 10. *Let R be a commutative ring with identity and I an ideal of R. Let $S' = \{(s, s) \mid s \in S\}$. Then the following assertions hold.*

(1) If $R \bowtie I$ is a weakly S'-Noetherian ring and I is an S-finite ideal of R, then R is a weakly S-Noetherian ring.
(2) If R is a weakly S-Noetherian ring and I is an S-Noetherian R-module contained in $J(R)$, then $R \bowtie I$ is a weakly S'-Noetherian ring.

We are closing this section with the following question.

Question 1. *In Lemma 1(2), Theorem 2(2) and Corollary 10(2), is the Jacobson radical condition essential?*

4. Conclusions

Let R be a commutative ring with identity and S a (not necessarily saturated) multiplicative subset of R. In this paper, we introduce the concept of weakly S-Noetherian rings in order to give an answer to a natural question that if every S-finite proper ideal of R is an S-Noetherian R-module, then R is an S-Noetherian ring. We find out that the class of weakly S-Noetherian rings contains weakly Noetherian rings and S-Noetherian rings. However, by constructing examples, we show that a weakly Noetherian ring and an S-Noetherian ring need not be a weakly S-Noetherian ring. Also, under an additional condition on the Jacobson radical, we study the transfer of the weakly S-Noetherian property to the Nagata's idealization and the amalgamated algebra along an ideal, which are nice examples of pullback constructions. Unfortunately, we could not verify whether the condition on Jacobson radical is essential. In the next work, we will investigate if the condition on the Jacobson radical is superfluous in Lemma 1(2), Theorem 2(2) and Corollary 10(2). We will also study another properties of weakly S-Noetherian rings.

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