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Contractive Inequalities for Some Asymptotically Regular Set-Valued Mappings and Their Fixed Points

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Abstract: The symmetry concept is a congenital characteristic of the metric function. In this paper, our primary aim is to study the fixed points of a broad category of set-valued maps which may include discontinuous maps as well. To achieve this objective, we newly extend the notions of orbitally continuous and asymptotically regular mappings in the set-valued context. We introduce two new contractive inequalities one of which is of Geraghty-type and the other is of Boyd and Wong-type. We proved two new existence of fixed point results corresponding to those inequalities.

Keywords: fixed point; asymptotically regular map; set-valued map; metric space; orbitally continuous map

1. Preliminaries

In 1968, Markin [1] extended Browder's fixed point theorem to its set-valued counterpart, whereas, in 1969, Nadler [2] proved the set-valued version of Banach's contraction principle with the help of the Hausdorff metric. In 1972, Assad and Kirk [3] proved some new set-valued fixed point existence results in a metric space (hereafter denoted by MS) which was complete and metrically convex.

Recently, Jleli et al. [4] have studied existence of fixed points for multi-valued maps under some Ćirić-type contractions. The Hardy–Rogers contraction for set-valued maps have been investigated recently by Chifu and Petrusel [5] and Debnath and de La Sen [6]. Fixed points for multi-valued weighted mean contractions have been studied by Bucur [7].

A great deal of information about recent developments in fixed point theory of single and set-valued maps may be found in the monographs by Kirk and Shahzad [8] and Pathak [9].

The following definition of a Pompeiu–Hausdorff metric plays a crucial role in set-valued analysis.

Let $\Gamma(X)$ denote the class of all non-empty closed and bounded subsets of a non-empty set X and $(\Gamma(X), \mathcal{PH})$ denote the Pompeiu–Hausdorff metric in a metric space (MS) (X, δ). The metric function $\mathcal{PH} : \Gamma(X) \times \Gamma(X) \to [0, \infty)$ is defined by

$$\mathcal{PH}(U, V) = \max\{\sup_{\xi \in V} \Delta(\xi, U), \sup_{\eta \in U} \Delta(\eta, V)\}, \text{ for all } U, V \in \Gamma(X),$$

where $\Delta(\eta, V) = \inf_{\xi \in V} \delta(\eta, \xi)$.

Definition 1. *[2] Let $R : X \to \Gamma(X)$ be a set-valued map. $\mu \in X$ is called a fixed point of R if $\mu \in R\mu$.*

The following results are important in the present context.

Lemma 1. *[10,11] Let (X, δ) be an MS and $U, V, W \in \Gamma(X)$. Then*

1. $\Delta(\mu, V) \leq \delta(\mu, \gamma)$ *for any* $\gamma \in V$ *and* $\mu \in X$;
2. $\Delta(\mu, V) \leq \mathcal{PH}(U, V)$ *for any* $\mu \in U$.

Lemma 2. *[2] Let $U, V \in \Gamma(X)$ and let $\eta \in U$, then for any $p > 0$, there exists $\xi \in V$ such that*

$$\delta(\eta, \xi) \leq \mathcal{PH}(U, V) + p.$$

However, there may not be a point $\xi \in V$ such that

$$\delta(\eta, \xi) \leq \mathcal{PH}(U, V).$$

If V is compact, then such a point ξ exists, i.e., $\delta(\eta, \xi) \leq \mathcal{PH}(U, V)$.

Lemma 3. *[2] Let $\{U_n\}$ be a sequence in $\Gamma(X)$ and $\lim_{n\to\infty} \mathcal{PH}(U_n, U) = 0$ for some $U \in \Gamma(X)$. If $\mu_n \in U_n$ and $\lim_{n\to\infty} \delta(\mu_n, \mu) = 0$ for some $\mu \in X$, then $\mu \in U$.*

The concept of $\mathcal{PH}$-continuity for set-valued maps is defined as follows.

Definition 2. *[12] Let (X, δ) be an MS. A set-valued map $R : X \to \Gamma(X)$ is said to be $\mathcal{PH}$-continuous at a point μ_0, if for each sequence $\{\mu_n\} \subset X$, such that $\lim_{n\to\infty} \delta(\mu_n, \mu_0) = 0$, we have $\lim_{n\to\infty} \mathcal{PH}(R\mu_n, R\mu_0) = 0$ (i.e., if $\mu_n \to \mu_0$, then $R\mu_n \to R\mu_0$ as $n \to \infty$).*

Or equivalently, R is said to be $\mathcal{PH}$-continuous at a point μ_0, if for every $\epsilon > 0$, there exists $\lambda > 0$ such that $\mathcal{PH}(R\mu, R\mu_0) < \epsilon$, whenever $\delta(\mu, \mu_0) < \lambda$.

Definition 3. *[2] Let $R : X \to \Gamma(X)$ be a set-valued map. R is said to be a set-valued contraction if $\mathcal{PH}(R\mu, R\nu) \leq \lambda\delta(\mu, \nu)$ for all $\mu, \nu \in X$, where $\lambda \in [0, 1)$.*

Remark 1.

1. *R is $\mathcal{PH}$-continuous on a subset S of X if it is continuous on every point of S.*
2. *If R is a set-valued contraction, then it is $\mathcal{PH}$-continuous.*

Orbital sequence is one of the important components in the investigation of fixed points for set-valued maps (see [13,14]).

Definition 4. *[12] Let (X, δ) be an MS and $R : X \to \Gamma(X)$ a set-valued map. An R-orbital (or, simply orbital) sequence of R at a point $\mu \in X$ is a set $O(\mu, R)$ of points in X defined by $O(\mu, R) = \{\mu_0 = \mu, \mu_{n+1} \in R\mu_n, n = 0, 1, 2, \ldots\}$.*

An open problem was posed by Rhoades [15] about the availability of contractive conditions that guarantee the existence of a fixed point but the mapping is not necessarily continuous at that fixed point. In [16], Górnicki considered a special class of mappings $R : X \to X$ satisfying the condition

$$\delta(R\mu, R\nu) \leq M \cdot \delta(\mu, \nu) + k \cdot \{\delta(\mu, R\mu) + \delta(\nu, R\nu)\} \text{ for all } \mu, \nu \in X, \tag{1}$$

where $0 \leq M < 1$ and $0 \leq k < \infty$ are fixed. The class of mappings satisfying condition (Equation (1)) generalizes Banach's contraction, Kannan-type contractions with $k < \frac{1}{2}$ and several other contractive inequalities, but the mappings under consideration are not necessarily continuous.

Work in a similar direction has been carried out by Pant [17] and Bisht [18]. Asymptotically regular maps play a very significant role in the investigations of discontinuity of a map at a fixed point. Fixed points of asymptotically regular multi-valued maps have been studied by Beg and Azam [19] and Singh et al. [20].

Recently, Górnicki [21] has shown that there are non-linear maps those admit unique fixed point but the maps need not be continuous at the fixed point. He replaced the constant M in condition (1) by control functions.

Inspired by the work of Górnicki [21], in the present paper we present the set-valued versions of his results. Most of the contractive conditions existing in literature produce fixed points but they force the map under consideration to be continuous as well. As such, the theory remains applicable to a restricted class of continuous functions. In the current paper, our aim is to contribute to the study of fixed points of a larger family of maps that includes discontinuous maps.

The rest of the paper is organized as follows. Section 2 contains a result using Geraghty-type [22] control function. Section 3 contains a fixed point result in which a Boyd and Wong-type [23] contractive inequality is introduced. Section 4 contains conclusions and future work.

2. Geraghty-Type Contractive Inequality

In this section, first we introduce the concepts of orbitally continuous and asymptotically regular set-valued maps and then present a Geraghty-type fixed point result.

The recent proofs due to Górnicki [21] will be taken as a framework and his proofs will be extended to their set-valued analogues using the function Δ and the Pompeiu–Hausdorff metric $\mathcal{PH}$.

Definition 5. *Let (X, δ) be an MS. A set-valued map $R : X \to \Gamma(X)$ is called orbitally continuous (in short, OC) at a point $\theta \in X$, if for any orbital sequence $\{\mu_n\} = O(\mu, R)$, $\{\mu_n\}$ converges to some $\theta \in X$ (i.e., $\mu_n \to \theta$) implies $R\mu_n \to R\theta$.*

If R is OC at all points of its domain, then it is called OC.

Definition 6. *Let (X, δ) be an MS. A set-valued map $R : X \to \Gamma(X)$ is said to be asymptotically regular (in short, AR) at a point $\mu_0 \in X$, if for any orbital sequence $\{\mu_n\} = O(\mu_0, R)$, we have*

$$\lim_{n\to\infty} \delta(\mu_n, \mu_{n+1}) = 0.$$

If R is AR at all points of its domain, then it is called AR.

Geraghty introduced a particular class of functions $\mathcal{G}_s$ to generalize Banach's fixed point theorem. Let $\mathcal{G}_s$ $(s > 0)$ be the class of mappings $\alpha : [0, \infty) \to [0, \frac{1}{s})$ satisfying the condition: $\alpha(\xi_n) \to \frac{1}{s}$ implies $\xi_n \to 0$. An example of such a map is $\alpha(\xi) = \frac{1}{s} \cdot (1 + \xi)^{-1}$ for all $\xi > 0$ and $\alpha(0) \in [0, \frac{1}{s})$.

Theorem 1. *Let (X, δ) be an MS and $R : X \to \Gamma(X)$ be an AR set-valued map such that $R(\mu)$ is compact for all $\mu \in X$. Suppose there exist $\alpha \in \mathcal{G}_s, 0 \le k < \infty$, such that for each $\mu, \nu \in X$,*

$$\mathcal{PH}(R\mu, R\nu) \le \alpha(\delta(\mu, \nu)) \cdot \delta(\mu, \nu) + k\{\Delta(\mu, R\mu) + \Delta(\nu, R\nu)\}. \tag{2}$$

If R is OC, then $Fix(R) \neq \phi$.

Proof. Fix $\mu_0 \in X$ and choose $\mu_1 \in R(\mu_0)$. Since each $R(\mu)$ is compact, by Lemma 2, we can choose $\mu_2 \in R(\mu_1)$ such that $\delta(\mu_2, \mu_1) \le \mathcal{PH}(R\mu_1, R\mu_0)$. Similarly, we select $\mu_3 \in R(\mu_2)$ such that $\delta(\mu_3, \mu_2) \le \mathcal{PH}(R\mu_2, R\mu_1)$. Continuing in this manner, we construct an orbital sequence $O(\mu_0, R)$ satisfying the inequality $\delta(\mu_{n+1}, \mu_n) \le \mathcal{PH}(R\mu_n, R\mu_{n-1})$. Without loss of generality, assume that $\mu_n \notin R\mu_n$ for all $n \ge 0$, otherwise we trivially obtain a fixed point.

First we prove that the orbital sequence $O(\mu_0, R)$ constructed as above is a Cauchy sequence. To the contrary, assume that $O(\mu_0, R)$ is not Cauchy. Then $\limsup_{n,m\to\infty} \delta(\mu_n, \mu_m) > 0$.

By the triangle inequality, we have

$$\delta(\mu_n, \mu_m) \le \delta(\mu_n, \mu_{n+1}) + \delta(\mu_{n+1}, \mu_{m+1}) + \delta(\mu_{m+1}, \mu_m). \tag{3}$$

Now we have

$$\begin{aligned}\delta(\mu_{n+1},\mu_{m+1}) &\leq \mathcal{PH}(R\mu_n, R\mu_m), (\text{ by Lemma 2}) \\ &\leq \alpha(\delta(\mu_n,\mu_m))\cdot\delta(\mu_n,\mu_m) + k\{\Delta(\mu_n, R\mu_n) + \Delta(\mu_m, R\mu_m)\}, (\text{ by (2)}) \\ &\leq \alpha(\delta(\mu_n,\mu_m))\cdot\delta(\mu_n,\mu_m) + k\{\delta(\mu_n, R\mu_{n+1}) + \delta(\mu_m, R\mu_{m+1})\}, (\text{ by Lemma 1}).\end{aligned} \tag{4}$$

Replacing Equation (4) in Equation (3), we have

$$\begin{aligned}&\delta(\mu_n,\mu_m) \leq \alpha(\delta(\mu_n,\mu_m))\cdot\delta(\mu_n,\mu_m) + (k+1)\{\delta(\mu_n,\mu_{n+1}) + \delta(\mu_m,\mu_{m+1})\} \\ \implies &\delta(\mu_n,\mu_m)[1-\alpha(\delta(\mu_n,\mu_m))] \leq (k+1)[\delta(\mu_n,\mu_{n+1}) + \delta(\mu_m,\mu_{m+1})] \\ \implies &\frac{\delta(\mu_n,\mu_m)}{\delta(\mu_n,\mu_{n+1}) + \delta(\mu_m,\mu_{m+1})} \leq \frac{k+1}{1-\alpha(\delta(\mu_n,\mu_m))}.\end{aligned} \tag{5}$$

Since R is AR, we have $\delta(\mu_n,\mu_{n+1}) \to 0$ and $\delta(\mu_m,\mu_{m+1}) \to 0$. Further, using the fact that $\limsup_{n,m\to\infty}\delta(\mu_n,\mu_m) > 0$, from the last inequality of Equation (5), we have that

$$\limsup_{n,m\to\infty}\frac{k+1}{1-\alpha(\delta(\mu_n,\mu_m))} = +\infty,$$

which in turn, implies that

$$\limsup_{n,m\to\infty}\alpha(\delta(\mu_n,\mu_m)) = 1.$$

But since $\alpha \in \mathcal{G}_s$, we obtain $\limsup_{n,m\to\infty}\delta(\mu_n,\mu_m) = 0$, which contradicts our initial hypothesis. Hence the orbital sequence $O(\mu_0, R)$ is Cauchy.

Since (X,δ) is complete, there exists $\eta \in X$ such that $\mu_n \to \eta$ as $n \to \infty$. Again since R is orbitally continuous, we have $R\mu_n \to R\eta$ as $n \to \infty$. But $\mu_{n+1} \in R\mu_n$ for all $n \geq 0$ and $R\mu_{n+1} \to R\eta$ as $n \to \infty$ (since $R\mu_n \to R\eta$). Thus, using Lemma 3, we may conclude that $\eta \in R\eta$. □

Example 1. *Consider $X = [0,1]$ with usual metric $\delta(\mu,\nu) = |\mu-\nu|$ for all $\mu,\nu \in X$. Define $R : X \to \Gamma(X)$ by*

$$R\mu = [0, \frac{\mu}{6}]$$

and the function $\alpha : [0,\infty) \to [0,1)$ by

$$\alpha(\mu) = e^{-\mu} \text{ for } \mu > 0 \text{ and } g(0) \in [0,1).$$

If we consider $\mu,\nu \in X$ such that $\mu \neq \nu$, then

$$\begin{aligned}\mathcal{PH}(\mathcal{R}\mu,\mathcal{R}\nu) &= \frac{|\mu-\nu|}{6} \\ &= \frac{1}{6}\delta(\mu,\nu) \\ &\leq e^{-\delta(\mu,\nu)}\cdot\delta(\mu,\nu) \\ &= \alpha(\delta(\mu,\nu))\cdot\delta(\mu,\nu).\end{aligned}$$

Clearly, the condition in Equation (2) is satisfied for any $k \geq 0$. Thus, all conditions of Theorem 1 are satisfied and $\mu = 0$ is a fixed point of R.

3. Boyd and Wong-Type Contractive Inequality

Our next result is inspired by the work of Boyd and Wong [23]. Let Ω denote the family of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

1. $\psi(t) < t$ for all $t > 0$,
2. ψ is upper semi-continuous from right (i.e., $t_n \to t > 0$ implies that $\limsup_{n\to\infty} \psi(t_n) \leq \psi(t)$).

Theorem 2. *Let (X, δ) be a complete MS and $R : X \to \Gamma(X)$ an AR set-valued map such that $R\mu$ is compact for all $\mu \in X$. Suppose there exist $\psi \in \Omega$, $0 \leq k < \infty$ such that for each $\mu, \nu \in X$,*

$$\mathcal{PH}(R\mu, R\nu) \leq \psi(\delta(\mu, \nu)) + k\{\Delta(\mu, R\mu) + \Delta(\nu, R\nu)\}. \tag{6}$$

If R is OC, then $Fix(R) \neq \phi$.

Proof. Fix $\mu_0 \in X$ and in a similar fashion as in the proof of Theorem 1, construct an orbital sequence $O(\mu_0, R)$ satisfying the inequality $\delta(\mu_{n+1}, \mu_n) \leq \mathcal{PH}(R\mu_n, R\mu_{n-1})$ for all $n \geq 0$. Without loss of generality, assume that $\mu_n \notin R\mu_n$ for all $n \geq 0$, otherwise we trivially obtain a fixed point.

We prove that $O(\mu_0, R)$ is a Cauchy sequence. Assume that $O(\mu_0, R)$ is not Cauchy. Then there exist $\epsilon > 0$ and positive integers n_i, m_i such that $m_i > n_i \geq i$ and

$$\delta(\mu_{n_i}, \mu_{m_i}) \geq \epsilon \text{ for } i = 1, 2, \ldots$$

Also, choosing m_i as small as desired, we can obtain

$$\delta(\mu_{n_i}, \mu_{m_i-1}) < \epsilon.$$

Hence for each $i \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon \leq \delta(\mu_{n_i}, \mu_{m_i}) &\leq \delta(\mu_{n_i}, \mu_{m_i-1}) + \delta(\mu_{m_i-1}, \mu_{m_i}) \\ &< \epsilon + \delta(\mu_{m_i-1}, \mu_{m_i}). \end{aligned} \tag{7}$$

Further, using the asymptotic regularity of R and taking limit in both sides of Equation (7) as $i \to \infty$, we have

$$\lim_{i\to\infty} \delta(\mu_{n_i}, \mu_{m_i}) = \epsilon.$$

Now, by the triangle inequality, we have

$$\delta(\mu_{n_i}, \mu_{m_i}) \leq \delta(\mu_{n_i}, \mu_{n_i+1}) + \delta(\mu_{n_i+1}, \mu_{m_i+1}) + \delta(\mu_{m_i+1}, \mu_{m_i}). \tag{8}$$

Thus

$$\begin{aligned} \delta(\mu_{n_i+1}, \mu_{m_i+1}) &\leq \mathcal{PH}(R\mu_{n_i}, R\mu_{m_i}), (\text{ by Lemma 2}) \\ &\leq \psi(\delta(\mu_{n_i}, \mu_{m_i})) + k\{\Delta(\mu_{n_i}, R\mu_{n_i}) + \Delta(\mu_{m_i}, R\mu_{m_i})\} \\ &\leq \psi(\delta(\mu_{n_i}, \mu_{m_i})) + k\{\delta(\mu_{n_i}, \mu_{n_i+1}) + \delta(\mu_{m_i}, \mu_{m_i+1})\}. \end{aligned} \tag{9}$$

From Equations (8) and (9), we have

$$\delta(\mu_{n_i}, \mu_{m_i}) \leq \psi(\delta(\mu_{n_i}, \mu_{m_i})) + (k+1)\{\delta(\mu_{n_i}, \mu_{n_i+1}) + \delta(\mu_{m_i}, \mu_{m_i+1})\}. \tag{10}$$

Since R is AR and ψ is upper semi-continuous, taking limit in both sides of Equation (10) as $i \to \infty$, we have

$$\epsilon = \lim_{i\to\infty} \delta(\mu_{n_i}, \mu_{m_i}) \leq \limsup_{i\to\infty} \psi(\delta(\mu_{n_i}, \mu_{m_i})) \leq \psi(\epsilon) < \epsilon,$$

which is a contradiction. Hence $O(\mu_0, R)$ is Cauchy.

Using the fact that R is OC and Lemma 3, similar arguments as in the proof of Theorem 1 show that there exists $\eta \in X$ such that $\eta \in R\eta$. $\square$

Example 2. *Consider $X = \{1, 2, 3, 8\}$ with the usual metric. Define $R : X \to \Gamma(X)$ by $R1 = \{1\}, R2 = \{8\}, R3 = \{1\}, R8 = \{3\}$.*

Also let the function $\psi \in \Omega$ be given by $\psi(\mu) = \frac{7}{8}\mu$ and suppose $k = 5$. Then for each $\mu, \nu \in X$, the condition Equation (6) is satisfied.

Further, it can be seen that R is AR, $R\mu$ is compact for each $\mu \in X$ and R is OC.

Thus, all conditions of Theorem 2 are satisfied and we observe that R has a fixed point $\mu = 1$.

4. Conclusions

In the present paper, we have extended the recent results of Górnicki [21] to their set-valued counterparts. Our main contribution in this paper is the defining two new contractive inequalities for set-valued maps. Our results are proved with the assumption that images of the set-valued mappings under consideration are compact. It would be interesting to investigate further if this assumption can be dropped.

As result of our work, fixed points of a larger family of mappings can be investigated which include discontinuous set-valued maps as well.

Study of common fixed points for such OC and AR set-valued maps is a suggested future work.

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