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# Nonexistence of Global Weak Solutions for a Nonlinear Schrödinger Equation in an Exterior Domain

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**Abstract:** We study the large-time behavior of solutions to the nonlinear exterior problem  $\mathcal{L}u(t, x) = \kappa|u(t, x)|^p$ ,  $(t, x) \in (0, \infty) \times D^c$  under the nonhomogeneous Neumann boundary condition  $\frac{\partial u}{\partial \nu}(t, x) = \lambda(x)$ ,  $(t, x) \in (0, \infty) \times \partial D$ , where  $\mathcal{L} := i\partial_t + \Delta$  is the Schrödinger operator,  $D = B(0, 1)$  is the open unit ball in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $D^c = \mathbb{R}^N \setminus D$ ,  $p > 1$ ,  $\kappa \in \mathbb{C}$ ,  $\kappa \neq 0$ ,  $\lambda \in L^1(\partial D, \mathbb{C})$  is a nontrivial complex valued function, and  $\partial \nu$  is the outward unit normal vector on  $\partial D$ , relative to  $D^c$ . Namely, under a certain condition imposed on  $(\kappa, \lambda)$ , we show that if  $N \geq 3$  and  $p < p_c$ , where  $p_c = \frac{N}{N-2}$ , then the considered problem admits no global weak solutions. However, if  $N = 2$ , then for all  $p > 1$ , the problem admits no global weak solutions. The proof is based on the test function method introduced by Mitidieri and Pohozaev, and an adequate choice of the test function.

**Keywords:** nonlinear Schrödinger equation; exterior domain; nonhomogeneous Neumann boundary condition; global weak solution

**MSC:** 35B44; 35B33

## 1. Introduction

We investigate the nonlinear exterior problem

$$\mathcal{L}u(t, x) = \kappa|u(t, x)|^p, \quad (t, x) \in (0, \infty) \times D^c, \quad (1)$$

where  $\mathcal{L} := i\partial_t + \Delta$  is the Schrödinger operator,  $D = B(0, 1)$  is the open unit ball in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $D^c = \mathbb{R}^N \setminus D$ ,  $p > 1$ , and  $\kappa \in \mathbb{C}$ ,  $\kappa \neq 0$ . Problem (1) is studied under the nonhomogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t, x) = \lambda(x), \quad (t, x) \in (0, \infty) \times \partial D, \quad (2)$$

where  $\lambda \in L^1(\partial D, \mathbb{C})$  is a nontrivial complex valued function and  $\partial \nu$  is the outward unit normal vector on  $\partial D$ , relative to  $D^c$ . Namely, we derive sufficient conditions so that Equations (1) and (2) admit no global weak solutions.

We mention below some motivations for studying the considered problem. Let us first fix some notations. Given a complex number  $z$ , the real part of  $z$  is denoted by  $\operatorname{Re} z$ , the imaginary part of  $z$  is denoted by  $\operatorname{Im} z$ , the conjugate of  $z$  is denoted by  $\bar{z}$ , and the modulus of  $z$  is denoted by  $|z|$ .

In the literature, there are many results related to the blow-up of solutions for nonlinear Schrödinger equations in the whole space  $\mathbb{R}^N$ . Glassey [1] studied the Cauchy problem

$$\begin{cases} i\partial_t u &= \Delta u + |u|^{p-1}u, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) &= \varphi(x), & x \in \mathbb{R}^N. \end{cases} \tag{3}$$

Under the assumptions

- (i)  $\int_{\mathbb{R}^N} \left( |\nabla \varphi|^2 - \frac{2}{p+1} |\varphi|^{p+1} \right) dx \leq 0,$
- (ii)  $\text{Im} \int_{\mathbb{R}^N} r \bar{\varphi} \varphi_r dx > 0, r = |x|,$
- (iii)  $p > 1 + \frac{4}{N},$

it was shown that  $\|\nabla u\|_2$  and  $\|u\|_\infty$  blow up in finite time. In [2], Ogawa and Tsutsumi proved that, if  $N \geq 2$  and

$$1 + \frac{4}{N} < p < \min \left\{ \frac{N+2}{N-2}, 5 \right\},$$

then, if the initial data  $\varphi \in H^1$  is radially symmetric and has negative energy, the solution to (3) in  $H^1$  blows up in finite time. For other related works, see, for example [3–6] and the references therein. Ikeda and Wakasugi [7] investigated the problem

$$\begin{cases} i\partial_t u + \Delta u &= \kappa |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) &= \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{4}$$

where  $\kappa \in \mathbb{C}, \kappa \neq 0$  and  $1 < p \leq 1 + \frac{2}{N}$ . Under certain assumptions on  $\varphi$ , they proved that the  $L^2$ -norm of the solution  $u$  of (4) blows up in finite time. In [8], Ikeda and Inui extended the results obtained in [7] to the case  $1 < p < 1 + \frac{4}{N}$ . For other related results, see, for example [9,10].

We mention that the nonlinearity  $\kappa |u|^p$  works differently from the nonlinearity  $\kappa |u|^{p-1}u$ . Indeed, Ikeda and Wakasugi in [7] showed that  $\kappa |u|^p$  does not act as a long range effect such as  $\kappa |u|^{p-1}u$  (note that the  $L^2$ -norm of solutions for the equation in (3) conserves for all  $t \in \mathbb{R}$ , see [7]).

On the other hand, it is well known that for many problems, the large-time behavior of solutions depends on the geometry of the domain, as well as the considered boundary conditions. As an example, let us consider the semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u &= |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) &= u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{5}$$

It is well known from a famous result of Fujita [11] that the critical exponent of (5) is

$$p_F = 1 + \frac{2}{N}, \tag{6}$$

i.e., if  $1 < p < p_F$  and  $u_0 > 0$ , problem (5) possesses no global positive solutions; if  $p > p_F$  and  $u_0$  is smaller than a small Gaussian, then (5) has global positive solutions. Consider now the same problem posed in the exterior domain  $D^c$  with a Neumann boundary condition on  $\partial D$ , i.e.,

$$\begin{cases} \partial_t u - \Delta u &= |u|^p, & (t, x) \in (0, \infty) \times D^c, \\ \frac{\partial u}{\partial \nu}(t, x) &= \lambda(x), & (t, x) \in (0, \infty) \times \partial D, \\ u(0, x) &= u_0(x), & x \in D^c. \end{cases} \tag{7}$$

In the case  $\lambda \equiv 0$ , it was shown by Levine and Zhang [12] that the critical exponent of problem (7) is equal to the Fujita critical exponent  $p_F$  defined by (6). However, if  $N \geq 3$  and  $\int_{\partial D} \lambda(x) d\sigma > 0$ ,

it was proven by Bandle, Levine, and Zhang [13] that the critical exponent of problem (7) jumps from  $p_F$  (the critical exponent of (5)) to a bigger number  $\frac{N}{N-2}$ . This shows the influence of the geometry of the domain and the considered boundary conditions on the critical behavior of solutions to (5).

In [14], Jleli and Samet studied the exterior problem (1) under the nonhomogeneous Dirichlet boundary condition

$$u(t, x) = \lambda(x), \quad (t, x) \in (0, \infty) \times \partial D \tag{8}$$

and the initial condition

$$u(0, x) = \varphi(x), \quad x \in D^c. \tag{9}$$

It was shown that, if  $1 < p < \frac{N}{N-2}$  ( $N \geq 3$ ) and

$$\operatorname{Re} \kappa \cdot \operatorname{Im} \int_{D^c} \varphi(x)H(x) dx < 0, \quad \operatorname{Re} \kappa \cdot \operatorname{Re} \int_{\partial D} \lambda(x) d\sigma < 0$$

or

$$\operatorname{Im} \kappa \cdot \operatorname{Re} \int_{D^c} \varphi(x)H(x) dx > 0, \quad \operatorname{Im} \kappa \cdot \operatorname{Im} \int_{\partial D} \lambda(x) d\sigma < 0,$$

where  $H$  is a harmonic function, then Equations (1), (8) and (9) admit no global weak solutions. A natural question is to ask whether the above result still holds if, instead of the nonhomogeneous Dirichlet boundary condition (8), we consider the nonhomogeneous Neumann boundary condition (2).

In this paper, motivated by the above mentioned facts, we study the exterior problem (1) under the nonhomogeneous Neumann boundary condition (2). The rest of the paper is organized as follows. In Section 2, we give the definition of the global weak solutions of (1) and (2), and we prove some preliminary results. In Section 3, we present and proof our main result.

## 2. Global Weak Solutions

First, we fix some notations. Let

$$\mathcal{Q} = (0, \infty) \times D^c$$

and

$$\Gamma = (0, \infty) \times \partial D.$$

Let

$$\Phi = \left\{ \varphi \in C^2_{\text{cpt}}(\mathcal{Q}, \mathbb{R}_+) : \frac{\partial \varphi}{\partial \nu} \Big|_{\Gamma} = 0 \right\},$$

where  $C^2_{\text{cpt}}(\mathcal{Q}, \mathbb{R}_+)$  is the space of nonnegative  $C^2$  functions compactly supported in  $\mathcal{Q}$ . Recall that  $D^c$  is closed and  $\Gamma \subset \mathcal{Q}$ .

**Definition 1.** A function  $u \in L^p_{\text{loc}}(\mathcal{Q}, \mathbb{C})$  is called a global weak solution to (1) and (2), if

$$\kappa \int_{\mathcal{Q}} |u|^p \varphi dx dt - \int_{\Gamma} \lambda(x)\varphi d\sigma dt = -i \int_{\mathcal{Q}} u \partial_t \varphi dx dt + \int_{\mathcal{Q}} u \Delta \varphi dx dt, \tag{10}$$

for all  $\varphi \in \Phi$ .

Let

$$\mathcal{S}_{GL} = \{u \in L^p_{\text{loc}}(\mathcal{Q}, \mathbb{C}) : u \text{ is a global weak solution to (1) and (2)}\}.$$

From (10), one observes that

**Lemma 1.** If  $u \in \mathcal{S}_{GL}$ , then

$$\operatorname{Re} \kappa \cdot \int_{\mathcal{Q}} |u|^p \varphi dx dt - \operatorname{Re} \int_{\Gamma} \lambda(x)\varphi d\sigma dt = \operatorname{Im} \int_{\mathcal{Q}} u \partial_t \varphi dx dt + \operatorname{Re} \int_{\mathcal{Q}} u \Delta \varphi dx dt$$

and

$$\operatorname{Im} \kappa \cdot \int_{\mathcal{Q}} |u|^p \varphi \, dx \, dt - \operatorname{Im} \int_{\Gamma} \lambda(x) \varphi \, d\sigma \, dt = -\operatorname{Re} \int_{\mathcal{Q}} u \partial_t \varphi \, dx \, dt + \operatorname{Im} \int_{\mathcal{Q}} u \Delta \varphi \, dx \, dt,$$

for any  $\varphi \in \Phi$ .

Let  $\psi \in C^\infty(\mathbb{R}^N)$  be a function satisfying

$$0 \leq \psi \leq 1; \quad \psi \equiv 1 \text{ in } D; \quad \psi(x) = 0 \text{ if } |x| \geq 2.$$

Let  $\zeta \in C^\infty(\mathbb{R})$  be a function satisfying

$$\zeta \geq 0; \quad \zeta \not\equiv 0; \quad \operatorname{supp}(\zeta) \subset (0, 1).$$

For  $0 < T < \infty$ , let

$$\psi_T(x) = \psi\left(\frac{x}{T}\right)^\tau, \quad x \in D^c$$

and

$$\zeta_T(t) = \zeta\left(\frac{t}{T^\rho}\right)^\tau, \quad t > 0,$$

where  $\tau > \frac{2p}{p-1}$  and  $\rho > 0$  are constants. Let

$$\varphi_T(t, x) = \zeta_T(t) \psi_T(x), \quad (t, x) \in \mathcal{Q}. \tag{11}$$

It can be easily seen that

**Lemma 2.** For all  $1 < T < \infty$ ,

$$\varphi_T \in \Phi.$$

It follows from Lemmas 1 and 2 that

**Lemma 3.** If  $u \in \mathcal{S}_{GL}$ , then

$$\operatorname{Re} \kappa \cdot \int_{\mathcal{Q}} |u|^p \varphi_T \, dx \, dt - \operatorname{Re} \int_{\Gamma} \lambda(x) \varphi_T \, d\sigma \, dt = \operatorname{Im} \int_{\mathcal{Q}} u \partial_t \varphi_T \, dx \, dt + \operatorname{Re} \int_{\mathcal{Q}} u \Delta \varphi_T \, dx \, dt \tag{12}$$

and

$$\operatorname{Im} \kappa \cdot \int_{\mathcal{Q}} |u|^p \varphi_T \, dx \, dt - \operatorname{Im} \int_{\Gamma} \lambda(x) \varphi_T \, d\sigma \, dt = -\operatorname{Re} \int_{\mathcal{Q}} u \partial_t \varphi_T \, dx \, dt + \operatorname{Im} \int_{\mathcal{Q}} u \Delta \varphi_T \, dx \, dt, \tag{13}$$

for any  $1 < T < \infty$ .

### 3. Main Result

In this section, we obtain sufficient conditions for which  $\mathcal{S}_{GL} = \emptyset$ . As in [14], the used technique is based on the test function method introduced by Mitidieri and Pohozaev (see, e.g., [15]). However, due to the boundary condition (2), the considered test function in our case is different to that used in [14].

For  $N \geq 3$ , let

$$p_c = \frac{N}{N-2}.$$

Define the sets  $H_i, i = 1, 2$ , by

$$H_1 = \left\{ (\kappa, \lambda) \in \mathbb{C} \setminus \{0\} \times L^1(\partial D, \mathbb{C}) : \operatorname{Re} \kappa \cdot \operatorname{Re} \int_{\partial D} \lambda(x) \, d\sigma < 0 \right\}$$

and

$$H_2 = \left\{ (\kappa, \lambda) \in \mathbb{C} \setminus \{0\} \times L^1(\partial D, \mathbb{C}) : \text{Im } \kappa \cdot \text{Im} \int_{\partial D} \lambda(x) d\sigma < 0 \right\}.$$

Our main result is the following.

**Theorem 1.** Let  $(\kappa, \lambda) \in H_1 \cup H_2$ .

(i) If  $N \geq 3$ , then

$$\mathcal{S}_{GL} = \emptyset, \quad \text{for all } 1 < p < p_c.$$

(ii) If  $N = 2$ , then

$$\mathcal{S}_{GL} = \emptyset, \quad \text{for all } 1 < p < \infty.$$

**Proof.** Let  $(\kappa, \lambda) \in H_1$ , and suppose that  $u \in \mathcal{S}_{GL}$ . Using (12), for sufficiently large  $0 < T < \infty$ , one has

$$\int_Q |u|^p \varphi_T dx dt - \sigma(\kappa) \text{Re} \int_{\Gamma} \lambda(x) \varphi_T d\sigma dt \leq |\sigma(\kappa)| \int_Q |u| |\partial_t \varphi_T| dx dt + |\sigma(\kappa)| \int_Q |u| |\Delta \varphi_T| dx dt, \quad (14)$$

where  $\sigma(\kappa) = (\text{Re } \kappa)^{-1}$ . On the other hand, using (11), one obtains

$$\begin{aligned} \int_{\Gamma} \lambda(x) \varphi_T(t, x) d\sigma dt &= \int_{(0, \infty) \times \partial D} \lambda(x) \xi_T(t) \psi_T(x) d\sigma dt \\ &= \int_{(0, \infty) \times \partial D} \lambda(x) \xi \left( \frac{t}{T^p} \right)^{\tau} \psi \left( \frac{x}{T} \right)^{\tau} d\sigma dt \\ &= \left( \int_0^{\infty} \xi \left( \frac{t}{T^p} \right)^{\tau} dt \right) \left( \int_{\partial D} \lambda(x) \psi \left( \frac{x}{T} \right)^{\tau} d\sigma \right) \\ &= T^p \left( \int_0^1 \xi(s)^{\tau} ds \right) \left( \int_{\partial D} \lambda(x) d\sigma \right), \end{aligned}$$

which yields

$$-\sigma(\kappa) \text{Re} \int_{\Gamma} \lambda(x) \varphi_T(t, x) d\sigma dt = C_1 T^p, \quad (15)$$

where

$$C_1 = \left( \int_0^1 \xi(s)^{\tau} ds \right) \left( -\sigma(\kappa) \text{Re} \int_{\partial D} \lambda(x) d\sigma \right).$$

Since  $(\kappa, \lambda) \in H_1$ , one has  $C_1 > 0$ . It follows from (14) and (15) that

$$I_T + C_1 T^p \leq |\sigma(\kappa)| \int_Q |u| |\partial_t \varphi_T| dx dt + |\sigma(\kappa)| \int_Q |u| |\Delta \varphi_T| dx dt, \quad (16)$$

where

$$I_T = \int_Q |u|^p \varphi_T dx dt.$$

Further, by Hölder’s inequality, one obtains

$$\int_Q |u| |\partial_t \varphi_T| dx dt \leq I_T^{\frac{1}{p}} \left( \int_Q [\varphi_T]^{\frac{-1}{p-1}} |\partial_t \varphi_T|^{\frac{p}{p-1}} dx dt \right)^{\frac{p-1}{p}}. \quad (17)$$

Similarly, one has

$$\int_Q |u| |\Delta \varphi_T| dx dt \leq I_T^{\frac{1}{p}} \left( \int_Q [\varphi_T]^{\frac{-1}{p-1}} |\Delta \varphi_T|^{\frac{p}{p-1}} dx dt \right)^{\frac{p-1}{p}}. \quad (18)$$

Using (16)–(18), one obtains

$$I_T + C_1 T^p \leq |\sigma(\kappa)| I_T^{\frac{1}{p}} \left( A_T^{\frac{p-1}{p}} + B_T^{\frac{p-1}{p}} \right), \quad (19)$$

where

$$A_T = \int_Q [\varphi_T]^{\frac{-1}{p-1}} |\partial_t \varphi_T|^{\frac{p}{p-1}} dx dt$$

and

$$B_T = \int_Q [\varphi_T]^{\frac{-1}{p-1}} |\Delta \varphi_T|^{\frac{p}{p-1}} dx dt.$$

Further, we shall estimate the terms  $A_T$  and  $B_T$  for sufficiently large  $T$ .

- Estimate of  $A_T$ : using (11), one obtains

$$\begin{aligned} A_T &= \left( \int_{D^c} \psi \left( \frac{x}{T} \right)^\tau dx \right) \left( \int_0^\infty [\xi_T(t)]^{\frac{-1}{p-1}} |\xi'_T(t)|^{\frac{p}{p-1}} dt \right) \\ &\leq \left( T^N \int_{|y|<2} \psi(y)^\tau dy \right) \left( T^{\frac{-\rho p}{p-1}} \tau^{\frac{p}{p-1}} \int_0^\infty \xi \left( \frac{t}{T^\rho} \right)^{\tau - \frac{p}{p-1}} \left| \xi' \left( \frac{t}{T^\rho} \right) \right|^{\frac{(\tau-1)p}{p-1}} dt \right) \\ &= \tau^{\frac{p}{p-1}} \left( \int_{|y|<2} \psi(y)^\tau dy \right) \left( \int_0^1 \xi(s)^{\tau - \frac{p}{p-1}} |\xi'(s)|^{\frac{(\tau-1)p}{p-1}} ds \right) T^{N - \frac{\rho}{p-1}}, \end{aligned}$$

i.e.,

$$A_T \leq C_2 T^{N - \frac{\rho}{p-1}}, \tag{20}$$

where

$$C_2 = \tau^{\frac{p}{p-1}} \left( \int_{|y|<2} \psi(y)^\tau dy \right) \left( \int_0^1 \xi(s)^{\tau - \frac{p}{p-1}} |\xi'(s)|^{\frac{(\tau-1)p}{p-1}} ds \right) > 0.$$

- Estimate of  $B_T$ : using (11), one has

$$B_T = \left( \int_0^\infty \xi_T(t) dt \right) \left( \int_{D^c} \psi_T(x)^{\frac{-1}{p-1}} |\Delta \psi_T(x)|^{\frac{p}{p-1}} dx \right). \tag{21}$$

On the other hand,

$$\begin{aligned} \int_0^\infty \xi_T(t) dt &= \int_0^\infty \xi \left( \frac{t}{T^\rho} \right)^\tau dt \\ &= T^\rho \int_0^1 \xi(s) ds. \end{aligned} \tag{22}$$

Furthermore, one has

$$\begin{aligned} \int_{D^c} \psi_T(x)^{\frac{-1}{p-1}} |\Delta \psi_T(x)|^{\frac{p}{p-1}} dx &= T^{N - \frac{2p}{p-1}} \int_{1<|y|<2} \psi(y)^{\frac{-\tau}{p-1}} |\Delta \psi(y)^\tau|^{\frac{p}{p-1}} dy \\ &= T^{N - \frac{2p}{p-1}} \tau^{\frac{p}{p-1}} \int_{1<|y|<2} \psi(y)^{\tau - \frac{2p}{p-1}} |(\tau - 1)|\nabla \psi(y)|^2 + \psi(y)\Delta \psi(y)|^{\frac{p}{p-1}} dy. \end{aligned} \tag{23}$$

Hence, using (21), (22), and (23), one deduces that

$$B_T \leq C_3 T^{\rho + N - \frac{2p}{p-1}}, \tag{24}$$

where

$$C_3 = \tau^{\frac{p}{p-1}} \left( \int_0^1 \xi(s) ds \right) \left( \int_{1<|y|<2} \psi(y)^{\tau - \frac{2p}{p-1}} |(\tau - 1)|\nabla \psi(y)|^2 + \psi(y)\Delta \psi(y)|^{\frac{p}{p-1}} dy \right) > 0.$$

Next, using (19), (20), and (24), it holds

$$I_T + C_1 T^\rho \leq C_4 |\sigma(\kappa)| I_T^{\frac{1}{p}} \left( T^{(N - \frac{\rho}{p-1}) \frac{p-1}{p}} + T^{(\rho + N - \frac{2p}{p-1}) \frac{p-1}{p}} \right), \tag{25}$$

where

$$C_4 = \max \left\{ C_2^{\frac{p-1}{p}}, C_3^{\frac{p-1}{p}} \right\} > 0.$$

Taking  $\rho = 2$  in (25), one obtains

$$I_T + C_1 T^2 \leq 2C_4 |\sigma(\kappa)| I_T^{\frac{1}{p}} T^{\frac{N(p-1)-2}{p}}. \tag{26}$$

Using Young’s inequality, it holds

$$2C_4 |\sigma(\kappa)| I_T^{\frac{1}{p}} T^{\frac{N(p-1)-2}{p}} \leq I_T + C_5 T^{N-\frac{2}{p-1}},$$

where

$$C_5 = \frac{p-1}{p} \left( 2p^{\frac{-1}{p}} C_4 |\sigma(\kappa)| \right)^{\frac{p}{p-1}} > 0.$$

Hence, by (26), one deduces that

$$0 < C_6 \leq T^{N-2-\frac{2}{p-1}}, \tag{27}$$

where

$$C_6 = \frac{C_1}{C_5}.$$

Suppose now that  $N \geq 3$  and  $1 < p < p_c$ . In this case, one has

$$N - 2 - \frac{2}{p-1} < 0.$$

Hence, letting  $T \rightarrow \infty$  in (27), we obtain  $0 < C_6 \leq 0$ , that is a contradiction. This proves part (i) of Theorem 1. Similarly, if  $N = 2$ , for all  $1 < p < \infty$ , one has

$$N - 2 - \frac{2}{p-1} = -\frac{2}{p-1} < 0.$$

Letting  $T \rightarrow \infty$  in (27), we obtain the same contradiction as in the above case, which proves part (ii) of Theorem 1.

Consider now the case  $(\kappa, \lambda) \in H_2$ , and suppose that  $u \in \mathcal{S}_{GL}$ . Using (13), for sufficiently large  $0 < T < \infty$ , one has

$$\int_{\mathcal{Q}} |u|^p \varphi_T dx dt - \mu(\kappa) \text{Im} \int_{\Gamma} \lambda(x) \varphi_T d\sigma dt \leq |\mu(\kappa)| \int_{\mathcal{Q}} |u| |\partial_t \varphi_T| dx dt + |\mu(\kappa)| \int_{\mathcal{Q}} |u| |\Delta \varphi_T| dx dt,$$

where  $\mu(\kappa) = (\text{Im } \kappa)^{-1}$ . Next, using similar techniques as above, a contradiction follows.  $\square$

**Remark 1.** Note that no assumptions on the initial condition are required in Theorem 1.

**Remark 2.** Note that the condition  $1 < p < p_c$  in the assertion (i) of Theorem 1 is optimal, in the sense that, if  $p > p_c$ , then there exists  $(\kappa, \lambda) \in H_1 \cup H_2$ , such that  $\mathcal{S}_{GL} \neq \emptyset$ . Indeed, for  $p > p_c = \frac{N}{N-2}$  ( $N \geq 3$ ), let

$$w(x) = -\varepsilon \rho^{-\delta}, \quad \rho = |x| \geq 1,$$

where

$$\delta = \frac{2}{p-1} \quad \text{and} \quad \varepsilon = [\delta(N-2-\delta)]^{\frac{1}{p-1}}. \tag{28}$$

Note that since  $p > p_c$ , one has  $\delta(N - 2 - \delta) > 0$ . On the other hand, using (28), one obtains

$$\begin{aligned}\Delta w(x) &= -\varepsilon \Delta \rho^{-\delta} \\ &= -\varepsilon \left[ \frac{d^2}{d\rho^2}(\rho^{-\delta}) + \frac{(N-1)}{\rho} \frac{d}{d\rho}(\rho^{-\delta}) \right] \\ &= -\varepsilon \left[ (-\delta)(-\delta-1)\rho^{-\delta-2} + \frac{(N-1)}{\rho} (-\delta\rho^{-\delta-1}) \right] \\ &= \varepsilon[\delta(N-2-\delta)]\rho^{-\delta-2} \\ &= \varepsilon^p \rho^{-\delta-2} \\ &= \varepsilon^p \rho^{\frac{-2p}{p-1}} \\ &= \varepsilon^p \rho^{-\delta p},\end{aligned}$$

which yields

$$\Delta w(x) = |w(x)|^p, \quad x \in D^c. \quad (29)$$

Moreover,

$$-\left. \frac{dw(x)}{d\rho} \right|_{\rho=1} = -\varepsilon,$$

which yields

$$\frac{\partial w}{\partial \nu}(x) = -\varepsilon, \quad x \in \partial D. \quad (30)$$

Next, taking

$$u(t, x) = w(x), \quad t \geq 0, x \in D^c,$$

using (29) and (30), one deduces that  $u$  is a stationary solution to Equations (1) and (2) with  $\kappa = 1$  and  $\lambda \equiv -\varepsilon$ . Observe that  $(k, \lambda) \in H_1 \cup H_2$  and  $u \in \mathcal{S}_{GL}$ .

#### 4. Conclusions

Nonlinear Schrödinger equations attracted the attention of many mathematicians, due to their significant applications in physics. Many efforts were made to identify the blow-up of the solution of different boundary value problems involving such a type of nonlinear equations. Hence, there is a variety of approaches in the literature to studying the dynamical properties of the blow-up of the solution and prove the existence/nonexistence of global weak solutions. Here, Theorem 1 complements the results with power-type nonlinearities, in the nonhomogeneous Neumann boundary case. As pointed out above, we do not impose any assumption on the initial condition. We establish the result using the approach originally developed by Mitidieri and Pohozaev [15], together with an adequate choice of the test function. We recall that the approach methodology is indirect, starting on the assumption of the existence of a global weak solution to (1) and (2). The test function in our case is different from that used in recent analogous papers (see, for example, the one in [14]).

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