

Article

Influence of Ramped Wall Temperature and Ramped Wall Velocity on Unsteady Magnetohydrodynamic Convective Maxwell Fluid Flow

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Abstract: This article provides a comprehensive analysis regarding effects of ramped wall temperature and ramped wall velocity on incompressible time-dependent magnetohydrodynamic flow of Maxwell fluid. The flow is due to free convection and bounded to an infinite vertical plate embedded in porous medium. Solutions of mass, shear stress, and energy fields are computed symmetrically by introducing some suitable non-dimensional parameters along with the Laplace transformation technique. The expression for the Nusselt number is also calculated. A comparison between solutions incorporating isothermal temperature and ramped wall temperature conditions is also executed to examine the profile differences. A graphical study is performed to highlight the influence of parameters on mass flow and energy transfer.

Keywords: Maxwell fluid; Laplace transformation; ramped wall; porous material

1. Introduction

Non-Newtonian fluids such as paints, toothpastes, plastics, and foodstuffs are attractive to researchers because of their significant practical utility in modern and emerging technologies. In recent years, forced and free convection flows of such fluids influenced by magnetohydrodynamic (MHD) force have been in high demand in MHD generators, energy generators, polymer fabrication, accelerators, power elevators, aerodynamic heating, and the purification of mineral oil [1,2]. Engineers and hydrologists have examined the fluid flows in porous materials ranging from sand packs to fused Pyrex glass in order to forecast their reactions in several types of reservoir [3]. The goal of anticipating the behaviors of all non-Newtonian fluids cannot be served by considering a single model. The nonlinear relation between shear rate and shear stress makes the flow equations more complex than the Navier–Stokes equation can explain, and enhances their order [4,5]. The handling of such flow equations is not trivial, because of nonlinear supplementary relations in the momentum equation.

However, a number of models exist to predict the features of such fluids. Three main categories of such models involve integral, differential, and rate type models. From a research perspective, rate type models are more practical as they anticipate both memory and elastic effects. Therefore, in the current work, a sub-category of rate type model known as the Maxwell model is considered. This model was first initiated by Maxwell to forecast the visco-elastic features of air [6]. The Maxwell model is an elementary rate model utilized for predicting fluids' rheological behavior and has the properties of both viscosity and elasticity.

An exact solution for the Maxwell model past an infinite vertical plate was obtained by Fetecau and Fetecau [7]. Further, Fetecau et al. [8] investigated the second problem of Stokes, considering Maxwell fluid flow over an infinite plate oscillating in its plane. Jordan et al. [9] calculated the exact solutions for first problem of Stokes, incorporating the Maxwell model. Noor [10] observed the impacts of thermophoresis and chemical reaction on MHD Maxwell fluid flow for a vertical stretching sheet. Ramesh and Gireesha provided the effect of heat absorption/consumption on a Maxwell nanofluid [11]. Cao et al. derived the fractional derivatives of a Maxwell nanofluid over a moving plate [12]. Sidra et al. conducted a study to examine the enhancement in heat transfer when carbon nanotubes were added to free convection Maxwell fluid flow [13]. Some significant studies involving the Maxwell model can be viewed in references [14–17].

All of the abovementioned studies are of great physical significance, but only incorporate either constant boundary conditions or uniform boundary conditions. To our best knowledge, no single article in the literature covers the influence of ramped wall velocity and ramped wall temperature conditions at the same time for a Maxwell fluid. However, the simultaneous use of these boundary conditions has many applications in the medical and chemical industries. For instance, diagnoses of cardiovascular diseases involve treadmill testing and ergometers, which function on the basis of ramped velocity conditions [18]. Additionally, the inspection of the working of a blood vessel system and the pumping of a heart, diagnostic analysis, and determining treatments involve applications of ramped velocity conditions. Myers and Belin [19] and Bruce [20] applied ramped velocity to evaluate the limitations of the exercise tolerance and working capacity of heart patients. Kundu declared that control temperature conditions have valuable outcomes when a cancer patient is treated with thermal therapy based on such conditions [21]. Later, Kundu provided five kinds of boundary conditions to improve the treatment of cancerous cells and to reduce the side effects of this treatment to almost nonexistence.

The credit for introducing ramped temperature conditions may be given to Malhotra [22], Hayday [23], and Schetz [24]. Ramped heating is an effective and efficient technique to control the temperature elevation under adiabatic conditions. An in-depth survey and comparative analysis of various heating methods in the chemical industry was given in [25]. Chandran et al. [26] examined the influence of ramped wall temperature conditions on convective viscous flow. Further, Seth et al. extended this work by including the porosity of the medium [27]. Seth et al. gave primary focus to instinctive and moving vertical plates to explore the effects of ramped temperature (at the wall) on mass and energy profiles, encountering physical phenomena like chemical reaction, Hall current, thermal diffusion, heat absorption, and immersion of the plate in porous medium [28–31]. An investigation was executed by Narahari et al. to observe the fluid flow with a heated wall for an infinite vertical plate [32]. Khan's work [33] on the MHD flow of a Jeffery fluid was extended by Zin et al. [34] for ramped wall temperature conditions.

In the literature, there is a shortage of articles dealing with flows subjected to ramped wall temperature and ramped wall velocity conditions, despite the valuable practical implications. One of the main bounds is properly controlling the resulting complex expressions. The idea of using both ramped conditions at the same time was first given by Ahmed and Dutta [35] to analyze the motion of Newtonian non-Grey fluid past an infinite vertical plate. Maqbool et al. [36] examined the effect of simultaneous ramped wall conditions on freely convective flow of a Jeffery fluid in the presence of a porous medium. Further, Mazhar et al. studied the influence of simultaneous ramped conditions

on Oldroyd-B fluid [37]. Keeping the aforementioned observations in view, the principal concern of this work is the simultaneous application of ramped temperature and velocity on the natural convective flow of an unsteady, incompressible Maxwell fluid for an infinite vertical plate embedded in porous medium under the effect of magnetohydrodynamic (MHD) force. The Laplace transformation is implemented to reach the solutions. The dependence of mass, shear stress, and energy solutions on different parameters is discussed with the aid of graphs. The rate of heat transfer (Nusselt number) is also taken into account.

2. Mathematical Modeling

Let us consider MHD, time dependent, incompressible flow of Maxwell fluid with natural convection over an infinite vertical plate (wall) immersed in porous medium. For this case, Cartesian coordinates are chosen and the plate is located in the $x - y$ plane. At the wall end, both velocity and temperature have time-dependent conditions up to some certain limit of time known as the characteristic time; after that time, both velocity and temperature attain constant values u_0 and T_∞ . The physical interpretation of the model's flow under observation here is expressed in Figure 1. The fundamental governing equations under the assumptions of small magnetic Reynolds number and Boussinesq's approximation are given as [38,39]

$$\rho \left(1 + \lambda_1 \frac{\partial}{\partial t'} \right) \frac{\partial u'}{\partial t'} = \mu \frac{\partial^2 u'}{\partial y'^2} - \left(1 + \lambda_1 \frac{\partial}{\partial t'} \right) \sigma B_0^2 u' - \frac{\mu \phi}{k^*} u' + \left(1 + \lambda_1 \frac{\partial}{\partial t'} \right) \rho g \beta (T' - T_\infty), \quad (1)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t'} \right) S' = \mu \frac{\partial u'}{\partial y'}, \quad (2)$$

$$\rho c_p \frac{\partial T'}{\partial t'} = k \frac{\partial^2 T'}{\partial y'^2}. \quad (3)$$

The adequate corresponding initial and boundary conditions are

$$u'(y', 0) = 0, \quad T'(y', 0) = T_\infty, \\ y' \geq 0: \quad u'_{t'}(y', 0) = 0, \quad u'_{y'}(y', 0) = 0, \quad (4)$$

$$u'(0, t') = \begin{cases} u_0 \frac{t'}{t_0} & 0 < t' \leq t_0 \\ u_0 & t' > t_0, \end{cases} \\ T'(0, t') = \begin{cases} T_\infty + (T_w - T_\infty) \frac{t'}{t_0} & 0 < t' \leq t_0 \\ T_w & t' > t_0, \end{cases} \quad (5)$$

$$t' > 0: \quad u'(y', t') \rightarrow 0, \quad T'(y', t') \rightarrow T_\infty, \quad \text{for } y' \rightarrow \infty. \quad (6)$$

Dimensionless quantities are provided as

$$u = \frac{u'}{u_0}, \quad \eta = \frac{y' u_0}{\nu}, \quad \lambda = \frac{\lambda_1 u_0^2}{\nu}, \quad S = \frac{S'}{\rho u_0^2}, \quad \tau = \frac{t' u_0^2}{\nu}, \quad t_0 = \frac{\nu}{u_0^2}, \quad \theta = \frac{T' - T_\infty}{T_w - T_\infty}.$$

The dimensionless forms of partial differential Equations (1)–(3) subjected to the above dimensionless quantities are obtained as

$$\left(1 + \lambda \frac{\partial}{\partial \tau} \right) \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} - \left(1 + \lambda \frac{\partial}{\partial \tau} \right) Mu - \frac{1}{K} u + \left(1 + \lambda \frac{\partial}{\partial \tau} \right) Gr \theta, \quad (7)$$

$$\left(1 + \lambda \frac{\partial}{\partial \tau} \right) S = \frac{\partial u}{\partial \eta}, \quad (8)$$

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2}, \quad (9)$$

where

$$Gr = \frac{g\beta\nu(T_w - T_\infty)}{u_0^3}, \quad M = \frac{\sigma B_0^2\nu}{\rho u_0^2}, \quad Pr = \frac{\mu c_p}{k}, \quad \frac{1}{K} = \frac{\phi\nu^2}{k^*u_0^2}.$$

The dimensionless forms of the initial and boundary conditions are obtained as

$$u(\eta, 0) = 0, \quad \theta(\eta, 0) = 0, \\ \eta \geq 0: \quad u_\tau(\eta, 0) = 0, \quad u_\eta(\eta, 0) = 0, \quad (10)$$

$$u(0, \tau) = \theta(0, \tau) = \begin{cases} \tau & 0 < \tau \leq 1 \\ 1 & \tau > 1, \end{cases} \quad (11)$$

$$\tau > 0: \quad u(\eta, \tau) \rightarrow 0, \quad \theta(\eta, \tau) \rightarrow 0 \quad \text{when} \quad \eta \rightarrow \infty. \quad (12)$$

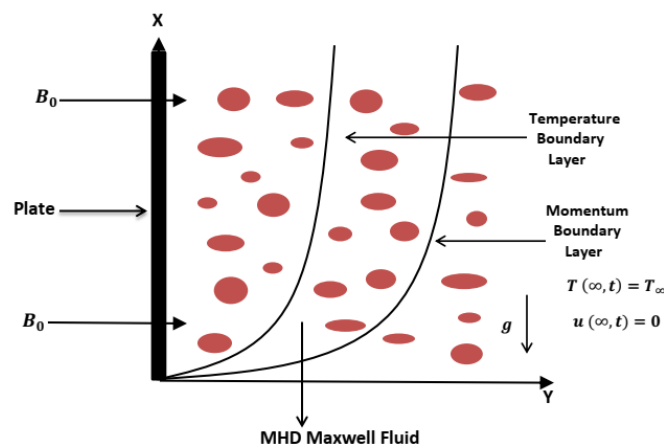


Figure 1. Geometrical presentation of flow.

3. Analytical Solutions

To derive the results of current model, the Laplace transformation [40] is an efficient tool due to its effective applicability for nonuniform boundary conditions. Other common methods like the homotopy analysis method, Adomian decomposition, perturbation method, and separation of variables do not serve the purpose here due to the nonuniform boundary conditions. We generated the Laplace transform pair in order to calculate the results of the current problem as an integral of the following form:

$$\bar{R}(\eta, s) = \int_0^\infty e^{-s\tau} R(\eta, \tau) d\tau = \mathcal{L}[R](\tau), \quad \tau \geq 0, \quad (13)$$

where $R \in \{u, S, \theta\}$. The above integral converges for $\alpha_0 < \text{Re}(s)$, where $s = \Psi + z\Omega$, α_0 is a real number and $z = \sqrt{-1}$. Transformation of the Laplace domain solutions back to the original time domain can be performed as:

$$R(\eta, \tau) = \frac{1}{2\pi z} \int_{BR} e^{s\tau} \bar{R}(\eta, s) ds = \mathcal{L}^{-1}[\bar{R}](s). \quad (14)$$

Implementation of the Laplace transform (13) on Equations (7)–(9) yields the ordinary differential equations as

$$\frac{d^2 \bar{u}}{d\eta^2} - (a + \lambda s^2 + bs) \bar{u} = -Gr(1 + \lambda s) \bar{\theta}, \quad (15)$$

$$(1 + \lambda s) \bar{S} = \frac{d\bar{u}}{dy}, \quad (16)$$

$$s\bar{\theta} = \frac{1}{Pr} \frac{d^2 \bar{\theta}}{d\eta^2}, \quad (17)$$

where

$$a = M + \frac{1}{K}, \quad b = 1 + \lambda M.$$

The initial and boundary conditions in the Laplace domain are deduced as

$$\bar{u}(\eta, 0) = 0, \quad \bar{\theta}(\eta, 0) = 0, \quad (18)$$

$$\bar{u}(0, s) = \bar{\theta}(0, s) = \frac{1 - e^{-s}}{s^2}, \quad (19)$$

$$\bar{u}(\eta, s) \rightarrow 0, \quad \bar{\theta}(\eta, s) \rightarrow 0 \quad \text{for } \eta \rightarrow \infty. \quad (20)$$

The solution of the energy Equation (17) corresponding to boundary conditions (3.7), (20) is derived as

$$\bar{\theta}(\eta, s) = \frac{e^{-\sqrt{Prs}\eta}}{s^2} - e^{-s} \left(\frac{e^{-\sqrt{Prs}\eta}}{s^2} \right) = \bar{H}(\eta, s) - e^{-s} \bar{H}(\eta, s). \quad (21)$$

Applying the inverse Laplace transform as (14) on the above equation leads to the following solution of the energy equation:

$$\theta(\eta, \tau) = H(\eta, \tau) - H(\eta, \tau - 1)g(\tau - 1), \quad (22)$$

where

$$H(\eta, \tau) = \left(\tau + \frac{\eta^2 Pr}{2} \right) \text{erfc} \left(\frac{\eta \sqrt{Pr}}{2\sqrt{\tau}} \right) - \frac{\eta \sqrt{Pr\tau}}{\sqrt{\pi}} e^{-\frac{\eta^2 Pr}{4\tau}},$$

with $g(\cdot)$ and $\text{erfc}(\cdot)$ denoting the standard Heaviside function [41] and complementary error function [42] respectively such as

$$g(z) = \begin{cases} 0 & z < 0 \\ 1 & z \geq 0, \end{cases}$$

$$\text{erfc}(y) = 1 - \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-z^2} dz.$$

Substituting the value of $\bar{\theta}$ from Equation (21) in Equation (15) results in the following form:

$$\frac{d^2 \bar{u}}{d\eta^2} - (a + \lambda s^2 + bs) \bar{u} = -Gr(1 + \lambda s) \left(\frac{1 - e^{-s}}{s^2} \right) e^{-\sqrt{Prs}\eta}. \quad (23)$$

Solution of above equation corresponding to boundary conditions (19) and (20) is computed as

$$\bar{u}(\eta, s) = \left(\frac{1 - e^{-s}}{s^2} \right) \bar{G}(\eta, s), \quad (24)$$

where

$$\bar{G}(\eta, s) = e^{-\sqrt{a+\lambda s^2+bs}\eta} + \frac{Gr(1+\lambda s)}{Prs - (a + \lambda s^2 + bs)} \left(e^{-\sqrt{a+\lambda s^2+bs}\eta} - e^{-\sqrt{Prs}\eta} \right). \quad (25)$$

Differentiating Equation (24) with respect to variable η yields

$$\frac{du}{d\eta} = \bar{F}(\eta, s) \left(\frac{1 - e^{-s}}{s^2} \right), \quad (26)$$

where

$$\bar{F}(\eta, s) = -\sqrt{c}e^{-\sqrt{c}\eta} - \frac{Gr(1+\lambda s)\sqrt{c}e^{-\sqrt{c}\eta}}{Prs - c} + \frac{Gr(1+\lambda s)\sqrt{Prs}e^{-\sqrt{Prs}\eta}}{Prs - c}, \quad (27)$$

with

$$c = a + \lambda s^2 + bs.$$

Plugging Equation (26) in Equation (16) gives the following solution of shear stress:

$$\bar{S}(\eta, s) = \frac{\bar{F}(\eta, s)}{1 + \lambda s} \left(\frac{1 - e^{-s}}{s^2} \right). \quad (28)$$

Since the mass and shear stress Laplace domain solutions in Equations (24) and (28) are multivalued relations of the Laplace parameter s , they are therefore inverted back to the original time domain with the help of a numerical Laplace inversion called as Durbin method [43].

The expression for the Nusselt number Nu is calculated as:

$$Nu = -\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0}, \quad (29)$$

$$Nu = \frac{2\sqrt{Pr}}{\sqrt{\pi}} \left[\sqrt{\tau} - \sqrt{\tau-1}g(\tau-1) \right]. \quad (30)$$

$$(31)$$

4. Special Cases

This section presents two special cases of this work.

4.1. Case 1

The mass and energy solutions of a Maxwell fluid for constant boundary conditions can be derived as:

$$\theta(\eta, \tau) = \operatorname{erfc} \left(\frac{\eta\sqrt{Pr}}{2\sqrt{\tau}} \right), \quad (32)$$

$$u(\eta, \tau) = \mathcal{L}^{-1} \left[\left(\frac{1 - e^{-s}}{s} \right) \bar{G}(\eta, s) \right], \quad (33)$$

where

$$\bar{G}(\eta, s) = e^{-\sqrt{a+\lambda s^2+bs}\eta} + \frac{Gr(1+\lambda s)}{Prs - (a+\lambda s^2+bs)} \left(e^{-\sqrt{a+\lambda s^2+bs}\eta} - e^{-\sqrt{Prs}\eta} \right).$$

4.2. Case 2

The momentum and temperature solution of a viscous fluid with ramped wall temperature can be derived when $\lambda_1 = 0$ [44]:

$$\theta(\eta, \tau) = H(\eta, \tau) - H(\eta, \tau - 1)g(\tau - 1), \quad (34)$$

$$u(\eta, \tau) = \mathcal{L}^{-1} \left[\frac{1}{s} e^{-\sqrt{s+a}\eta} + \left(\frac{Gr}{Prs - (s+a)} \frac{1 - e^{-s}}{s^2} \right) \left(e^{-\sqrt{s+a}\eta} - e^{-\sqrt{Prs}\eta} \right) \right]. \quad (35)$$

5. Parametric Study

In this section, the physical significance of various pertinent parameters such as the relaxation time (λ), magnetic parameter (M), porosity (K), Grashof number (Gr), Prandtl number (Pr), and dimensionless time (τ) in momentum, shear stress, and temperature boundary layer is described with the aid of graphs. These graphs of mass, shear stress, and energy profiles for the Maxwell material are comprised of two types of solution: (1) Solution with ramped wall temperature conditions indicated by solid lines. (2) Solution with isothermal temperature conditions indicated by dashed lines.

Figure 2 describes the variation in momentum profile with alteration in values of M . It can be seen that the mass boundary layer thickness reduces when the value of M increases. This is physically justified by the existence of the Lorentz force which arises due to the imposed magnetic field. This Lorentz force together with allied forces act as a dragging force which leads to a reduction of fluid velocity. This Lorentz force gets weaker far away from the plate, and the fluid eventually comes to rest. Furthermore, the velocity profile is higher in the case of isothermal temperature conditions in comparison with ramped temperature conditions.

The impact of Gr on the mass profile for ramped and isothermal plates is sketched in Figure 3. It can be seen that the mass boundary layer thickness increased with increasing Gr values. This is because Gr is the fraction of buoyancy and viscous forces. An increase in Gr values means a reduction of viscosity and increased strength of buoyancy force. This strong buoyancy force near the plate acts as a driving force, which ultimately increases the height of velocity curve. Eventually, away from the plate, this force along other aiding forces loses its strength and the fluid gradually calms down. Moreover, the velocity is lower for the ramped condition in contrast to the isothermal condition.

Figure 4 illustrates the velocity distribution for increasing values of K . An increase in K enhances fluid velocity in both isothermal and ramped conditions and the mass boundary layer thickness is higher in the case of isothermal temperature condition. The factor providing enough ground for such enhancement is that when the material's porosity is intensified, the viscous force decreases and consequently the fluid accelerates.

In Figure 5, the influence of Pr on the momentum profile for isothermal and ramped temperature conditions is observed. The fluid velocity reduces as Pr increases. As Pr is the weightage of momentum diffusivity and thermal diffusivity, an increase in Pr corresponds to enhanced viscosity, which makes the fluid thicker. Consequently, the fluid moves slowly and finally comes to rest away from the plate. The momentum boundary layer thickness has large values in the case of isothermal temperature in contrast to ramped wall temperature conditions.

The key role of relaxation time in the flow of fluid for ramped wall and isothermal wall conditions is presented in Figure 6. An increase in the Maxwell parameter leads to a decrease in velocity. In the physical sense, relaxation describes the return of a perturbed system to a state of equilibrium. As an increased relaxation time implies that the fluid will take extra time to become calm, this readily justifies

the decrease in velocity. A comparative analysis presented later in this paper found that fluid velocity had a higher profile in the case of isothermal temperature condition.

The influence of Pr on temperature profiles is given in Figure 7. The energy profile decays with increasing Pr values because of small thermal conductivity. For small Pr values, the fluid receives a large amount of heat and so the thickness of the temperature boundary layer eventually increases. In the case of an isothermal plate, the temperature solution has higher profile. Figure 8 reveals that increase in dimensionless time elevates the temperature profiles for both ramped and isothermal wall conditions.

The significant relationship between the Maxwell parameter (λ) and shear stress is demonstrated in Figure 9. An increase in λ results in decreasing shear stress magnitude. This is justified by the significant decay in internal viscosity due to small friction. Additionally, it is interesting to note that shear stress has greater values in the case of ramped wall temperature in contrast to isothermal temperature.

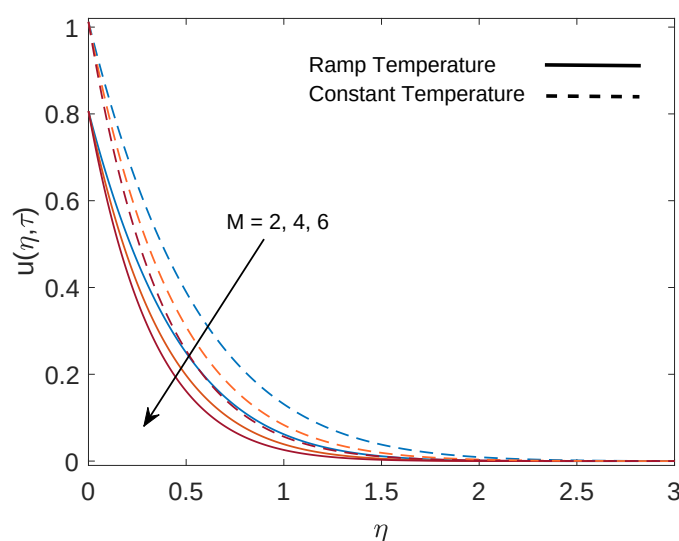


Figure 2. Effect of different values of M on the momentum profile for $Gr = 1$, $\lambda = 0.2$ and $K = 0.5$.

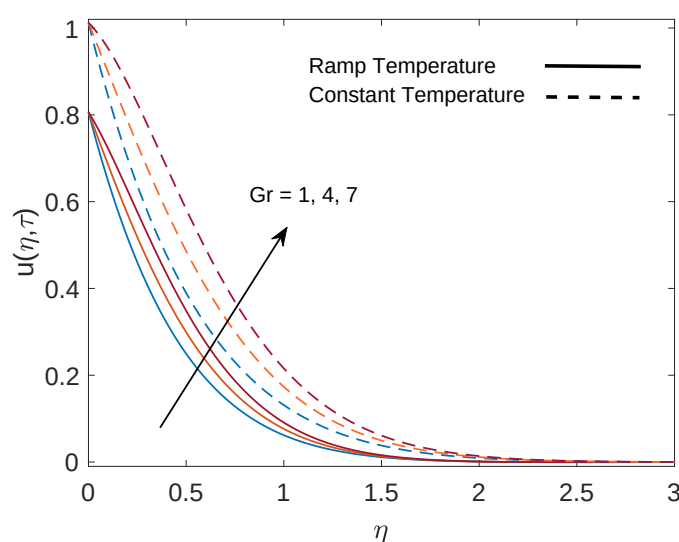


Figure 3. Effect of different values of Gr on momentum profile for $M = 2$, $\lambda = 0.2$ and $K = 0.5$.

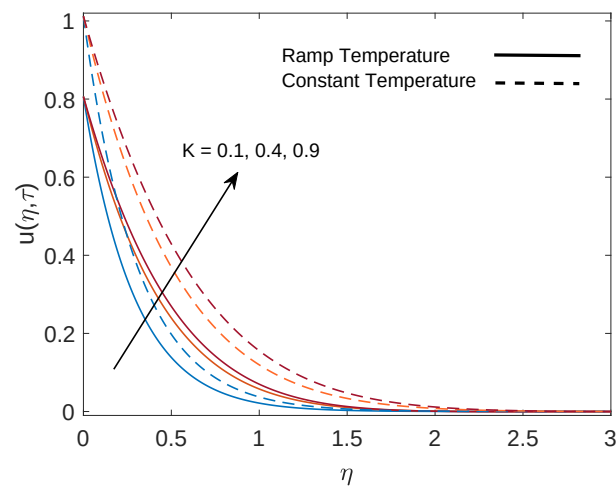


Figure 4. Effect of different values of K on momentum profile for $Gr = 1$, $\lambda = 0.2$ and $M = 2$.

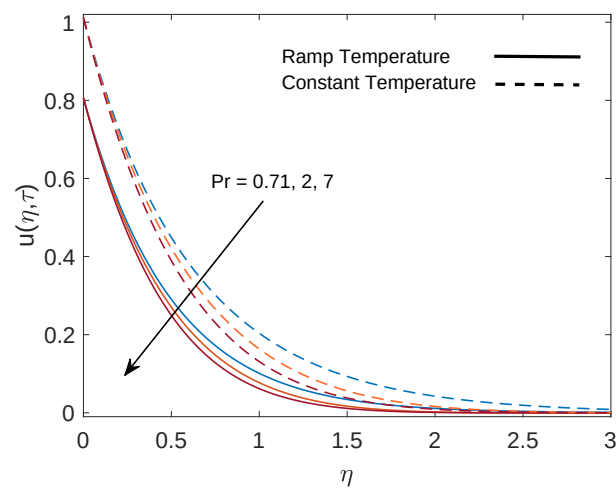


Figure 5. Effect of different values of Pr on momentum profile for $Gr = 1$, $\lambda = 0.2$, $M = 2$, and $K = 0.5$.

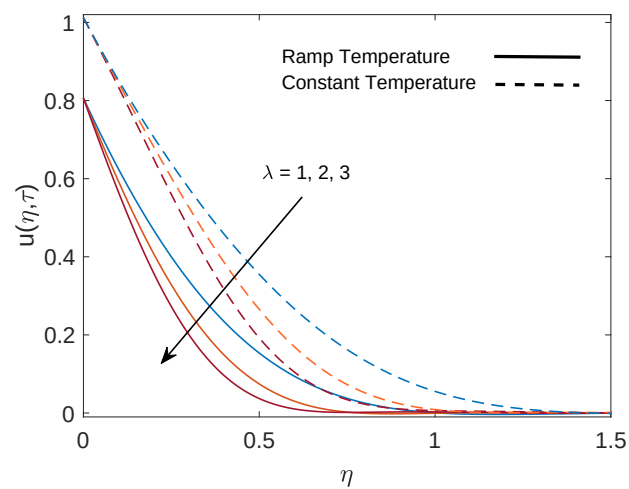


Figure 6. Effect of different values of λ on momentum profile for $Gr = 1$, $M = 2$ and $K = 0.5$.

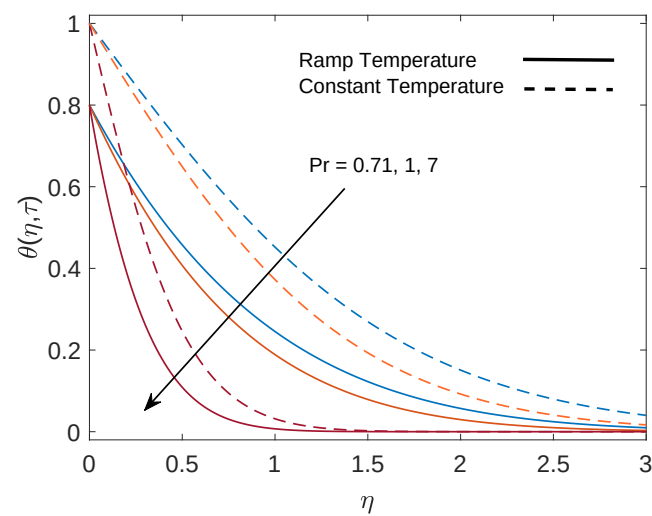


Figure 7. Effect of different values of Pr on the energy profile.

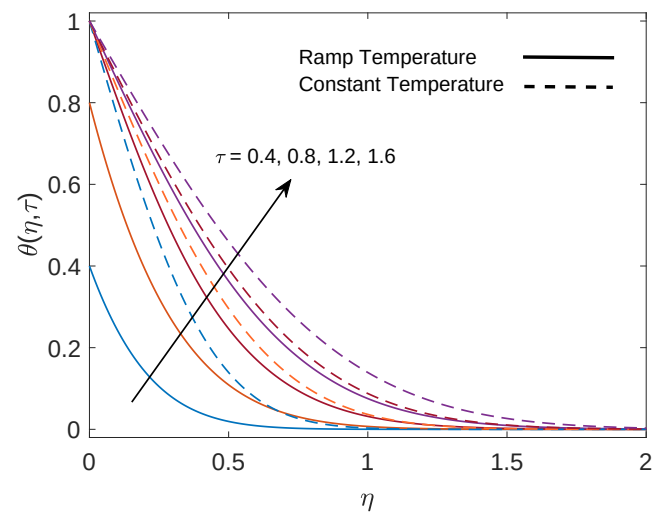


Figure 8. Effect of different values of τ on the energy profile.

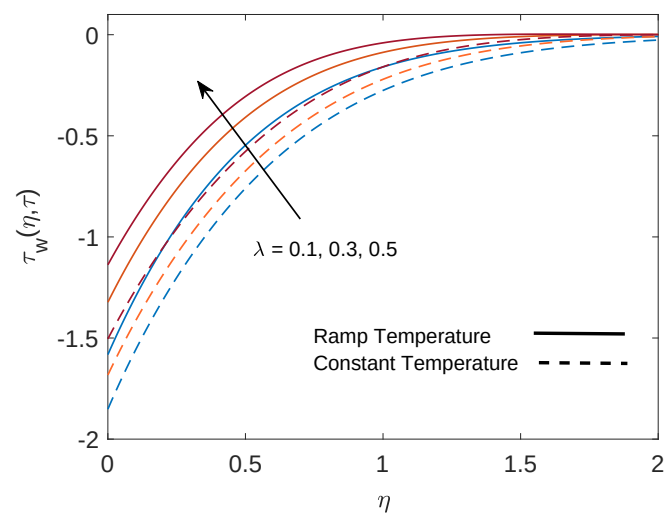


Figure 9. Skin friction for different values of λ .

6. Conclusions

The prime purpose of this work was to examine the influence of the simultaneous use of ramped wall temperature and velocity conditions on unsteady, MHD naturally convective flow of a Maxwell fluid when an infinite vertical plate is embedded in a porous material. It is significant to state that the literature is limited when it comes to the simultaneous use of these conditions, though physically these conditions have a key significance. It is analytically difficult to calculate the solutions of a Maxwell fluid incorporating both ramped wall conditions at the same time. In the current work, the Laplace transformation is first applied as a principal solution-finding tool, and later complex combinations of the Laplace parameter appearing because of simultaneous ramped conditions are handled with the help of a numerical Laplace inversion named as Durbin method. An expression for the rate of heat transfer (Nusselt number) is also calculated. The behavioral study of momentum, shear stress, and temperature profiles subjected to pertinent parameters was conducted and discussed with the aid of graphs. Moreover, a comparison between two solutions (i.e., solution with ramped wall temperature condition and solution with isothermal temperature condition) is also drawn for several parameters.

The significant findings of this analysis are:

- Momentum boundary layer thickness increases with increasing values of the parameters Gr (due to the strong buoyancy force) and K (due to the decrease in viscous force). Meanwhile, it reduces with increasing values of M (due to the strong drag force) and λ (due to the increase in resistance).
- An elevation in the relaxation time λ leads to a decrease in velocity on the plate (in terms of skin friction).
- The Nusselt number describes that the heat transfer rate enhances with increasing thermal diffusivity. For higher Pr values, increased resistance leads to a lower rate of heat transfer whereas the fluid has higher thermal conductivity for small values of Pr [45,46].

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Nomenclature

ρ	Fluid density
λ	Maxwell parameter
τ	Dimensionless time
μ	Dynamic viscosity
u	Fluid velocity
η	Space variable
σ	Electrical conductivity
β	Coefficient of thermal expansion
ϕ	Porosity parameter
k^*	Permeability parameter
g	Acceleration due to gravity
T	Fluid temperature
T_∞	Free stream temperature

S	Shear stress
k	Thermal conductivity
c_p	Heat capacitance
T_w	Wall temperature
Gr	Grashof number
M	Magnetic parameter
K	Dimensionless porosity parameter
Pr	Prandtl number
s	Laplace parameter
$\mathcal{L}$	Laplace transform coefficient

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