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On (p, q)-Sumudu and (p, q)-Laplace Transforms of the Basic Analogue of Aleph-Function

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Abstract: In this paper, we introduce the definitions of Sumudu and Laplace transforms of first and second kind in quantum calculus by using functions of several variables. On account of the general nature of the (p, q)-analogue of Aleph-function, a large number of new and known results for these transforms were obtained. Also, we obtain some interesting relationships and identities for these transforms. We also derive some correlations among Aleph function and the above-mentioned integral transforms in quantum calculus.

Keywords: integral transforms; basic hypergeometric functions; q-calculus

MSC: 33D05; 33D15; 33D60

1. Introduction

There are numerous integral transforms to solve differential equations. The Laplace transform is the most popular and widely used in several branches of engineering and applied mathematics among these transforms. In the sequence of the Laplace, Fourier, Mellin, and Hankel transforms, in 1993 Watugala [1,2] introduced a new integral transform, named the Sumudu transform, and further applied it to the solution of ordinary differential equation in control engineering problems. This transform has a property that it has larger frequency domain than Laplace transform. That is why they attracted the attention of so many researchers [3–5]. Two of the most frequently used formulas in the area of integral transforms are the classical Laplace and Sumudu transform and their corresponding q-analogues. They were successively applied in the theory of differential/difference equations, quantum mechanics, special functions, and calculating integrals [6].

The theory of q-analysis and the concepts of difference equations dates back to eighteenth century by Euler and developed by Gauss and Ramnaujan. The number of studies in q-analysis have been grown rapidly in many ares of mathematics such as number theory, orthogonal polynomials, in ordinary, and partial differential equations and also in physical disciplines such as quantum mechanics and relativity [7–9].

In the past century, many authors have generalized H-function. In a recent paper, Sudland et al. [10] introduced a generalization of Saxena's I-function [11], which is also a generalization of Fox's H-function. This function is known as Aleph function. In their paper, Saxena and Pogany [12] studied fractional integration formulae for the Aleph functions. Fractional driftless Fokker–Planck equation were studied by Sudland et al. [13], a special case of the Aleph function. This subject has tremendous applications in the field of applied mathematics [14–19]. Motivated by these applications, various researchers are working on these operators to evaluate q-analogue of various integral transforms and special functions.

The Aleph function [12,20] is defined as follows.

$$\begin{aligned}\aleph(z) &= \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]\\ &= \frac{1}{2\pi\omega}\int\limits_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)z^{-s}ds \end{aligned} \tag{1}$$

where $z \neq 0, \omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i[\prod\limits_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\prod\limits_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)]}.$$

The q-gamma function was first introduced by Thomae and later by Jackson. The q-analogue of gamma function which is defined by Jackson [21] is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, 0 < q < 1. \tag{2}$$

Jackson gave the general definition which is given below

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt \tag{3}$$

where

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n \tag{4}$$

Jackson also defined an integral, i.e.,

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n \tag{5}$$

q-analogues of gamma and beta functions are defined as follows (see [22]),

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} x^{\alpha-1}E_q(q(1-q)x)d_qx \ \ (\alpha > 0). \tag{6}$$

$$\Gamma_q(\alpha) = K(A;\alpha)\int_0^{\frac{\infty}{A(1-q)}} x^{\alpha-1}e_q(-(1-q)x)d_qx \ \ (\alpha > 0). \tag{7}$$

$$B_q(t;s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}d_qx \ \ (t,s > 0). \tag{8}$$

where

$$K(A;t) = A^{t-1}\frac{(-q/A;q)_\infty}{(-q^t/A;q)_\infty}\frac{(-A;q)_\infty}{(-Aq^{t-1};q)_\infty} \quad (t \in \mathbb{R}) \tag{9}$$

P. Njionou Sadjang [23] introduced the so-called the shifted factorial as follows.

$$(x \ominus a)^n_{p,q} = (x-a)(px-aq)(p^2x-aq^2)\dots(xp^{n-1}-aq^{n-1})$$
$$(x \oplus a)^n_{p,q} = (x+a)(px+aq)(p^2x+aq^2)\dots(xp^{n-1}+aq^{n-1})$$

These definitions are extended as follows.

$$(a \ominus b)^n_{p,q} = \prod_{k=0}^{\infty}(ap^k - bq^k)$$
$$(a \oplus b)^n_{p,q} = \prod_{k=0}^{\infty}(ap^k + bq^k)$$

Let x be a complex number, the (p,q)-Gamma function is defined by P. Njionou Sadjang [24]

$$\Gamma_{p,q}(x) = \frac{(p\ominus q)^\infty_{p,q}}{(p^x \ominus q^x)^\infty_{p,q}}(p-q)^{1-x}, 0<q<p$$

If we put $p=1$, then $\Gamma_{p,q}$ reduces to Γ_q.

The (p,q)-Gamma function fulfills the following fundamental relation.

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x)$$

If x is a nonnegative integer, it follows from above that

$$\Gamma_{p,q}(x+1) = [x]_{p,q}!$$

It can be also easily seen from the definition that

$$\Gamma_{p,q}(x+1) = \frac{(p\ominus q)^x_{p,q}}{(p-q)^x_{p,q}}$$

P. Njionou Sadjang [24] also defined the (p,q)-Beta function as

$$B_{p,q}(x,y) = \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}$$

The (p,q)-derivative of the function $f(x)$ is defined as follows [23,24],

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, x \neq 0$$

where $D_{p,q}f(0) = f'(0)$, provided that $f(x)$ is differentiable at $x=0$

The (p,q)-numbers $[n]_{p,q}$ and (p,q) factorials $[n]_{p,q}!$ are defined [23] as:

$$[n]_{p,q} = \frac{p^n - q^n}{(p-q)}$$

and

$$[n]_{p,q}! = [1]_{p,q}[2]_{p,q}\dots[n]_{p,q}$$

respectively.

Also it happens that $D_{p,q}(x^n) = [n]_{p,q} x^{n-1}$.

The (p,q)-analogue of the exponential function are given [25] as follows.

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n$$

For $0 < q < p$, the (p,q)-Gamma function of first kind is defined in [25] as follows,

$$\Gamma_{p,q}(z) = p^{\frac{z(z-1)}{2}} \int_0^{\infty} t^{z-1} E_{p,q}(-qt) d_{p,q}t. \quad Re(z) > 0.$$

For $0 < q < p$, the (p,q)-Gamma function of second kind is defined in [25] as follows,

$$\gamma_{p,q}(z) = q^{\frac{z(z-1)}{2}} \int_0^{\infty} t^{z-1} e_{p,q}(-pt) d_{p,q}t. \quad Re(z) > 0.$$

Remark 1. *$D_{p,q}(x)$ reduces to Hahn Derivative $d_q f(x)$ iff $p \to 1$.*

Remark 2. *$[n]_{p,q} = [n]_q$ (Hahn Basic Number) iff $p \to 1$. where $[n]_q = \frac{1-q^n}{1-q}, q \neq 1$.*

Note 1.

$$D^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu-n+1)} x^{\mu-n}, Re(\mu)+1 > 0$$

$$I^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu+n+1)} x^{\mu+n}, Re(\mu)+1 > 0$$

Classical Laplace transform:

Suppose $f(t)$ is a real-valued function defined over the interval $(0,\infty)$. The Laplace transform of $f(t)$ is defined by

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \quad Re(s) > 0. \tag{10}$$

The Laplace transform is said to exist if the integral (10) is convergent for some values of s.

***q*-Laplace transform:**

Hahn [26] defined the q-image of classical Laplace transform by means of the following q-integrals as

$$L_q[f(s)] = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q^{qsx} f(x) d_q x, \; Re(s) > 0. \tag{11}$$

$$or \;\; L_q[f(s)] = \frac{1}{(1-q)} \int_0^{\infty} e_q^{-sx} f(x) d_q x, \; Re(s) > 0. \tag{12}$$

where e_q^{-sx} and E_q^{qsx} are the q-analogues of exponential function [26].

The Laplace transform of the power function is defined as

$$L[t^{\mu}] = \frac{\Gamma(\mu+1)}{s^{\mu+1}}. \tag{13}$$

The q-Laplace transform of the power function is defined as in [27,28]

$$L_q[t^{\mu}] = \frac{\Gamma_q(\mu+1)(1-q)^{\mu}}{s^{\mu+1}}. \tag{14}$$

For a given function f(t), Sadjang [25] defined (p,q)-Laplace transform of the first kind as

$$F_1(s) = L_{p,q}f(t)(s) = \int_0^{\infty} f(t)E_{p,q}(-qts)dp,qt, \quad s > 0.$$

Also for $\alpha > -1$, Sadjang [25] defined the (p,q)-Laplace transform of first kind of the power function as

$$L_{p,q}t^{\alpha} = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}} \quad s > 0.$$

For a given function f(t), Sadjang [25] defined (p,q)-Laplace transform of the second kind as

$$F_2(s) = \mathbb{L}_{p,q}f(t)(s) = \int_0^{\infty} f(t)e_{p,q}(-pts)dp,qt, \quad s > 0.$$

Also for $\alpha > -1$, Sadjang [25] defined the (p,q)-Laplace transform of second kind of the power function as

$$\mathbb{L}_{p,q}t^{\alpha} = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}} \quad s > 0.$$

Classical Sumudu Transform:

Over the set of function

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, \ \texttt{if} \ t \in (-1)^j \times [0,\infty) \right\}$$

the Sumudu transform is defined by [2]

$$G(u) = S[f(t)] = \int_0^{\infty} f(ut)e^{-t}dt, u \in (-\tau_1, \tau_2). \tag{15}$$

q-Sumudu Transform:

Albayrak et al. [29] defined the q-analogues of the Sumudu transform by means of the following.

$$S_q[f(t);s] = \frac{1}{(1-q)} \int_0^1 E_q(qt)f(st)d_qt, \ \ s \in (-\tau_1, \tau_2)$$

$$\texttt{or} \ \ S_q[f(t);s] = \frac{1}{(1-q)s} \int_0^s E_q(\frac{q}{s}t)f(t)d_qt, \ \ s \in (-\tau_1, \tau_2) \tag{16}$$

$$S_q[x^{\alpha-1};s] = s^{\alpha-1}(1-q)^{\alpha-1}\Gamma_q(\alpha) \tag{17}$$

$$\texttt{Also,} \quad (1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}.$$

The (p,q)-Sumudu transform of first kind for a given function $f(t)$ is defined by Sadjang [30] as follows,

$$G_1(s) = S_{p,q}f(t)(s) = \frac{1}{s}\int_0^\infty f(t)E_{p,q}(\frac{-qt}{s})dp,qt$$
$$= \int_0^\infty f(st)E_{p,q}(-qt)dp,qt, \quad s>0.$$

Sadjang [30] also defined the (p,q)-Sumudu transform of first kind of the power function as

$$S_{p,q}t^\alpha = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}}s^\alpha \quad s>0.$$

The (p,q)-Sumudu transform of second kind for a given function $f(t)$ is defined by Sadjang [30] as follows,

$$G_2(s) = \$_{p,q}f(t)(s) = \frac{1}{s}\int_0^\infty f(t)e_{p,q}(\frac{-pt}{s})dp,qt$$
$$= \int_0^\infty f(st)e_{p,q}(-qt)dp,qt, \quad s>0.$$

Sadjang [30] also defined the (p,q)-Sumudu transform of first kind of the power function as

$$\$_{p,q}t^\alpha = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}}}s^\alpha \quad s>0.$$

2. The (p,q)-Analogue of Aleph-Function

Dutta and Arora [31] gave the definition of q-analogue of Aleph-function as follows,

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod\limits_{j=1}^{m}G(q^{b_j+B_js})\prod\limits_{j=1}^{n}G(q^{1-a_j-A_js})\pi z^{-s}d_qs}{\sum_{i=1}^r\tau_i[\prod\limits_{j=m+1}^{q_i}G(q^{1-b_{ji}-B_{ji}s})\prod\limits_{j=n+1}^{p_i}G(q^{a_{ji}+A_{ji}s})]G(q^s)G(q^{1-s})sin\pi s}, \tag{18}$$

where $z\neq 0, \omega=\sqrt{-1}$ and

$$G(q^\alpha) = \prod_{n=0}^{\infty}(1-q^{\alpha+n})^{-1} = \frac{1}{(q^\alpha;q)_\infty}. \tag{19}$$

And the convergence conditions are defined in [31].

Altaf et al. [17] defined q-analogue of Aleph-function by applying q-Gamma function as follows,

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma_q(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma_q(1-a_j-A_js)z^{-s}d_qs}{\sum_{i=1}^r\tau_i\left[\prod\limits_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}-B_{ji}s)\prod\limits_{j=n+1}^{p_i}\Gamma_q(a_{ji}+A_{ji}s)\right]\Gamma_q(s)\Gamma_q(1-s)sin\pi s} \tag{20}$$

The convergence conditions remains same as defined in [31].

Remark 3. *Put $\tau_i = 1$ in (20), we will get the special case for the basic analogue of the I-function [32,33]:*

$$I^{m,n}_{p_i,q_i;r}\left[z;q\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & (a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m} & \dots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma_q(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma_q(1-a_j-A_js)z^{-s}d_qs}{\sum_{i=1}^{r}\left[\prod\limits_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}-B_{ji}s)\prod\limits_{j=n+1}^{p_i}\Gamma_q(a_{ji}+A_{ji}s)\right]\Gamma_q(s)\Gamma_q(1-s)sin\pi s}. \tag{21}$$

Taking $r = 1$ in (21) we have known results [34] as

$$H^{m,n}_{P,Q}\left[z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma_q(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma_q(1-a_j-A_js)z^{-s}d_qs}{\prod\limits_{j=m+1}^{Q}\Gamma_q(1-b_j-B_js)\prod\limits_{j=n+1}^{P}\Gamma_q(a_j+A_js)\Gamma_q(s)\Gamma_q(1-s)sin\pi s}. \tag{22}$$

The existence conditions will remain same as in above for Equation (20).

Alok et al. [35] defined (p,q)-analogue of Aleph-function as follows,

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i[\prod\limits_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod\limits_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}ds$$

where L is contour of integration running from $-i\infty$ to $+i\infty$ in such a manner so that all poles of $\Gamma_{p,q}(b_j+B_js); 1 \le j \le m$ are to right of the path and those of $\Gamma_{p,q}(1-a_j-A_js); 1 \le j \le n$, are to left. The integral converges as defined in [35].

Special Cases:

Taking $\tau_i = 1$, in the above equation we get the (p,q)-analogue of I-function defined by Altaf et al. [36] as follows.

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j + B_j s) \prod_{j=1}^{n} \Gamma_{p,q}(1 - a_j - A_j s) \pi z^{-s}}{\sum_{i=1}^{r} [\prod_{j=m+1}^{q_i} \Gamma_{p,q}(1 - b_{ji} - B_{ji} s) \prod_{j=n+1}^{p_i} \Gamma_{p,q}(a_{ji} + A_{ji} s) \Gamma_{p,q}(s) \Gamma_{p,q}(1-s) sin\pi s]} ds \quad \textbf{(A)}$$

Taking $r = 1$, we will get the (p,q)-analogue of Fox's H-function defined by Altaf et al. [36] as follows.

$$H_{P,Q}^{m,n} \left[\left(z; (p,q) \middle| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right) \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j + B_j s) \prod_{j=1}^{n} \Gamma_{p,q}(1 - a_j - A_j s) \pi z^{-s} ds}{\prod_{j=m+1}^{Q} \Gamma_{p,q}(1 - b_j - B_j s) \prod_{j=n+1}^{P} \Gamma_{p,q}(a_j + A_j s) \Gamma_{p,q}(s) \Gamma_{p,q}(1-s) sin\pi s} \quad \textbf{(B)}$$

If we take $A_j = B_j = 1$, in the above equation, we will get the (p,q)-analogue of Meijer's G-Function defined by Swati Pathak et al. [37] as follows,

$$G_{P,Q}^{m,n} \left[\left(z; (p,q) \middle| \begin{matrix} a_1, a_2, \ldots, a_P \\ b_1, b_2, \ldots, b_Q \end{matrix} \right) \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma_{p,q}(b_j + s) \prod_{j=1}^{n} \Gamma_{p,q}(1 - a_j - s) \pi z^{-s} ds}{\prod_{j=m+1}^{Q} \Gamma_{p,q}(1 - b_j - s) \prod_{j=n+1}^{P} \Gamma_{p,q}(a_j + s) \Gamma_{p,q}(s) \Gamma_{p,q}(1-s) sin\pi s}$$

Put $p = 1$, the above results of (p,q)-analogue change into well known results of basic analogues of Aleph-function, I-function, H-function and G-function [17,38].

3. (p,q)-Sumudu Transforms of First and Second Kind of the (p,q) Analogue of $\aleph$-Function

In this section we derive (p,q)-Sumudu transforms of first and second kind of the (p,q)-analogue of Aleph-function with the help of (p,q)-Gamma function and Mellin–Barnes-type integral [39]. We use here notation

$$\aleph_{p_i,q_i,\tau_i;r}^{m,n}[z;(p,q)|.] \text{ for } \aleph_{p_i,q_i,\tau_i;r}^{m,n} \left[z;(p,q) \middle| \begin{matrix} (a_j, A_j)_{1,n} & \ldots & [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m} & \ldots & [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i} \end{matrix} \right].$$

3.1. (p,q)-Sumudu Transform of First and Second Kind for $\aleph_{p_i,q_i,\tau_i;r}^{m,n}[z;(p,q)|.]$:

Theorem 1. *Let $z, \rho \in \mathbb{C}, u \in (-\tau_1, \tau_2)$ and $Re(s) > 0$ and $|\frac{p}{q}| > 1$. Further, let the constants A_j,B_j,A_{ji},B_{ji} be real and positive, a_j,b_j,a_{ji},b_{ji} are complex numbers $(i = 1, 2, \ldots, p_i; j = 1, 2, \ldots, q_i)$, $\tau_i > 0$ for $i = 1, 2, \ldots, r$. The conditions as given in above for (p,q)-Aleph function also holds. Then, the (p,q)-Sumudu transform of first kind of the $\aleph$-function exists and the following relation holds,*

$$S_{p,q} \left[t^{\rho-1} \aleph_{p_i,q_i,\tau_i;r}^{m,n} \left[z;(p,q) \middle| \begin{matrix} (a_j, A_j)_{1,n} & \ldots & [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m} & \ldots & [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i} \end{matrix} \right] \right]$$

$$= [p^{\frac{-\rho}{2}} u]^{\rho-1} \aleph_{p_i,q_i+1,\tau_i;r}^{m+1,n} \left[z;(p,q) \middle| \begin{matrix} (a_j, A_j)_{1,n} & \ldots & [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (\rho, 0)(b_j, B_j)_{1,m} & \ldots & [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i} \end{matrix} \right]. \quad (23)$$

Proof. To prove (23), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of (23) in terms of Mellin–Barnes contour integral and apply (p,q)-Sumudu transform integral (say I).

$$I = \int_0^\infty (ut)^{\rho-1} \aleph^{m,n}_{p_i,q_i,\tau_i;r} \left[z;(p,q) \middle| \begin{matrix} (a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i} \end{matrix} \right] E_{p,q}(-qt)d_{p,q}t.$$

$$\begin{aligned} &= \int_0^\infty (ut)^{\rho-1} \frac{1}{2\pi\omega} \int_L \frac{\prod_{j=1}^m \Gamma_{p,q}(b_j+B_js) \prod_{j=1}^n \Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r \tau_i [\prod_{j=m+1}^{q_i} \Gamma_{p,q}(1-b_{ji}-B_{ji}s) \prod_{j=n+1}^{p_i} \Gamma_{p,q}(a_{ji}+A_{ji}s)]\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s} \\ &\times \{E_{p,q}(-qt)d_{p,q}td_{p,q}s\} \end{aligned} \tag{24}$$

$$\begin{aligned} I &= u^{\rho-1} \left[\frac{1}{2\pi\omega} \int_L \frac{\prod_{j=1}^m \Gamma_{p,q}(b_j+B_js) \prod_{j=1}^n \Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r \tau_i [\prod_{j=m+1}^{q_i} \Gamma_{p,q}(1-b_{ji}-B_{ji}s) \prod_{j=n+1}^{p_i} \Gamma_{p,q}(a_{ji}+A_{ji}s)]\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s} \right. \\ &\times \left. \int_0^\infty (t)^{\rho-1} E_{p,q}(-qt)d_{p,q}t \right] d_{p,q}s. \end{aligned}$$

For the second integral using (p,q)-Gamma function

$$S_{p,q}(x^\alpha) = \frac{\Gamma_{p,q}(\alpha+1)s^\alpha}{p^{\frac{\alpha(\alpha+1)}{2}}} \quad (\alpha > 0),$$

we arrive at

$$I = p^{\frac{-\rho(\rho-1)}{2}} u^{\rho-1} \left[\frac{1}{2\pi\omega} \int_L \frac{\prod_{j=1}^m \Gamma_{p,q}(b_j+B_js) \prod_{j=1}^n \Gamma_{p,q}(1-a_j-A_js)\pi z^{-s} \Gamma_{p,q}(\rho-0.s)d_{p,q}s}{\sum_{i=1}^r \tau_i [\prod_{j=m+1}^{q_i} \Gamma_{p,q}(1-b_{ji}-B_{ji}s) \prod_{j=n+1}^{p_i} \Gamma_{p,q}(a_{ji}+A_{ji}s)]\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s} \right]$$

$$= [up^{\frac{-\rho}{2}}]^{\rho-1} \aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r} \left[z;(p,q) \middle| \begin{matrix} (a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (\rho,0),(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i} \end{matrix} \right].$$

By interpreting the Mellin–Barnes counter integral thus obtained in terms of the (p,q)-analogue of $\aleph$-function, we obtain the result (23). This completes the proof of Theorem 1. □

Theorem 2. *Let $z,\rho \in \mathbb{C}, u \in (-\tau_1,\tau_2)$ and $Re(s) > 0$ and $|\frac{p}{q}| > 1$. Further, let the constants A_j,B_j,A_{ji},B_{ji} are real and positive, a_j,b_j,a_{ji},b_{ji} are complex numbers ($i = 1,2,\dots,p_i; j = 1,2,\dots,q_i$), $\tau_i > 0$ for $i = 1,2,\dots,r$. The conditions as given in above for (p,q)-Aleph function also holds. Then, the (p,q)-Sumudu transform of second type of the $\aleph$-function exists and the following relation holds,*

$$\mathbb{S}_{p,q} \left[t^{\rho-1} \aleph^{m,n}_{p_i,q_i,\tau_i;r} \left[z;(p,q) \middle| \begin{matrix} (a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i} \end{matrix} \right] \right]$$

$$= [q^{\frac{-\rho}{2}}u]^{\rho-1} \aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r} \left[z;(p,q) \middle| \begin{matrix} (a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (\rho,0)(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i} \end{matrix} \right]. \tag{25}$$

Proof. In order to prove (25), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of (25) in terms of Mellin–Barnes contour integral and apply (p,q)-Sumudu transform integral (say I).

$$I = \int_0^\infty (ut)^{\rho-1} \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right] e_{p,q}(-pt)d_{p,q}t.$$

$$\begin{aligned}&= \int_0^\infty (ut)^{\rho-1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)]\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s}\\&\times \quad \{e_{p,q}(-pt)d_{p,q}td_{p,q}s\}\end{aligned} \tag{26}$$

$$\begin{aligned}I &= u^{\rho-1}\left[\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)]\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s}\right.\\&\times \left.\int_0^\infty (t)^{\rho-1}E_{p,q}(-pt)d_{p,q}t\right]d_{p,q}s.\end{aligned}$$

For the second integral using (p,q)-gamma function

$$\mathbb{S}_{p,q}(x^\alpha) = \frac{\gamma_{p,q}(\alpha+1)s^\alpha}{q^{\frac{\alpha(\alpha+1)}{2}}} \qquad Re(\alpha) > 0,$$

we arrive at

$$I = q^{\frac{-\rho(\rho-1)}{2}}u^{\rho-1}\left[\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}\gamma_{p,q}(\rho-0.s)d_{p,q}s}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)]\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s}\right]$$

$$= [uq^{\frac{-\rho}{2}}]^{\rho-1}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left[z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0),(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right].$$

By interpreting the Mellin–Barnes counter integral thus obtained in terms of the (p,q)-analogue of $\aleph$-function, we obtain the result (25). This completes the proof of Theorem 2. □

Special Cases:

Now, if we put $p=1$ in theorem 1 then it reduces to q-Sumudu transform of Aleph-function [38], and if we put $p=1$ and $q=1$ in theorem 1 then it reduces to Sumudu transform of Aleph-function [40].

If we put τ_i= 1 $(i=1,2,\dots,r)$ in (23) and take (A) into account, then we arrive at the following result in terms of the (p,q)-analogue of I-function [36] as follows.

Corollary 1.

$$S_{p,q}\left[t^{\rho-1}I^{m,n}_{p_i,q_i;r}\left[z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n} & \dots & (a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m} & \dots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right]\right]$$

$$= [uq^{\frac{-\rho}{2}}]^{\rho-1} I^{m+1,n}_{p_i,q_i+1;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\(\rho,0),(b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \quad (27)$$

The existence conditions for (27) are the same as given in Theorem 1.

If we put τ_i=1, $i = 1,2,\ldots,r$, $p_i = P, q_i = Q$ and set $r = 1$ in (23) and take (B) into account, then we arrive at the following result in terms of (p,q)-analogue of H-function [36] as follows.

Corollary 2.

$$S_{p,q}\left[t^{\rho-1}H^{m,n}_{P,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(b_Q,B_Q)\end{matrix}\right.\right]\right] = [u(1-q)]^{\rho-1}H^{m,n+1}_{P+1,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(\rho,0),(b_Q,B_Q)\end{matrix}\right.\right] \quad (28)$$

Now if we put $p = 1$ in theorem 2 then it reduces to q-Sumudu transform of Aleph-function [38],

and if we put $p = 1$ and $q = 1$ in theorem 2 then it reduces to Sumudu transform of Aleph-function [40].

If we put τ_i= 1 ($i = 1,2,\ldots,r$) in (23) and take (A) into account, then we arrive at the following result in terms of the (p,q)-analogue of I-function [36] as follows.

Corollary 3.

$$\mathbb{S}_{p,q}\left[t^{\rho-1}I^{m,n}_{p_i,q_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= [q^{\frac{-\rho}{2}}u]^{\rho-1} I^{m+1,n}_{p_i,q_i+1;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \quad (29)$$

The existence conditions for (29) are the same as given in Theorem 2.

If we put τ_i=1, $i = 1,2,\ldots,r$, $p_i = P, q_i = Q$ and set $r = 1$ in (23) and take (B) into account, then we arrive at the following result in terms of (p,q)-analogue of H-function [36] as follows.

Corollary 4.

$$\mathbb{S}_{p,q}\left[t^{\rho-1}H^{m,n}_{P,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(b_Q,B_Q)\end{matrix}\right.\right]\right] = [u(1-q)]^{\rho-1}H^{m,n+1}_{P+1,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(\rho,0),(b_Q,B_Q)\end{matrix}\right.\right] \quad (30)$$

Next, we find the (p,q)-Laplace transform of first and second kind for (p,q)-analogue of $\aleph$-function.

3.2. (p,q)-Laplace Transform of First and Second Kind for $\aleph^{m,n}_{p_i,q_i,\tau_i;r}[z;(p,q)|.]$:

Theorem 3. *Let z,$\rho \in \mathbb{C}$ and $Re(s) > 0$, $Re(u) > 0$ and $|\frac{p}{q}| > 1$. Further, let the constants A_j,B_j,A_{ji},B_{ji} are real and positive, a_j,b_j,a_{ji},b_{ji} are complex numbers ($i = 1,2,\ldots,p_i; j = 1,2,\ldots,q_i$), $\tau_i > 0$ $\texttt{for}$ $i = 1,2,\ldots,r$. The conditions as given in above for (p,q)-Aleph function also holds. Then, the (p,q)-Laplace transform of first kind of the $\aleph$-function exists and the following relation holds,*

$$L_{p,q}\left[t^{\rho-1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} & \ldots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= u^{-\rho}p^{\frac{\rho(1-\rho)}{2}}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0),(b_j,B_j)_{1,m} & \ldots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]. \quad (31)$$

Proof. To prove (31), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of (31) in terms of Mellin–Barnes contour integral and apply (p,q)-Laplace transform integral (say I).

$$I = \int_0^\infty t^{\rho-1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right] E_{p,q}(-qtu)d_{p,q}t$$

$$= \int_0^\infty t^{\rho-1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\right]\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s}$$
$$\times \quad \{E_{p,q}(-qtu)d_{p,q}td_{p,q}s\}$$

$$= \frac{1}{2\pi\omega}\int_L \Bigg[\frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\right]\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s}$$
$$\times \quad \int_0^\infty (t)^{\rho-1}E_{p,q}(-qtu)d_{p,q}t\Bigg]d_{p,q}s. \tag{32}$$

We know that the (p,q)-Laplace transform of first kind of certain elementary function is given by

$$L_{p,q}(x^n;s) = \int_0^\infty E_{p,q}(-sqx)x^n d_{p,q}x = \frac{\Gamma_{p,q}(n+1)}{p^{\frac{n(n+1)}{2}}s^{n+1}}, \tag{33}$$
$$Re(s) > 0, Re(n+1) > 0\ (n \in \mathbb{N}\cup\{0\}) \text{ and } |\frac{q}{p}| < 1.$$

Then, by using (33) in (32), we arrive at the following.

$$I = \left[\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\Gamma_{p,q}(\rho-0.s)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s\right]p^{\frac{\rho(\rho-1)}{2}}u^\rho}\right]d_{p,q}s$$

$$= u^{-\rho}(p)^{\frac{\rho(1-\rho)}{2}} \times$$

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\Gamma_{p,q}(\rho-0.s)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s\right]}d_{p,q}s$$

$$= u^{-\rho}(p)^{\frac{\rho(1-\rho)}{2}}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (\rho,0),(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right].$$

By interpreting the Mellin–Barnes counter integral, thus obtained in terms of the (p,q)-analogue of $\aleph$-function, we obtain the result (31). This completes the proof of Theorem 3. □

Theorem 4. *Let $z,\rho \in \mathbb{C}$ and $Re(s) > 0, Re(u) > 0$ and $|\frac{p}{q}| > 1$. Further, let the constants A_j,B_j,A_{ji},B_{ji} are real and positive, a_j,b_j,a_{ji},b_{ji} are complex numbers $(i = 1,2,\dots,p_i; j = 1,2,\dots,q_i)$, $\tau_i > 0$ `for` $i = 1,2,\dots,r$.*

The conditions as given in above for (p,q)-Aleph function also holds. Then, the (p,q)-Laplace transform of second kind of the $\aleph$-function exists and the following relation holds,

$$\mathbb{L}_{p,q}\left[t^{\rho-1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= u^{-\rho}q^{\frac{\rho(1-\rho)}{2}}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (\rho,0),(b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]. \quad (34)$$

Proof. To prove (34), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of (34) in terms of Mellin–Barnes contour integral and apply (p,q)-Laplace transform integral (say I).

$$I=\int_0^\infty t^{\rho-1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]e_{p,q}(-ptu)d_{p,q}t$$

$$= \int_0^\infty t^{\rho-1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^m \gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)\right]\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s}$$
$$\times\quad \{e_{p,q}(-ptu)d_{p,q}td_{p,q}s\}$$

$$= \frac{1}{2\pi\omega}\int_L\Bigg[\frac{\prod_{j=1}^m \gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)\right]\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s}$$
$$\times\quad \int_0^\infty (t)^{\rho-1}e_{p,q}(-ptu)d_{p,q}t\Bigg]d_{p,q}s. \quad (35)$$

We know that the (p,q)-Laplace transform of second type of certain elementary function is given by

$$\mathbb{L}_{p,q}(x^n;s)=\int_0^\infty e_{p,q}(-spx)x^nd_{p,q}x=\frac{\gamma_{p,q}(n+1)}{q^{\frac{n(n+1)}{2}}s^{n+1}}, \quad (36)$$
$$Re(s)>0, Re(n+1)>0\ (n\in\mathbb{N}\cup\{0\}) \text{ and } |\frac{q}{p}|<1.$$

Then, by using (36) in (35), we arrive at the following.

$$I=\left[\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^m \gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\gamma_{p,q}(1-a_j-A_js)\gamma_{p,q}(\rho-0.s)\pi z^{-s}}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s\right]q^{\frac{\rho(\rho-1)}{2}}u^\rho}\right]d_{p,q}s$$

$$= u^{-\rho}(q)^{\frac{\rho(1-\rho)}{2}}\times$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^m \gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\gamma_{p,q}(1-a_j-A_js)\gamma_{p,q}(\rho-0.s)\pi z^{-s}}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)\gamma_{p,q}(s)\gamma_{p,q}(1-s)sin\pi s\right]}d_{p,q}s$$

$$= u^{-\rho}(q)^{\frac{\rho(1-\rho)}{2}} \aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (\rho,0),(b_j,B_j)_{1,m} & \ldots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right].$$

By interpreting the Mellin–Barnes counter integral thus obtained in terms of the (p,q)-analogue of $\aleph$-function, we obtain the result (34). This completes the proof of Theorem 4. □

Special Cases:

Now if we put $p = 1$ in theorem 3 then it reduces to q-Laplace transform of Aleph-function [38], and if we put $p = 1$ and $q = 1$ in theorem 3 then it reduces to Laplace transform of Aleph-function [40].

If we put τ_i= 1 ($i = 1, 2, \ldots, r$) in (31) and take (A) into account, then we arrive at the following result in terms of the (p,q)-analogue of I-function [36] as follows.

Corollary 5.

$$L_{p,q}\left[t^{\rho-1} I^{m,n}_{p_i,q_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= u^{-\rho}q^{\frac{\rho(1-\rho)}{2}} I^{m+1,n}_{p_i,q_i+1;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\ (\rho,0),(b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \tag{37}$$

The existence conditions for (37) are the same as given in Theorem 3.

If we put $\tau_i = 1, i = 1, 2, \ldots, r$, $p_i = P, q_i = Q$ and set $r = 1$ in (31) and take (B) into account, then we arrive at the following result in terms of (p,q)-analogue of H-function [36].

Corollary 6.

$$L_{p,q}\left[t^{\rho-1} H^{m,n}_{P,Q}\left[\left(z;(p,q)\left|\begin{matrix}(a_P,A_P)\\ (b_Q,B_Q)\end{matrix}\right.\right)\right]\right]$$

$$= u^{-\rho}q^{\frac{\rho(1-\rho)}{2}} H^{m,n+1}_{P+1,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\ (\rho,0),(b_Q,B_Q)\end{matrix}\right.\right] \tag{38}$$

Now if we put $p = 1$ in theorem 4 then it reduces to q-Laplace transform of Aleph-function [38], and if we put $p = 1$ and $q = 1$ in theorem 4 then it reduces to Laplace transform of Aleph-function [40].

If we put τ_i= 1 ($i = 1, 2, \ldots, r$) in (34) and take (A) into account, then we arrive at the following result in terms of the (p,q)-analogue of I-function [36] as follows,

Corollary 7.

$$\mathbb{L}_{p,q}\left[t^{\rho-1} I^{m,n}_{p_i,q_i;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= u^{-\rho}p^{\frac{\rho(1-\rho)}{2}} I^{m+1,n}_{p_i,q_i+1;r}\left[z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n} & \ldots & (a_{ji},A_{ji})_{n+1,p_i}\\ (\rho,0),(b_j,B_j)_{1,m} & \ldots & (b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]. \tag{39}$$

The existence conditions for (39) are the same as given in Theorem 4.

If we put $\tau_i = 1, i = 1, 2, \ldots, r$, $p_i = P, q_i = Q$ and set $r = 1$ in (34) and take (B) into account, then we arrive at the following result in terms of (p,q)-analogue of H-function [36].

Corollary 8.

$$\mathbb{L}_{p,q}\left[t^{\rho-1}H^{m,n}_{P,Q}\left[\left(z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(b_Q,B_Q)\end{matrix}\right.\right)\right]\right]$$

$$=u^{-\rho}p^{\frac{\rho(1-\rho)}{2}}H^{m,n+1}_{P+1,Q}\left[z;(p,q)\left|\begin{matrix}(a_P,A_P)\\(\rho,0),(b_Q,B_Q)\end{matrix}\right.\right] \tag{40}$$

4. Results and Discussions

In this work, we focus on the definition of Laplace and Sumudu transforms in quantum calculus. Certain properties for these transforms were obtained. We have also introduced the (p,q)-analogue of Aleph function and derive some interesting results. Therefore, the main results can play an important role in the theory of integral transforms and special functions. We conclude our investigation by remarking that several new techniques can be obtained from the main results by appropriate choices of hypergeometric functions.

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References

1. Watugala, G.K. Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *Math. Engrgy Ind.* **1998**, *6*, 319–329. [CrossRef]
2. Watugala, G.K. Sumudu Transform: A new integral transform to solve differential equations and control engineering problems. *Integr. Educ.* **1993**, *24*, 35–43. [CrossRef]
3. Asiru, M.A. Sumudu transform and the solution of integral equations of convolution type. *Int. J. Math. Educ. Sci. Technol.* **2001**, *32*, 906–910. [CrossRef]
4. Belgacem, F.B.M.; Karaballi, A.A.; Kalla, S.L. Analytical investigations of the Sumudu transform and applications to integral production equations. *Math. Probl. Eng.* **2003**, *3*, 103–118. [CrossRef]
5. Saxena, R.K.; Ram, J.; Kumar, D. Alternative Derivation of generalized fractional kinetic equations. *J. Fract. Calc. Appl.* **2013**, *4*, 322–334.
6. Brahim, K.; Raihi, L. Two dimensional Mellin transform in quantum calculus. *Acta Math. Sci.* **2018**, *38*, 546–560. [CrossRef]
7. Abu-Risha, M.H.; Annaby, M.H.; Ismail, M.E.H.; Mansour, Z.S. Linear q-difference equations. *Z. Anal. Anwend.* **2007**, *26*, 481–494. [CrossRef]
8. Bangerezako, G. Variational calculus on q-nonuniform lattices. *J. Math. Anal. Appl.* **2005**, *306*, 161–179. [CrossRef]
9. Koelink, E. *Quantum Groups and q-Special Functions, Report 96–10*; Universiteit van Amsterdam: Amsterdam, The Netherlands, 1996.
10. Sudland, N.; Baumann, B.; Nonnenmacher, T.F. Who knows about the Aleph-function? *Fract. Calc. Appl. Anal.* **1998**, *1*, 401–402.
11. Saxena, V.P. *The I-Function*; Anamaya Publishers: New Delhi, India, 2008.

12. Saxena, R.K.; Pogany, T.K. On fractional integration formulae for Aleph functions. *Appl. Math. Comput.* **2011**, *218*, 985–990. [CrossRef]
13. Sudland, N.; Baumann, B.; Nonnenmacher, T.F. Fractional driftless Fokker–Planck equation with power law diffusion coefficients. In *Computer Algebra in Scientific Computing (CASC Konstanz 2001)*; Gangha, V.G., Mayr, E.W., Vorozhtsov, W.G., Eds.; Springer: Berlin, Germany, 2001; pp. 513–525.
14. Ahmad, A.; Ahmad, F.; Tripath, S.K. Application of Sumudu Transform in Two-Parameter Fractional Telegraph Equation. *Appl. Math. Inf. Sci. Lett.* **2016**, *3*, 1–9. [CrossRef]
15. Ahmad, A.; Jain, R.; Jain, D.K.; Ahmad, F. The Classical Sumudu Transform and its q-Image Involving Mittag-Leffler Function. *Rev. Investig. Oper.* **2018**, *39*, 545–548.
16. Ahmad, A.; Jain, R.; Jain, D.K.; Ahmad, F. The Classical Sumudu Transform and its q-Image of the Most Generalized Hypergeometric and Wright-Type Hypergeometric functions. *Int. J. Recent Innov. Trends Comput. Commun.* **2017**, *5*, 405–413.
17. Ahmad, A.; Jain, R.; Jain, D.K. q-Analogue of Aleph-function and its Transformation Formulae with q-Derivative. *J. Stat. Appl. Probab.* **2017**, *6*, 1–9. [CrossRef]
18. Chakrabarti, R.; Jagannathan, R. A (p, q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A: Math. Gen.* **1991**, *24*, L711 . [CrossRef]
19. Odzijewicz, A. Quantum algebraa and q-special functions related to coherent states maps of the disc. *Commun. Math. Phys.* **1998**, *192*, 183–215. [CrossRef]
20. Saxena, R.K.; Pogany, T.K. Mathieu-type Series for the $\aleph$-function occurring in Fokker–Planck Equation. *EJPAM* **2010**, *3*, 980-988.
21. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
22. Kac, V.G.; Sole, A.D. On Integral representations of q-gamma and q-beta functions. *Rend. Mat. Acc. Lincei* **2005**, *9*, 11–29.
23. Sadjang, P. N. On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. *arXiv* **2013**, arXiv:1309.3934.
24. Sadjang, P.N. On the (p, q)-Gamma and the (p, q)-Beta functions. *arXiv* **2015**, arXiv:1506.07394v1.
25. Sadjang, P.N. On two (p,q)-Analogues of the Laplace Transform. *arXiv* **2017**, arXiv:1703.01906v1 .
26. Hahn, W. Beitrage Zur Theorie der Heineshan Reihen. Die 24 integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation. *Math. Nachrichten* **1949**, *2*, 340–379. [CrossRef]
27. Purohit, S.D.; Yadav, R.K. On q-Laplace transforms of certain q-hypergeometric polynomials. *Proc. Natl. Acad. Sci. India* **2006**, *III*, 235–242.
28. Purohit, S.D.; Yadav, R.K.; Vyas, V.K. On applications of q-Laplace transforms to a basic analogue of the I-function. *Rev. Bull. Calcutta Math Soc.* **2010**.
29. Albayrak, D.; Purohit, S.D.; Ucar, F. On q-analogues of Sumudu transform. *Analele Univ. "Ovidius"-Constanta-Ser. Mat.* **2013**, *21*, 239–259. [CrossRef]
30. Sadjang, P.N. On (p,q)-analogues of the Sumudu transform. *arXiv* **2019**, arXiv:submit/2592285.
31. Dutta, B.K.; Arora, L.K. On a Basic Analogue of Generalized H-function. *Int. J. Math. Engg. Sci.* **2011**, *1*, 21–30.
32. Jain, D.K.; Jain, R.; Ahmad, F. Some Transformation Formula for Basic Analogue of I-function. *Asian J. Math. Stat.* **2012**, *5*, 158–162.
33. Saxena, R.K.; Modi, G.C.; Kalla, S.L. A basic analogue of Fox's H-function. *Rev. Tecn. Fac. Ingr. Univ. Zullia* **1983**, *6*, 139–143.
34. Saxena, R.K.; Rajendra, K. Basic analogue of the Generalized H-function. *Le Math.* **1995**, *50*, 263–271.
35. Jain, A.; Bhat, A.A.; Jain, R.; Jain, D.K. Certain Results of (p,q)-analogue of Aleph-function with (p,q)-Derivative. *Communicated* **2019**. [CrossRef]
36. Ahmad, A.; Jain, R.; Jain, D.K. Certain Results of (p,q)-analogue of I-Function with (p,q)-Derivative. In Proceedings of the MSSCID-2017, 20th Annual Conference of VPI, Jaipur, India, 24–26 November 2017.
37. Pathak, S.; Jain, R. Some identities involving (p,q)-analogue of Meijer's G-function. *J. Indian Acad. Math. Indore*. **2017**, *39*, 137–146.
38. Ahmad, A.; Jain, R.; Jain, D.K. q-Sumudu and q-Laplace Transforms of the basic analogue of Aleph-Function. *Far East J. Math. Sci.* **2019**, *118*, 189–211.

39. Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-Function: Theory and Applications*; Springer: New York, NY, USA, 2010.
40. Saxena, R.K.; Ram, J.; Kumar, D. Sumudu and Laplace Transforms of the Aleph-Function. *Casp. J. Appl. Math. Ecol. Econ.* **2013**, *1*, 19–28.