

Article

Zero-Energy Modes, Fractional Fermion Numbers and The Index Theorem in a Vortex-Dirac Fermion System

Takashi Yanagisawa

National Institute of Advanced Industrial Science and Technology 1-1-1 Umezono, Tsukuba, Ibaraki 305-8568, Japan; t-yanagisawa@aist.go.jp

Received: 14 January 2020; Accepted: 17 February 2020 ; Published: 2 March 2020



Abstract: Physics of topological materials has attracted much attention from both physicists and mathematicians recently. The index and the fermion number of Dirac fermions play an important role in topological insulators and topological superconductors. A zero-energy mode exists when Dirac fermions couple to objects with soliton-like structure such as kinks, vortices, monopoles, strings, and branes. We discuss a system of Dirac fermions interacting with a vortex and a kink. This kind of systems will be realized on the surface of topological insulators where Dirac fermions exist. The fermion number is fractionalized and this is related to the presence of fermion zero-energy excitation modes. A zero-energy mode can be regarded as a Majorana fermion mode when the chemical potential vanishes. Our discussion includes the case where there is a half-flux quantum vortex associated with a kink in a magnetic field in a bilayer superconductor. A normalizable wave function of fermion zero-energy mode does not exist in the core of the half-flux quantum vortex. The index of Dirac operator and the fermion number have additional contributions when a soliton scalar field has a singularity.

Keywords: topological insulator; vortex, fractional quantization; Dirac Hamiltonian; layered superconductor; index theorem; zero-energy mode; Majorana fermion; fractional fermion number

1. Introduction

Recently, topological materials have been attracted much attention in physics. New interesting topological properties will emerge in the study of quantum systems from the viewpoint of topology. In topological materials, Dirac fermions sometimes exist on the surface or in the bulk. The index of Dirac operators plays an important role in the study of topological systems [1]. The Dirac index is related to the η invariant introduced by Atiyah, Patodi, and Singer [2–5]. The η invariant has also relation with the fermion number that can be fractional in a soliton–Dirac fermion system. New low-lying excitation modes would appear when fermions interact with soliton-like objects such as domain walls, vortices, kinks, and monopoles [6–9]. There also exist zero-energy bosonic modes on solitons [9–11], and thus both bosonic and fermionic zero-energy modes will emerge in the presence of solitons. These exotic quantum states carry fermionic quantum numbers that can be fractional [12–15]. The existence of Majorana zero modes has also been examined in doped topological materials [16,17].

We expect that the quantization depends on a topological structure. In superconductors, the magnetic flux is quantized as integer times the unit quantum flux ϕ_0 . There are, however, exceptions when superconductors have multi components or form some geometric structure. A fractional-flux quantum vortex (FFQV) may exist in a multicomponent or multi-layer superconductor. In fact, an FFQV has been observed in Nb thin film superconducting bilayers recently [18]. This may raise a question about quantization.

In this paper, we investigate zero-energy modes in a vortex–fermion system and a fractional vortex–fermion system. The zero-energy mode is a Majorana fermion mode in a Dirac semi-metal with vanishing chemical potential. The inclusion of non-zero chemical potential would change the nature of excitation modes. If a bilayer system including superconductors and a topological insulator is synthesized, the Dirac fermion on the surface of the topological insulator may cause a zero-energy mode in a vortex. There are several superconductors that are suggested to be a topological superconductor of Dirac electrons [19–21]. They are, for example, $\text{FeTe}_{1-x}\text{Se}_x$ [19] and $\text{CaKFe}_4\text{As}_4$ [20,21]. In $(\text{Bi}_{1-x}\text{Sb}_x)_2\text{Te}_3$, a surface Dirac electronic state is suggested to be realized. A vortex–Dirac fermion system may be formulated on a surface of a junction of superconductors and a topological insulator. Our discussion will include the case where there is a half-flux quantum vortex (HFQV) that is associated with a kink in a bilayer superconductor in a magnetic field. A normalizable single-valued or two-valued fermion zero-energy mode does not exist in the core of HFQV.

The index has been defined for Dirac operators. The index of a Dirac operator is closely related to the fermion number and η invariant. The fermion number can be fractional in a Dirac system. The Dirac index will have an additional contribution if a scalar field has a singularity like a vortex. The paper is organized as follows. In Section 2, we examine fermion zero-energy modes in a vortex–Dirac fermion system. We show that we can identify the fermion zero-energy mode as a Majorana mode when the chemical potential $\mu = 0$. In Section 3, we discuss the index of a Dirac operator and fractional fermion number in a vortex–Dirac fermion system. We give a summary in the last section.

2. Fermion Zero-Energy Modes And Solitons

2.1. A Vortex-Dirac Fermion Model

When Dirac fermions couple to a soliton, there may appear localized fermion zero modes in a soliton. Let us consider Dirac fermions in $(1 + 2)$ dimensions where Dirac fermions interact with a scalar field. The Lagrangian is given by [6]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - qA_\mu)\psi - \frac{1}{2}ig\phi\bar{\psi}\psi^c + \frac{1}{2}ig^*\phi^*\bar{\psi}^c\psi, \quad (1)$$

where ψ is a two-component spinor and q is the coupling to the gauge field. We use the notation $\bar{\psi} = \psi^\dagger\gamma^0$. Usually we choose $q = e$ or $q = 2e$, where e is the electron charge. We will choose $q = 2e$ so that the index of the Dirac operator becomes an integer, as the Dirac index is the difference of the dimensions of vector spaces. This will be described in Section 3. This is related to the property that the magnetic flux is quantized as an integer times the quantum unit $\phi_0 = h/2|e| = \pi\hbar/|e|$. A_μ is the abelian gauge field and $F_{\mu\nu}$ is the field strength given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. ψ^c is the charge conjugate spinor given as $\psi^c = C\bar{\psi}^T$ where C is the charge conjugation matrix and T indicates the transposition. g is the coupling constant. Dirac matrices are chosen as

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = -i\sigma_1, \quad (2)$$

and

$$C = i\gamma^0\gamma^2 = i\sigma_2. \quad (3)$$

We use the Minkowski metric $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1)$. For the representation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (4)$$

the interaction term is written as

$$\mathcal{L}_{int} \equiv -\frac{i}{2}g\phi\bar{\psi}\psi^c + \frac{i}{2}g\phi^*\bar{\psi}^c\psi = ig\phi\psi_1^*\psi_2^* - ig\phi^*\psi_2\psi_1. \quad (5)$$

\mathcal{L}_{int} indicates the pairing interaction between ψ_1 and ψ_2 . Thus, \mathcal{L} in Equation (1) represents a superconductor model in $(1 + 2)$ dimensions.

We assume that $A^0 = 0$ and

$$A^i(x, y) = \epsilon^{ij} \hat{\mathbf{r}}_j \frac{1}{2e} a(r), \quad (6)$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ with $\mathbf{r} = (x, y)$ and $r = |\mathbf{r}|$. $a(r)$ is a function of the radial variable r . The scalar field ϕ corresponds to the gap function and we assume the form with the vorticity Q :

$$\phi(\mathbf{r}) = e^{iQ\theta} f(r), \quad (7)$$

where θ is the angle variable $\theta = \tan^{-1}(y/x)$ and $f(r)$ is a function of r . In the conventional case Q takes an integer value. In this paper we also consider the case where Q could take a non-integer value [22]. We assume the asymptotic behaviors for $f(r)$ and $a(r)$ as follows,

$$f(r) \rightarrow f_\infty \quad (r \rightarrow \infty) \quad (8)$$

$$\rightarrow f_0 r^{|Q|} \quad (r \rightarrow 0) \quad (9)$$

$$a(r) \rightarrow -Q/r \quad (r \rightarrow \infty) \quad (10)$$

$$\rightarrow 0 \quad (r \rightarrow 0). \quad (11)$$

Here, f_∞ and f_0 are constants. We assume that $gf(r) \geq 0$. Then the magnetic flux is given by

$$\Phi = - \int d^2x F_{12} = \int dx dy F_{xy} = \frac{\pi}{e} Q, \quad (12)$$

for $F_{xy} = \partial_x A_y - \partial_y A_x$, where we set $A_x = A^1$ and $A_y = A^2$. We use the unit $\hbar = c = 1$ in this paper.

Let us consider fermion zero-energy modes in this system. The equation of motion for ψ is given by

$$i\partial_t \psi = \sigma_j \left(-i\partial_j - qA^j \right) - g\phi\sigma_2\psi^*. \quad (13)$$

The equation for the zero-energy mode is written as

$$\sigma_j (-i\partial_j - A^j) \psi - g\phi\sigma_2\psi^* = 0. \quad (14)$$

We set $D_j = \partial_j - ieA^j$ to obtain

$$D_1 + iD_2 = e^{i\theta} \left(\partial_r + i\frac{1}{r}\partial_\theta - a(r) \right), \quad (15)$$

$$D_1 - iD_2 = e^{-i\theta} \left(\partial_r - i\frac{1}{r}\partial_\theta - a(r) \right), \quad (16)$$

for $x = r \cos \theta$ and $y = r \sin \theta$. A solution ψ is written in the form

$$\psi = \begin{pmatrix} e^B \chi_1 \\ e^{-B} \chi_2 \end{pmatrix}, \quad (17)$$

where

$$B = \int_0^r dr' a(r'). \quad (18)$$

χ_1 and χ_2 should satisfy

$$e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \chi_1 + g f e^{iQ\theta} \chi_1^* = 0, \quad (19)$$

$$e^{-i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \chi_2 - g f e^{iQ\theta} \chi_2^* = 0. \quad (20)$$

When Q is an integer, there are $|Q|$ normalizable solutions [6]. This is easily shown by using the following Fourier decomposition:

$$\chi_1 = e^{i(Q-1)\theta/2} \sum_{\ell} e^{i\ell\theta} \chi_{1\ell}, \quad (21)$$

$$\chi_2 = e^{i(Q+1)\theta/2} \sum_{\ell} e^{i\ell\theta} \chi_{2\ell}. \quad (22)$$

We adopt that $\chi_{1\ell}$ and $\chi_{2\ell}$ are real. Then, we have the equations for $\chi_{1\ell}$ as

$$\left(\partial_r - \frac{(Q-1)/2 + \ell}{r} \right) \chi_{1\ell} + g f \chi_{1,-\ell} = 0, \quad (23)$$

$$\left(\partial_r - \frac{(Q-1)/2 - \ell}{r} \right) \chi_{1,-\ell} + g f \chi_{1\ell} = 0. \quad (24)$$

Similarly, the equations for $\chi_{2\ell}$ are

$$\left(\partial_r + \frac{(Q+1)/2 + \ell}{r} \right) \chi_{2\ell} + g f \chi_{2,-\ell} = 0, \quad (25)$$

$$\left(\partial_r + \frac{(Q+1)/2 - \ell}{r} \right) \chi_{2,-\ell} + g f \chi_{2\ell} = 0. \quad (26)$$

The following conditions should be satisfied so that $\chi_{1\ell}$ and $\chi_{1,-\ell}$ are regular at the origin,

$$-(Q-1)/2 \leq \ell \leq (Q-1)/2. \quad (27)$$

This indicates that $Q \geq 1$ and the allowed values of ℓ are as follows. For $Q = 1$, we have $\ell = 0$. For $Q = 2$, $\ell = \pm 1/2$. For $Q = 3$, ℓ takes $-1, 0, 1$, and so on. This is shown in Table 1. Therefore, there are Q solutions for χ_1 . The condition for χ_2 reads $(Q+1)/2 \leq \ell \leq -(Q+1)/2$. Thus, Q should be negative and ℓ is in the range

$$-(|Q|-1)/2 \leq \ell \leq (|Q|-1)/2, \quad Q \leq -1. \quad (28)$$

Therefore, χ_2 vanishes when ℓ is in the range of Equation (27), and instead χ_1 vanishes when Q and ℓ satisfy Equation (28).

Table 1. Allowed values of ℓ for positive integers Q . ℓ takes half-integers when Q is an even integer. m indicates a power of $\chi_{1\ell}$ for small $r \sim 0$.

Q	ℓ	$m \equiv (Q-1)/2 + \ell$
1	0	0
2	$-1/2, 1/2$	0, 1
3	$-1, 0, 1$	0, 1, 2
4	$-3/2, -1/2, 1/2, 3/2$	0, 1, 2, 3
5	$-2, -1, 0, 1, 2$	0, 1, 2, 3, 4

When ℓ is non-zero, a pair of $\chi_{1\ell}$ and $\chi_{1,-\ell}$ or $\chi_{2\ell}$ and $\chi_{2,-\ell}$ contribute to a gapless mode. When f vanishes, $\chi_{1\ell}$ is given by $\chi_{1\ell} \simeq r^{(Q-1)/2+\ell}$. $\chi_{1\ell}$ satisfies the second-order differential equation

$$\partial_r^2 \chi_{1\ell} + \frac{1-Q}{r} \partial_r \chi_{1\ell} \left(\frac{Q^2}{4} - \left(\ell - \frac{1}{2} \right)^2 \right) \frac{1}{r^2} \chi_{1\ell} + g f'(r) \chi_{1,-\ell} - (g f)^2 \chi_{1\ell} = 0. \quad (29)$$

This is the second-order differential equation with a regular singular point [23,24] if $f(r)$ is a regular function. When r is large, we neglect $1/r$ term in the equation to have

$$\chi_{1\ell} \simeq \chi_{1,-\ell} \simeq \exp \left(- \int_0^r g f(r') dr' \right). \quad (30)$$

For small r , since $f(r) \rightarrow 0$ as $r \rightarrow 0$, the behavior of solutions is determined by the indicial equation given by

$$k^2 - Qk + \left(\frac{Q}{2} \right)^2 - \left(\ell - \frac{1}{2} \right)^2 = 0. \quad (31)$$

There are two solutions for this equation:

$$k_1 = \frac{Q-1}{2} + \ell, \quad k_2 = \frac{Q+1}{2} - \ell. \quad (32)$$

As $k_1 - k_2 \geq 0$ if and only if $\ell \geq 1/2$, $\chi_{1\ell}$ exhibits the power behavior

$$\chi_{1\ell} \simeq r^{\frac{Q-1}{2}+\ell} \varphi(r), \quad (33)$$

for $\ell \geq 1/2$, where $\varphi(r)$ is a non-singular function. The power $(Q-1)/2 + \ell$ coincides with that derived from Equation (23) in the limit $f(r) \rightarrow 0$.

Let us examine the relation between the spinor ψ and the Majorana spinor. The zero-energy mode with $\ell = 0$ for a positive odd integer Q is given by

$$\chi_{1\ell=0} = r^{(Q-1)/2} \exp \left(- \int_0^r g f(\rho) d\rho \right), \quad (34)$$

and $\chi_2 = 0$. From this solution the Majorana fermion is formulated as

$$\psi_M = \psi + \psi^c = \begin{pmatrix} \xi \\ -\xi^* \end{pmatrix}, \quad (35)$$

where

$$\xi = e^B \chi_{1\ell=0}. \quad (36)$$

Thus, the fermion zero-energy mode can be regarded as the Majorana mode. The same argument applies for the zero-energy modes with $\ell \neq 0$. The Majorana spinor is also made from ψ for $\ell \neq 0$ since $\chi_{2\ell}$ vanishes for $Q > 0$. Thus there can be $|Q|$ Majorana modes in general.

When Q is a half-integer, $m \equiv (Q-1)/2 + \ell$ must be also a half-integer so that the wave function is a single-valued or two-valued function. For $Q = 1/2$, no value of ℓ is allowed. Thus there is no normalizable and two-valued solution of the zero-energy modes for $Q = 1/2$. For $Q = 3/2$, we have $\ell = \pm 1/4$ or $m = 0, 1/2$. It appears that there are $[Q]$ single-valued solutions for positive Q where $[Q]$ indicates the integer part of Q (Gauss symbol). For half-integer Q , we must have

$$[Q] > m > -\frac{1}{2}, \quad 2m \in \mathbf{Z}, \quad 2Q \in \mathbf{Z}. \quad (37)$$

For negative vorticity $Q < 0$, we replace Q by $|Q|$. In fact, in the case of half-flux vortex with $Q = 1/2$ we have a solution

$$\chi_1 = h(r)e^{-i\theta/4}, \quad (38)$$

and $\chi_2 = 0$. For this ansatz we obtain

$$h(r) = r^{-\frac{1}{4}} \exp \left(- \int_0^r dr' g f(r') \right). \quad (39)$$

This solution has a singularity at $r \sim 0$ but can be normalized. This solution, however, is not accepted because χ_1 is not a single-valued function. In the system with a half-flux quantum vortex, a wave function should be a single-valued or two-valued function [25]. We show allowed values of ℓ and m in Table 2. There are $2[Q] = 2Q - 1$ solutions for $Q > 0$ when including two-valued solutions.

Table 2. Allowed values of ℓ for positive odd half-integers Q . 2ℓ takes half-integers in this case.

Q	ℓ	$m \equiv (Q-1)/2 + \ell$
$\frac{1}{2}$	No solutions	No solutions
$\frac{3}{2}$	$-\frac{1}{4}, \frac{1}{4}$	$0, \frac{1}{2}$
$\frac{5}{2}$	$-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$	$0, \frac{1}{2}, 1, \frac{3}{2}$
$\frac{7}{2}$	$-\frac{5}{4}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$	$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$

2.2. Effect of the Chemical Potential

We have discussed a Dirac semi-metal with vanishing chemical potential $\mu = 0$ so far. In this subsection we examine a Dirac metal by introducing the chemical potential. The wave function is a sum of the positive and negative frequency parts:

$$\psi = e^{-iEt/\hbar} \psi_+(\mathbf{r}) + e^{iEt/\hbar} \psi_-(\mathbf{r}). \quad (40)$$

The eigen-equation reads

$$[\sigma_j(p_j - qA^j) - \mu] \psi_+ - g\phi\sigma_2\psi_-^* = E\psi_+, \quad (41)$$

$$[\sigma_j(p_j - qA^j) - \mu] \psi_- - g\phi\sigma_2\psi_+^* = -E\psi_-. \quad (42)$$

We put

$$\psi_+ = \begin{pmatrix} e^B \psi_1 \\ e^{-B} i \psi_2 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} e^B \chi_1 \\ e^{-B} i \chi_2 \end{pmatrix}. \quad (43)$$

We neglect the magnetic field by assuming that the Ginzburg-Landau parameter κ is large, the equations for ψ_+ and ψ_- are represented as

$$e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \psi_1 + g\phi\chi_1^* = -(E + \mu)\psi_2, \quad (44)$$

$$e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_\theta \right) \psi_2 + g\phi\chi_2^* = (E + \mu)\psi_1, \quad (45)$$

$$e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \chi_1 + g\phi\psi_1^* = (E - \mu)\chi_2, \quad (46)$$

$$e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_\theta \right) \chi_2 + g\phi\psi_2^* = -(E - \mu)\chi_1. \quad (47)$$

This set of equations is formally equivalent to the Bogoliubov-de Gennes equation used for superconducting graphene with two valleys [26–28].

We examine the zero-eigenvalue solution. For $E = 0$, we have a solution with $\chi_1 = \psi_1$ and $\chi_2 = \psi_2$. Then, the equations read

$$e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \psi_1 + g\phi\psi_1^* = -\mu\psi_2, \quad (48)$$

$$e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_\theta \right) \psi_2 + g\phi\psi_2^* = \mu\psi_1. \quad (49)$$

We use the representation

$$\psi_1 = e^{i(Q-1)\theta/2} \sum_{\ell} e^{i\ell\theta} \psi_{1\ell}, \quad (50)$$

$$\psi_2 = e^{i(Q+1)\theta/2} \sum_{\ell} e^{i\ell\theta} \psi_{2\ell}. \quad (51)$$

The equations are given as

$$\left(\partial_r - \frac{(Q-1)/2 + \ell}{r} \right) \psi_{1\ell} + gf\psi_{1,-\ell} = -\mu\psi_{2\ell}, \quad (52)$$

$$\left(\partial_r + \frac{(Q+1)/2 + \ell}{r} \right) \psi_{2\ell} + gf\psi_{2,-\ell} = \mu\psi_{1\ell}. \quad (53)$$

In the limit $f \rightarrow 0$, $\psi_{1\ell}$ and $\psi_{2\ell}$ are given by Bessel functions:

$$\psi_{1\ell}(r)|_{f \rightarrow 0} = J_{\frac{Q-1}{2} + \ell}(|\mu|r), \quad (54)$$

$$\psi_{2\ell}(r)|_{f \rightarrow 0} = J_{\frac{Q+1}{2} + \ell}(|\mu|r). \quad (55)$$

For $\ell = 0$, the solution in the presence of gf is easily obtained as

$$\psi_{1\ell=0}(r) = \exp \left(- \int_0^r gf(r') dr' \right) J_{\frac{Q-1}{2}}(|\mu|r), \quad (56)$$

$$\psi_{2\ell=0}(r) = \exp \left(- \int_0^r gf(r') dr' \right) J_{\frac{Q+1}{2}}(|\mu|r). \quad (57)$$

In the limit $r \rightarrow 0$, since $f(r) \rightarrow 0$, $\psi_{1\ell}$ and $\psi_{2\ell}$ approach Bessel functions shown above. For large r , $r \rightarrow \infty$, we may neglect $1/r$ terms so that we have

$$\partial_r \psi_{1\ell} + gf\psi_{1,-\ell} \simeq -\mu\psi_{2\ell}, \quad (58)$$

$$\partial_r \psi_{2\ell} + gf\psi_{2,-\ell} \simeq \mu\psi_{1\ell}. \quad (59)$$

As the equations for $\psi_{j\ell}$ ($j = 1, 2$) are independent of ℓ , we assume that $\psi_{j\ell} = \psi_{j,-\ell}$. Then, the asymptotic behaviors for large r are

$$\psi_{1\ell} \simeq \cos(\mu r) \exp \left(- \int_0^r gf(r') dr' \right), \quad (60)$$

$$\psi_{2\ell} \simeq \sin(\mu r) \exp \left(- \int_0^r gf(r') dr' \right), \quad (61)$$

or we have

$$\psi_{1\ell} \simeq \sin(\mu r) \exp \left(- \int_0^r gf(r') dr' \right), \quad (62)$$

$$\psi_{2\ell} \simeq -\cos(\mu r) \exp \left(- \int_0^r gf(r') dr' \right). \quad (63)$$

2.3. Dirac Fermions and Soliton Fields

Let us consider a model of Dirac fermions that couple with scalar fields. If scalar fields have a soliton-like structure, a zero-energy mode would exist. We consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - qA_\mu)\psi - g\bar{\psi}(\sigma_2\phi_1 + \sigma_1\phi_2)\psi, \quad (64)$$

where ϕ_1 and ϕ_2 are real scalar fields. The interaction term is written as

$$\mathcal{L}_{int} = ig\bar{\psi}\sigma_3M\psi, \quad (65)$$

with

$$M = \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}, \quad (66)$$

where $\phi = \phi_1 + i\phi_2$.

The equation for the zero-energy modes is

$$[\sigma_1(-i\partial_1 - eA^1) + \sigma_2(-i\partial_2 - eA^2)]\psi + gM\psi = 0. \quad (67)$$

We set the Fermi velocity $v_F = 1$ for simplicity. In a similar way, the wave function is written in the form

$$\psi = \begin{pmatrix} e^B\chi_1 \\ e^{-B}\chi_2 \end{pmatrix}, \quad (68)$$

where

$$B = \int_0^r a(r')dr'. \quad (69)$$

The equation for (χ_1, χ_2) reads

$$e^{i\theta} \left(\partial_r + \frac{i}{r} \partial_\theta \right) \chi_1 + g\phi^* \chi_1 = 0, \quad (70)$$

$$e^{-i\theta} \left(\partial_r - \frac{i}{r} \partial_\theta \right) \chi_2 + g\phi \chi_2 = 0. \quad (71)$$

The gap function is parametrized as

$$\phi(\mathbf{r}) = e^{-in\theta} |\phi(r)| \equiv e^{-in\theta} f(r). \quad (72)$$

We assume that $gf(r) > 0$.

$$\chi_1(\mathbf{r}) = \sum_{\ell \in \mathbf{Z}} e^{i\ell\theta} u_\ell(r), \quad (73)$$

$$\chi_2(\mathbf{r}) = \sum_{\ell \in \mathbf{Z}} e^{i\ell\theta} w_\ell(r), \quad (74)$$

where ℓ takes all the integer values. We set $w_\ell = iv_\ell$, and then the equations for fermion zero-energy modes with $E = 0$ read

$$\left(\partial_r - \frac{\ell}{r} \right) u_\ell(r) + gf(r)u_{\ell-n+1}(r) = 0, \quad (75)$$

$$\left(\partial_r + \frac{\ell}{r} \right) v_\ell(r) + gf(r)v_{\ell+n-1}(r) = 0. \quad (76)$$

For the vorticity $n = 1$, we have

$$\left(\partial_r - \frac{\ell}{r}\right) u_\ell(r) + g f(r) u_\ell(r) = 0, \quad (77)$$

$$\left(\partial_r + \frac{\ell}{r}\right) v_\ell(r) + g f(r) v_\ell(r) = 0. \quad (78)$$

The solutions are written as

$$u_\ell = a_\ell r^\ell \exp\left(-\int_0^r g f(r') dr'\right), \quad (79)$$

$$v_\ell = b_\ell r^{-\ell} \exp\left(-\int_0^r g f(r') dr'\right), \quad (80)$$

where a_ℓ and b_ℓ are normalization constants. For $\ell = 0$

$$u_0(r) = v_0(r) = \exp\left(-\int_0^r g f(r') dr'\right). \quad (81)$$

We must have $u_{-\ell} = v_\ell$ when ℓ is replaced by $-\ell$. The normalizable wave function that is regular at the origin is written as

$$\chi_1 = \sum_{\ell \geq 0} e^{i\ell\theta} a_\ell r^\ell \exp\left(-\int_0^r g f(r') dr'\right), \quad (82)$$

$$\chi_2 = i \sum_{\ell \leq 0} e^{i\ell\theta} a_{-\ell} r^{-\ell} \exp\left(-\int_0^r g f(r') dr'\right). \quad (83)$$

This indicates that χ_2/i is the complex conjugate of χ_1 :

$$\chi_2 = i\chi_1^*. \quad (84)$$

When we neglect the magnetic field, ψ is given as

$$\psi = \begin{pmatrix} \chi_1 \\ i\chi_1^* \end{pmatrix}. \quad (85)$$

By multiplying ψ by a phase factor $e^{i\pi/4}$, ψ is written in the form

$$\psi = \begin{pmatrix} e^{i\pi/4} \chi_1 \\ -e^{-i\pi/4} \chi_1^* \end{pmatrix} \equiv \begin{pmatrix} \xi \\ -\xi^* \end{pmatrix}, \quad (86)$$

where we set $\xi = e^{i\pi/4} \chi_1$. Therefore we have obtained the Majorana spinor satisfying

$$\psi = \psi^c. \quad (87)$$

We reached the conclusion that the fermion zero-energy mode is represented by the Majorana spinor.

3. Dirac Operator and Fractional Fermion Number

3.1. Index of the Dirac Operator

Let us consider the Dirac Hamiltonian given as

$$H = \sigma_j(-i\partial_j - qA^j) + gM + \sigma_3 m, \quad (88)$$

where M is the matrix of the gap function in Equation (67) and the mass m is a constant. H is written as

$$H = \begin{pmatrix} m & D + g\Delta \\ D^\dagger + g\Delta^* & -m \end{pmatrix}, \quad (89)$$

where

$$D = -i\frac{\partial}{\partial x} - qA_x - \frac{\partial}{\partial y} + iqA_y. \quad (90)$$

We put

$$\mathcal{D}_\Delta = \begin{pmatrix} 0 & D + g\Delta \\ D^\dagger + g\Delta^* & 0 \end{pmatrix}. \quad (91)$$

As \mathcal{D}_Δ anticommutes with σ_3 , we can define the index by

$$\text{Ind}(\mathcal{D}_\Delta) := \text{Tr}_{\mathcal{D}_\Delta \psi = 0} \sigma_3. \quad (92)$$

Here, the trace Tr is evaluated in the space $\text{Ker} \mathcal{D}_\Delta = \{\psi | \mathcal{D}_\Delta \psi = 0\}$. This definition means

$$\text{Ind}(\mathcal{D}_\Delta) = \dim \text{Ker} D_\Delta^\dagger - \dim \text{Ker} D_\Delta, \quad (93)$$

where $D_\Delta = D + g\Delta$ and $D_\Delta^\dagger = D^\dagger + g\Delta^*$. The index is represented as by introducing the cutoff:

$$\text{Ind}(\mathcal{D}_\Delta) = \lim_{\Lambda \rightarrow \infty} \text{Tr} \sigma_3 e^{-\mathcal{D}_\Delta^2 / \Lambda^2}. \quad (94)$$

Then $\text{Ind}(\mathcal{D}_\Delta)$ is calculated as

$$\begin{aligned} \text{Ind}(\mathcal{D}_\Delta) &= \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \text{tr} \langle k | \sigma_3 e^{-\mathcal{D}_\Delta^2 / \Lambda^2} | k \rangle \\ &= \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \int d^d x \text{tr} e^{-ik \cdot x} e^{-\mathcal{D}_\Delta^2 / \Lambda^2} e^{ik \cdot x}, \end{aligned} \quad (95)$$

where the tr indicates the trace operation with respect to 2×2 matrices. We use the formula

$$e^{-ik \cdot x} f(\partial_\mu) e^{ik \cdot x} \psi = f(\partial_\mu + ik_\mu) \psi, \quad (96)$$

for a function f , so that we have

$$\text{Ind}(\mathcal{D}_\Delta) = \lim_{\Lambda \rightarrow \infty} \int \frac{d^d k}{(2\pi)^d} \int d^d x \text{tr} \left(\sigma_3 e^{-\mathcal{D}_\Delta^2 / \Lambda^2} \Big|_{\partial_\mu \rightarrow \partial_\mu + ik_\mu} \right). \quad (97)$$

\mathcal{D}_Δ^2 is given as

$$\mathcal{D}_\Delta^2 = \begin{pmatrix} (D + g\Delta)(D^\dagger + g\Delta^*) & 0 \\ 0 & (D^\dagger + g\Delta^*)(D + g\Delta) \end{pmatrix}. \quad (98)$$

The matrix elements are evaluated as

$$\begin{aligned} (D + g\Delta)(D^\dagger + g\Delta^*) \Big|_{\partial_\mu \rightarrow \partial_\mu + ik_\mu} &= k_x'^2 + k_x' g\Delta_1 + g\Delta_1 k_x' + \Delta_1^2 + k_y'^2 - k_y' g\Delta_2 - g\Delta_2 k_y' + \Delta_2^2 \\ &\quad - q[\partial_x, A_y] - q[A_x, \partial_y] - i[k_x', \Delta_2] - i[k_y', \Delta_1] \\ &= (k_x' + \Delta_1)^2 + (k_y' - \Delta_2)^2 - q(\partial_x A_y - \partial_y A_x) - (\partial_x \Delta_2) - (\partial_y \Delta_1), \end{aligned} \quad (99)$$

where we set $k'_j = k_j - i\partial_j - qA_j$ for $j = x$ and y . In two-space dimensions $d = 2$, this results in

$$\begin{aligned}\text{Ind}(\not{D}_\Delta) &= \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \int d^2x \Lambda^2 \text{tr} \sigma_3 e^{F/\Lambda^2} \\ &= \frac{1}{2\pi} \int d^2x (qF_{xy} + \partial_x \Delta_2 + \partial_y \Delta_1) \\ &= \text{Ind}(\not{D}) + \text{Ind}(\Delta) \\ &= \frac{e}{\pi} \Phi + \text{Ind}(\Delta),\end{aligned}\quad (100)$$

where $F_{xy} = \partial_x A_y - \partial_y A_x$ and

$$F = \begin{pmatrix} qF_{xy} + \partial_x \Delta_2 + \partial_y \Delta_1 & 0 \\ 0 & -qF_{xy} - \partial_x \Delta_2 - \partial_y \Delta_1 \end{pmatrix}. \quad (101)$$

We defined

$$\text{Ind}(\Delta) = \frac{1}{2\pi} \int d^2x (\partial_x \Delta_2 + \partial_y \Delta_1) = -\frac{1}{2\pi} \int d^2x \text{rot} \vec{\Delta}, \quad (102)$$

for $\vec{\Delta} = (\Delta_1, -\Delta_2)$. This formula indicates that the Dirac index becomes non-zero if a scalar field is singular even when no magnetic field is applied. When $\Delta = \Delta_1 + i\Delta_2$ is not singular in two-space dimensions, the integral concerning the gap functions vanishes. In this case we have $\text{Ind}(\not{D}_\Delta) = \text{Ind}(\not{D})$:

$$\text{Ind}(\not{D}_\Delta) = \text{Ind}(\not{D}) = \frac{q}{2\pi} = \frac{e}{\pi} \Phi. \quad (103)$$

When the vorticity is $n = 1$, Φ is given by the unit flux $\Phi = \pi/e = -\phi_0$ where $\phi_0 = \pi/|e|$ ($\hbar = 1$). This leads to

$$\text{Ind}(\not{D}_\Delta) = 1. \quad (104)$$

Then, we have

$$\dim \text{Ker} D_\Delta^\dagger - \dim \text{Ker} D_\Delta = 1. \quad (105)$$

In fact, for positive angular momentum ℓ , we have a zero-energy normalizable solution ψ satisfying $D_\Delta^\dagger \psi = 0$ for the Hamiltonian H with $m = 0$, and a solution for $D_\Delta \psi = 0$ is not normalizable due to a singularity at the origin. As the solution of $D_\Delta^\dagger \psi = 0$ is also an eigenstate of σ_3 , this zero-mode can be regarded as a Majorana fermion.

3.2. Fractional Fermion Number

Let us consider the fermion number defined by

$$N = \int d^2x : \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) := \frac{1}{2} \int d^2x [\psi^\dagger(\mathbf{r}), \psi(\mathbf{r})], \quad (106)$$

for $\mathbf{r} = (x, y)$ where $:\cdots:$ indicates the normal ordering. N is related to the eta invariant defined as

$$\eta_H(s) = \sum_\lambda \text{sign}(\lambda) |\lambda|^{-s}, \quad (107)$$

where λ 's are eigenvalues of H . The fermion number N is given as [29]

$$N = -\frac{1}{2} \eta_H(0). \quad (108)$$

There is the relation between $\eta_{\not{D}}$ and $\text{Ind}(\not{D})$ [29,30]:

$$\eta_{\not{D}}(0) = -\text{Ind}(\not{D}). \quad (109)$$

This is generalized to

$$\eta_{\mathcal{D}_\Delta}(0) = -\text{Ind}(\mathcal{D}_\Delta). \quad (110)$$

Then the fermion number in the massless limit is

$$N = \frac{q}{4\pi} \Phi = \frac{e}{2\pi} \Phi. \quad (111)$$

When the flux Φ is $-n$ times the unit flux quantum, we have the fractional fermion number

$$N = \frac{1}{2}n. \quad (112)$$

When m is finite, N is given by

$$N = \frac{q}{4\pi} \frac{m}{|m|} \Phi = -\frac{q}{8\pi} \frac{m}{|m|} \int d^2x \epsilon^{ij} F_{ij}. \quad (113)$$

N is written as

$$N = \int d^2x j^0. \quad (114)$$

by introducing the fermion current j^μ . This suggests that the additional effective action is formulated as

$$\Delta S = \text{sign}(m) \frac{q^2}{16\pi} \epsilon^{\mu\nu\sigma} \int d^3x F_{\mu\nu} A_\sigma, \quad (115)$$

because of $\delta S / \delta A_\mu = -q \langle \bar{\psi} \gamma^\mu \psi \rangle = -q \langle j^\mu \rangle$ for the action S . Therefore the Chern–Simons term is induced in a Dirac–vortex system. This may be realized on the surface of a junction of a superconductor and a topological insulator.

3.3. Fractional Vortex and Dirac Index

Let us turn to the case of fractional-flux quantum vortex, that is, the fractional vorticity Q , especially the case of half-flux quantum vortex. The index $\text{Ind}(\mathcal{D})$ equals Q for $\Phi = -Q\phi_0$:

$$\text{Ind}(\mathcal{D}) = Q. \quad (116)$$

$\text{Ind}(\mathcal{D})$ should be an integer since the index is only the difference of dimensions of vector spaces. $\text{Ind}(\mathcal{D}_\Delta)$ has a contribution from the gap function because the phase of Δ has a singularity on the kink [31]. The half-flux quantum vortex exists associated with the kink in the phase space, where the kink is a one-dimensional object. We here give a comment on the kink in a multiband superconductor. The kink state may be unstable because of the energy cost when the field changes rapidly. In other words, the superconducting current flows between the layer, which may cause a force to the magnetic flux vortex. This may bring about a new effect on the zero modes in the vortex. We, however, neglect this effect in this paper.

We adopt that the gap function is given as

$$\Delta(\mathbf{r}) = \Delta_0(r) e^{-i\phi(\theta)}, \quad (117)$$

where $\phi(\theta)$ has a step-function-like singularity,

$$\phi(\theta) = \frac{1}{2}\theta + \pi H(\theta), \quad (118)$$

near the origin $-\pi < \theta < \pi$. $H(\theta)$ indicates the Heaviside step function. We assume $\Delta_0(r) = \Delta_\infty \tanh(r/\xi)$. Then, we calculate

$$\begin{aligned} \int dx dy (\partial_x \Delta_2 + \partial_y \Delta_1) &= - \int dx dy \Delta_0(r) (\cos \phi \cdot \phi'(\theta) \partial_x \theta + \sin \phi \cdot \phi'(\theta) \partial_y \theta) \\ &= - \int_0^R dr \int_{-\pi}^{\pi} d\theta \Delta_0(r) \phi'(\theta) (-\cos \phi \sin \theta + \sin \phi \cos \theta) \\ &= -\pi \Delta_\infty \sin \phi(0) \xi \ln \cosh(R/\xi). \end{aligned} \quad (119)$$

We take the cutoff R so that $\ln \cosh(R/\xi) \simeq 1$ and $\phi(0) = \pi/2$.

As $\xi \simeq 1/\Delta_\infty (= \hbar v_F / \Delta_\infty)$, we have

$$\int dx dy (\partial_x \Delta_2 + \partial_y \Delta_1) = -\pi. \quad (120)$$

This indicates

$$\text{Ind}(\not{D}_\Delta) = \text{Ind}(\not{D}) + \text{Ind}(\Delta) = Q - \frac{1}{2}. \quad (121)$$

Thus, $\text{Ind}(\not{D}_\Delta)$ becomes an integer with the contribution from the kink for the half-flux vortex.

3.4. Fermion Number and Kinks

The existence of a fermion zero-energy mode is related to the fractional fermion number. Let us examine the (1 + 2)-dimensional model of Dirac fermions that couples to a scalar field with kink structure. The Lagrangian is given as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \gamma^\mu (i\partial_\mu - qA_\mu) \psi - \bar{\psi} (m + \gamma^1 \phi_1) \psi, \quad (122)$$

where ϕ_1 is a real scalar field. We assume that ϕ_1 represents a kink solution with the asymptotic behavior,

$$\phi_1(x) \rightarrow v \text{ as } x \rightarrow \infty, \quad (123)$$

$$\phi_1(x) \rightarrow -v \text{ as } x \rightarrow -\infty. \quad (124)$$

The kink is a one-dimensional object depending on one variable and is situated outside the region where the vortex exists. Then the fermion number is a sum of two contributions from vortex and kink:

$$N = \text{Ind}(\not{D}) + N_{\text{kink}}. \quad (125)$$

N_{kink} is given by the Goldstone–Wilczek formula:

$$N_{\text{kink}} = -\frac{1}{2\pi} \left(\tan^{-1} \left(\frac{\phi_1(\infty)}{m} \right) - \tan^{-1} \left(\frac{\phi_1(-\infty)}{m} \right) \right). \quad (126)$$

Then, in the limit $m \rightarrow 0$ for $v > 0$, we have

$$N = \text{sign}(m) \left[\frac{q}{4\pi} \Phi - \frac{1}{2} \right] = -\text{sign} \left(\frac{1}{2} Q + \frac{1}{2} \right), \quad (127)$$

where the flux is given as $\Phi = -Q\phi_0$. For $v < 0$,

$$N = -\text{sign} \left(\frac{1}{2} Q - \frac{1}{2} \right), \quad (128)$$

4. Summary

We have investigated fermion zero-energy modes and the index of the Dirac operator in vortex–Dirac fermion systems in $(1 + 2)$ dimensions. Dirac fermions play an important role in many electron systems such as topological insulators, topological superconductors, graphene [32–34], and also Kondo systems [35–37]. A vortex–Dirac fermion system may be realized on the surface of a topological insulator in a junction of superconductors and topological insulators. We have shown that a fermion zero-energy mode exists in a vortex–fermion system and in a soliton–fermion system. The zero-energy modes are described by Majorana fermions in a Dirac semi-metal ($\mu = 0$). The quasi-particle energy level $\epsilon_n = (n + 1/2)\hbar\omega_0$ in the vortex core of conventional superconductors shifts to $\epsilon_n = n\hbar\omega_0$ in Dirac superconductors. We have also shown that there is no fermion zero mode in a vortex with fractional vorticity less than unity, as wave function has a singularity at the origin or becomes a multi-valued function. There is a contribution to the index of a Dirac operator when the scalar field has a soliton-like structure with singularity. Last, we give a comment that we neglected the non-equilibrium dynamics that are caused by the superconducting current flow between the layer brought about by the kink in a superconducting bilayer.

Funding: This work was supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan (Grant No. 17K05559).

Acknowledgments: A part of the computations was supported by the Supercomputer Center of the Institute for Solid State Physics, the University of Tokyo.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

2D	two-dimensional
TI	topological insulator

References

1. Qi, X.L.; Hughes, T.L.; Zhang, S.C. Topological field theory of time-reversal invariant insulators. *Phys. Rev. B* **2008**, *78*, 195424. [\[CrossRef\]](#)
2. Atiyah, M.; Patodi, V.; Singer, I. Spectral asymmetry and Riemannian geometry. *Bull. Lond. Philos. Soc.* **1973**, *5*, 229. [\[CrossRef\]](#)
3. Atiyah, M.; Patodi, V.; Singer, I. Spectral asymmetry and Riemannian geometry I. *Proc. Camb. Philos. Soc.* **1975**, *77*, 42. [\[CrossRef\]](#)
4. Atiyah, M.; Patodi, V.; Singer, I. Spectral asymmetry and Riemannian geometry II. *Proc. Camb. Philos. Soc.* **1975**, *78*, 405. [\[CrossRef\]](#)
5. Atiyah, M.; Patodi, V.; Singer, I. Spectral asymmetry and Riemannian geometry III. *Proc. Camb. Philos. Soc.* **1976**, *79*, 71. [\[CrossRef\]](#)
6. Jackiw, R.; Rossi, P. Zero modes of the vortex-fermion system. *Nucl. Phys. B* **1981**, *190*, 681. [\[CrossRef\]](#)
7. Callan, C.G.; Harvey, J.A. Anomalies and fermion zero modes on strings and domain walls. *Nucl. Phys. B* **1985**, *250*, 427. [\[CrossRef\]](#)
8. Weinberg, E.J. *Classical Solutions in Quantum Field Theory*; Cambridge University Press: Cambridge, UK, 2012.
9. Manton, N.S.; Sutcliffe, P. *Topological Solitons*; Cambridge University Press: Cambridge, UK, 2004.
10. Rajaraman, R. *Solitons and Instantons*; North-Holland: Amsterdam, The Netherlands, 1982.
11. Yanagisawa, T.; Hase, I.; Tanaka, Y. Massless and quantized modes of kinks in the phase space of superconducting gaps. *Phys. Lett. A* **2018**, *382*, 3483. [\[CrossRef\]](#)
12. Jackiw, R.; Rebbi, C. Solitons with fermion number $1/2$. *Phys. Rev. D* **1976**, *13*, 3398. [\[CrossRef\]](#)
13. Su, W.P.; Schrieffer, R.; Heeger, A.J. Solitons in polyacetylene. *Phys. Rev. Lett.* **1979**, *42*, 1698. [\[CrossRef\]](#)
14. Su, W.P.; Schrieffer, R.; Heeger, A.J. Soliton excitations in polyacetylene. *Phys. Rev. B* **1980**, *22*, 2099. [\[CrossRef\]](#)

15. Goldstone, J.; Wilczek, F. Fractional quantum numbers on solitons. *Phys. Rev. Lett.* **1981**, *47*, 986. [[CrossRef](#)]
16. Hosur, P.; Ghaemi, P.; Mong, R.S.K.; Vishwanath, A. Majorana modes at the ends of superconductor vortices in doped topological insulators. *Phys. Rev. Lett.* **2011**, *107*, 097001. [[CrossRef](#)] [[PubMed](#)]
17. Jiang, K.; Dai, X.; Wang, Z. Quantum anomalous vortex and Majorana zero modes in iron-based superconductor Fe(Te, Se). *Phys. Rev. X* **2019**, *9*, 011033. [[CrossRef](#)]
18. Tanaka, Y.; Yamamori, H.; Yanagisawa, T.; Nishio, T.; Arisawa, S. Experimental formation of a fractional vortex in a superconducting bi-layer. *Physica C* **2018**, *548*, 44. [[CrossRef](#)]
19. Kong, L.; Zhu, S.; Papaj, M.; Chen, H.; Cao, L.; Isobe, H.; Xing, Y.; Liu, W.; Wang, D.; Fan, P.; et al. Half-integer level shift of vortex bound states in an iron-based superconductor. *Nat. Phys.* **2019**, *15*, 1181. [[CrossRef](#)]
20. Iyo, A.; Kawashima, K.; Kinjo, T.; Nishio, T.; Ishida, S.; Fujihisa, H.; Gotoh, Y.; Kihou, K.; Eisaki, H.; Yoshida, Y. New-structure-type Fe-based superconductors: CaFe_4As_4 (A = K, Rb, Cs) and SrFe_4As_4 (A = Rb, Cs). *J. Am. Chem. Soc.* **2016**, *138*, 3410. [[CrossRef](#)]
21. Liu, W.; Cao, L.; Zhu, S.; Kong, L.; Wang, G.; Papaj, M.; Zhang, P.; Liu, Y.; Chen, H.; Li, G.; et al. A new Majorana platform in an Fe-As superconductor. *arXiv* **2019**, arXiv:1907.00904.
22. Yanagisawa, T. Fermion zero-energy modes and fractional fermion numbers in a fractional vortex-fermion model. In Proceedings of the 32nd International Symposium on Superconductivity, Kyoto, Japan, 3–5 December 2019. *J. Phys. Conf. Ser.* **2020**.
23. Pontryagin, L.S. *Ordinary Differential Equations*; Pergamon: New York, NY, USA, 1962.
24. Coddington, E.A.; Levinson, N. *Theory of Ordinary Differential Equations*; McGraw-Hill Education: New York, NY, USA, 1984.
25. Yanagisawa, T.; Higashi, Y.; Hase, I. Fractional skyrmion and absence of low-lying Andreev bound states in a micro fractional-flux quantum vortex. *J. Phys. Soc. Jpn.* **2019**, *88*, 104704. [[CrossRef](#)]
26. Beenakker, C.W. Specular Andreev reflection in graphene. *Phys. Rev. Lett.* **2006**, *97*, 067007. [[CrossRef](#)]
27. Jackiw, R.; Pi, S.Y. Persistence of zero modes in a gauged Dirac model for bilayer graphene. *Phys. Rev. B* **2008**, *78*, 132104. [[CrossRef](#)]
28. Khaymovich, I.M.; Kopnin, N.B.; Mel'nikov, A.S.; Shereshevskii, I.A. Vortex core states in superconducting graphene. *Phys. Rev. B* **2009**, *79*, 224506. [[CrossRef](#)]
29. Niemi, A.J. Topological solitons in a hot and dense Fermi gas. *Nucl. Phys. B* **1985**, *253*, 14. [[CrossRef](#)]
30. Niemi, A.J.; Semenoff, G.W. Index theorems on open infinite manifolds. *Nucl. Phys. B* **1986**, *269*, 131. [[CrossRef](#)]
31. Yanagisawa, T.; Tanaka, Y.; Hase, I.; Yamaji, K. Vortices and chirality in multi-band superconductors. *J. Phys. Soc. Jpn.* **2012**, *81*, 024712. [[CrossRef](#)]
32. McClure, J.W. Diamagnetism of graphite. *Phys. Rev.* **1956**, *104*, 666. [[CrossRef](#)]
33. Slonczewski, J.C.; Weiss, P. R. Band structure of graphite. *Phys. Rev.* **1958**, *109*, 272. [[CrossRef](#)]
34. Ando, T. Theory of electronic states and transport in carbon nanotubes. *J. Phys. Soc. Jpn.* **2005**, *74*, 777. [[CrossRef](#)]
35. Yanagisawa, T. Kondo effect in the presence of spin-orbit coupling. *J. Phys. Soc. Jpn.* **2012**, *81*, 094713. [[CrossRef](#)]
36. Yanagisawa, T. Kondo effect in Dirac systems. *J. Phys. Soc. Jpn.* **2015**, *84*, 074705. [[CrossRef](#)]
37. Yanagisawa, T. Dirac fermions and Kondo effect. *J. Phys. Conf. Ser.* **2015**, *603*, 012014. [[CrossRef](#)]

