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Higher Dimensional Static and Spherically Symmetric Solutions in Extended Gauss–Bonnet Gravity

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Abstract: We study a theory of gravity of the form $f(\mathcal{G})$ where $\mathcal{G}$ is the Gauss–Bonnet topological invariant without considering the standard Einstein–Hilbert term as common in the literature, in arbitrary $(d + 1)$ dimensions. The approach is motivated by the fact that, in particular conditions, the Ricci curvature scalar can be easily recovered and then a pure $f(\mathcal{G})$ gravity can be considered a further generalization of General Relativity like $f(R)$ gravity. Searching for Noether symmetries, we specify the functional forms invariant under point transformations in a static and spherically symmetric spacetime and, with the help of these symmetries, we find exact solutions showing that Gauss–Bonnet gravity is significant without assuming the Ricci scalar in the action.

Keywords: alternative theories of gravity; Gauss–Bonnet invariant; spherical symmetry; solar system tests

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1. Introduction

Even though Einstein’s General Relativity (GR) and the related cosmological model, Λ CDM, have been successful according to a wide range of observations (solar system tests, supernova type Ia, large scale structure, cosmic microwave background and more), there are some shortcomings that have to be addressed in view of a final theory of gravity and a self-consistent cosmological model [1–4]. The nature of the “dark sector” of the Universe, i.e., dark matter and dark energy, the huge discrepancy between the theoretical value of the cosmological constant with the observed one, as well as the inability to find a TeV-scale supersymmetry are some of the puzzles of modern physics. These issues, together with the unknown quantum nature of gravitational interaction, singularities, coincidence problem in cosmology and more, initiated the hunt for an alternative description of gravity (see e.g., [5–8], and references therein).

During the last two decades, there have been several approaches aimed at finding a more general description of the gravitational interaction. Generalization of the Einstein–Hilbert action, like $f(R)$ theories [9–12], addition of extra fields, like Brans–Dicke theory [13], Horndeski theory [14–16], Tensor–Vector–Scalar theory, bimetric theories [17], massive gravity [18], non-local theories [19,20], higher-dimensional gravitational theories in the framework of tangent Lorentz bundles [21], as well as reformulations such as the Teleparallel Equivalent of General Relativity (TEGR) [22] and its

modifications [23], are some of the approaches studied in detail, not only at cosmological scales, but also at astrophysical ones.

On the other hand, in the quantum regime, there have been many attempts to find a quantum formulation of gravity leading to higher dimensional theories, like Kaluza–Klein model, DGP model, Einstein–Dilaton–Gauss–Bonnet gravity, as well as generalizations such as the Lovelock gravity [24]. In the low energy effective action of string/M-theory, a specific curvature invariant naturally appears. It is a Gauss–Bonnet (GB) scalar [25] given by

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \quad (1)$$

where R , $R_{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ are the Ricci scalar, the Ricci tensor and the Riemann tensor respectively. This term is a topological invariant in $3 + 1$ dimensions (or less). Practically, this means that a linear term in $\mathcal{G}$, in the Einstein–Hilbert action, would not affect the equations of motion. However, in the literature, an addition of an arbitrary function $f(\mathcal{G})$ has been proposed [26]. Specifically, the theory given by the action $\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + f(\mathcal{G}) \right]$ has been extensively studied. In [27], cosmologically viable models are considered by studying the stability of a late-time de-Sitter solution and the existence of radiation and matter epochs. In [28], possible power-law scaling solutions have been taken into account by developing the scalar-tensor equivalent of the above theory. In particular, in [29], authors study cosmological perturbations and show that density perturbations cause instabilities. In [30], the author shows that the above theory is ruled out as a possible explanation of the late-time acceleration by solar system tests. In [31], the Gauss–Bonnet term is added to a $f(R)$ five-dimensional Lagrangian and a static spherically symmetric solution is studied. In [32] the authors study the energy bounds for Gauss–Bonnet gravity in an AdS_7 background. Finally in [33], a mimetic version of the above theory is considered and they find, besides accelerating behaviors, solutions that unify the inflation era together with dark energy. In addition, dark matter can be described in the framework of this model. In [34], the Newtonian and post-Newtonian limit of these models is studied in detail.

For almost half a century, higher dimensional theories of gravity have been studied in many different contexts in the literature [35]. The aforementioned puzzling phenomena in gravity can sometimes be explained by invoking extra dimensions [36–39]. Braneworlds and other higher dimensional modifications of Einstein’s theory, e.g., Lovelock theory [24] have been considered as possible extensions in the hunt for a self-consistent theory of gravity.

All of the above researches deal with a theory that safely recovers GR in the background or in some limit. This means that if one switches off the effect of the GB contribution, i.e., $f(\mathcal{G}) \rightarrow 0$, then the action reduces to the Einstein–Hilbert and one recovers GR. This happens because GR has to be restored in view of observations and experimental tests. In this paper, we propose a scenario where GR is not in the background “a priori” and gravity is given only by quadratic curvature invariants and specifically by an arbitrary function of the GB term. However, GR can be restored as a particular case of $f(\mathcal{G})$ gravity and the further degrees of freedom related to $R_{\mu\nu}$ and $R^\alpha{}_{\lambda\mu\nu}$ can be neglected with respect to R . This happens if particular symmetries are adopted like in homogeneous and isotropic cosmology or in other specific cases.

Here, we consider a spherically symmetric background and search for Noether Symmetries, in general, $(d + 1)$ dimensions. Specifically, we use the so-called Noether symmetry approach [40–42], which has been extensively used in the literature as a geometric criterion to select forms of the arbitrary functions in several alternative gravity theories that are invariant under point transformations (see for example, [40,43–46]).

Here, we adopt the above approach for $f(\mathcal{G})$ gravity in spherical symmetry for arbitrary $(d + 1)$ dimensions. It is interesting to point out that, the only forms of $f(\mathcal{G})$ selected by symmetries are power-law functions. With these symmetries, it is possible to find exact spherically symmetric solutions, which for specific values of the power-law, provide the same prediction as GR. It means that standard GR can be recovered from $f(\mathcal{G})$ gravity.

This paper is organized as follows: in Section 2, we present $f(\mathcal{G})$ gravity and derive the field equations. Furthermore, we construct the point-like Lagrangian in a spherically symmetric minisuperspace. Section 3 is devoted to the Noether symmetry approach and, in Section 4, we use it to find the forms of $f(\mathcal{G})$ selected by the Noether symmetries. Moreover, in Section 4, we find exact spherically symmetric solutions by making use of the symmetries. Finally, in Section 5, we draw conclusions and discuss future perspectives. Throughout the paper the metric signature is $(+---)$ and physical units $\hbar = c = k_B = 8\pi G = 1$ are adopted.

2. The Gauss–Bonnet Gravity in Spherical Symmetry

A general Gauss–Bonnet gravity theory is given by the action

$$S = \int \sqrt{|g|} f(\mathcal{G}) d^{d+1}x, \quad (2)$$

where $\mathcal{G}$ is the Gauss–Bonnet invariant given by Equation (1). In four dimensions (i.e., $d = 3$), a linear term $\mathcal{G}$ in the action is trivial because, as a topological invariant, it turns into a surface term and the related integral is null. As already mentioned in the Section 1, up to now, people studied $f(\mathcal{G})$ theories in $d = 3$, adding a Ricci scalar in the action (2), in order to recover General Relativity for $f(\mathcal{G}) \rightarrow 0$. In our case, we consider pure $f(\mathcal{G})$ theories and we claim that GR can be recovered without considering the Einstein–Hilbert term a priori in the action.

By varying (2) with respect to the metric, we get the field equations

$$\begin{aligned} \frac{1}{2} g_{\mu\nu} f - \left(2RR_{\mu\nu} - 4R_{\mu\alpha} R^{\alpha}_{\nu} + 2R_{\mu}^{\alpha\beta\gamma\delta} R_{\nu\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\mu\alpha\nu\beta} \right) f_{\mathcal{G}} + \\ + \left[2R\nabla_{\mu}\nabla_{\nu} + 4G_{\mu\nu}\square - 4(R^{\rho}_{\nu}\nabla_{\mu} + R^{\rho}_{\mu}\nabla_{\nu})\nabla_{\rho} + 4g_{\mu\nu}R^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma} - 4R_{\mu\alpha\nu\beta}\nabla^{\alpha}\nabla^{\beta} \right] f_{\mathcal{G}} = 0, \end{aligned} \quad (3)$$

where $f_{\mathcal{G}}$ is the derivative of f with respect to $\mathcal{G}$. It is worth using also the trace of Equation (3), that is

$$\frac{d+1}{2} f - 2\mathcal{G}f_{\mathcal{G}} - 2(d-2) (R\square - 2R^{\mu\nu}\nabla_{\mu}\nabla_{\nu}) f_{\mathcal{G}} = 0. \quad (4)$$

This can be seen as the equation of motion for the new scalar degree of freedom introduced in this theory. It is already known [47] that, the theory (2) in $d = 3$ is a part of the Horndeski action and thus contains an extra scalar degree of freedom.

Spherical Symmetry

Let us consider now a static and spherically symmetric ansatz for the metric, that reads

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 d\Omega_{d-1}^2, \quad (5)$$

where $d\Omega_{d-1}^2 = \sum_{j=1}^{d-1} d\theta_j^2 + \sin^2\theta_j d\phi^2$ is the metric element of the $(d-1)$ -sphere for a spacetime labeled by coordinates $x^{\mu} = (t, r, \theta_1, \theta_2, \dots, \theta_{d-2}, \phi)$. Before proceeding, an important comment is necessary here; we assume that the metric (5) is not dynamical, which means that the Birkhoff's theorem should be valid for these models. This is not proven and we take it for granted in theories such as (2). However, there are a lot of references in the literature claiming to have found cases where a generalization of the Birkhoff's theorem could exist [48–53].

The Gauss–Bonnet term (1) in arbitrary $(d+1)$ dimensions takes the form

$$\mathcal{G} = \frac{(d-1)(d-2)}{r^4} [4r^2 P(P^2 - 1)P'' + 8(d-3)rP(P^2 - 1)P' + 4r^2 (3P^2 - 1)P'^2 + (d-3)(d-4)(P^4 - 2P^2 + 1)], \quad (6)$$

where the prime stands for derivatives with respect to the radial coordinate and we set for simplicity $\theta_j = \pi/2$. Note that for $d \leq 2$ (i.e., in less than four dimensions), the above scalar vanishes identically, while for $d = 3$, it becomes a topological surface term.

In order to calculate the point-like Lagrangian of the theory for (5), we introduce a Lagrange multiplier as [45,46,54]

$$\mathcal{S} = \int d^{d+1}x \, r^{d-1} PQ [f(\mathcal{G}) - \lambda (\mathcal{G} - \tilde{\mathcal{G}})] , \quad (7)$$

with $\tilde{\mathcal{G}}$ being the Gauss–Bonnet term in spherical symmetry (6) and λ the Lagrange multiplier given by varying the action with respect to $\mathcal{G}$, i.e., $\lambda = \partial_{\mathcal{G}} f$. Substituting $\tilde{\mathcal{G}}$ and integrating out the second derivatives, we obtain

$$\begin{aligned} \mathcal{L}(r, P, Q, \mathcal{G}) = & \, r^{d-1} PQ [f - \mathcal{G} f_{\mathcal{G}}] + \\ & + \frac{(d-1)(d-2)r^{d-5}(Q^2-1)}{Q^4} \{ (d-3)P f_{\mathcal{G}} [(d-4)Q(Q^2-1) + 4rQ'] + 4r^2 Q P' \mathcal{G}' f_{\mathcal{G}\mathcal{G}} \} , \end{aligned} \quad (8)$$

where $f_{\mathcal{G}}$ and $f_{\mathcal{G}\mathcal{G}}$ are the first and second derivatives of f with respect to $\mathcal{G}$. This is the point-like and canonical Lagrangian of our theory in a static and spherically symmetric spacetime. Its configuration space is $\mathcal{Q} = \{P, Q, \mathcal{G}\}$, and the tangent space $\mathcal{TQ} = \{P, P', Q, Q', \mathcal{G}, \mathcal{G}'\}$.

3. The Noether Symmetry Approach

Let us briefly introduce some basic notions of the so called Noether symmetry approach [40,41]. Noether symmetries are a subclass of Lie point symmetries applied in dynamical systems described by a Lagrangian density. Noether's theorem states that if

$$X = \xi \partial_t + \eta^i \partial_{q^i} , \quad (9)$$

is a generator of infinitesimal point transformations, then the Lagrangian density is invariant under X if and only if

$$X^{[1]} \mathcal{L} + \dot{\xi} \mathcal{L} = \dot{g} , \quad (10)$$

with g being a function of the affine parameter t and the generalized coordinates q^i and $X^{[1]}$ is the first prolongation of X .

The n -prolongation of the generator has the form

$$X^{[n]} = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + \eta^{i[1]} \frac{\partial}{\partial \dot{q}^i} + \dots + \eta^{i[n]} \frac{\partial}{\partial \frac{d^n q^i}{dt^n}} , \quad (11)$$

with

$$\eta^{i[n]} = \frac{d\eta^{i[n-1]}}{dt} - \xi \frac{d^n q^i}{dt^n} . \quad (12)$$

The parameter t represents any affine parameter and it is chosen depending on the symmetry of the spacetime. Then we have

$$X^{[1]} = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + \eta^{i[1]} \frac{\partial}{\partial \dot{q}^i} = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i \dot{\xi}) \frac{\partial}{\partial \dot{q}^i} . \quad (13)$$

It is easy to extend the above to a general Lagrangian density that depends on x^μ parameters. Specifically, the prolongation (13) becomes

$$X^{[1]} = \xi^\mu \partial_\mu + \eta^i \frac{\partial}{\partial q^i} + (\partial_\mu \eta^i - \partial_\mu q^i \partial_\nu \xi^\nu) \frac{\partial}{\partial (\partial_\mu q^i)} \quad (14)$$

and Noether's theorem (10)

$$X^{[1]} \mathcal{L} + \partial_\mu \xi^\mu \mathcal{L} = \partial_\mu g^\mu . \quad (15)$$

In more details, let us consider the following transformation

$$\mathcal{L}(x^\mu, q^i, \partial_\mu q^i) \rightarrow \mathcal{L}(\tilde{x}^\mu, \tilde{q}^i, \partial_\mu \tilde{q}^i), \quad (16)$$

where transformation of x^μ and q^i are given by:

$$\begin{cases} \tilde{x}^\mu = x^\mu + \epsilon \zeta^\mu(x^\mu, q^i) + O(\epsilon^2), \\ \tilde{q}^i = q^i + \epsilon \eta^i(x^\mu, q^i) + O(\epsilon^2). \end{cases} \quad (17)$$

The derivatives of the generalized coordinates q^i transform as

$$\frac{d\tilde{q}^i}{d\tilde{x}^\mu} = \frac{dq^i + \epsilon d\eta^i}{dx^\mu + \epsilon d\zeta^\mu} = \left(\frac{dq^i}{dx^\mu} + \epsilon \frac{d\eta^i}{dx^\mu} \right) \left(1 + \epsilon \frac{d\zeta^\mu}{dx^\mu} \right)^{-1} \sim \left(\frac{dq^i}{dx^\mu} + \epsilon \frac{d\eta^i}{dx^\mu} \right) \left(1 - \epsilon \frac{d\zeta^\mu}{dx^\mu} \right), \quad (18)$$

which at first order in ϵ has the form

$$\frac{d\tilde{q}^i}{d\tilde{x}^\mu} = \frac{dq^i}{dx^\mu} + \epsilon \left(\frac{d\eta^i}{dx^\mu} - \frac{dq^i}{dx^\mu} \frac{d\zeta^\mu}{dx^\mu} \right) + O(\epsilon^2) = \partial_\mu q^i + \epsilon \left(\partial_\mu \eta^i - \partial_\mu q^i \partial_\nu \zeta^\nu \right) + O(\epsilon^2). \quad (19)$$

From Equations (17) and (19) we can construct the generator of these transformations, that reads

$$X = \zeta^\mu \partial_\mu + \eta^i \frac{\partial}{\partial q^i}. \quad (20)$$

Now, if the transformations (17) and (19) hold, the equations of motion, i.e., the Euler–Lagrange equations, are invariant, and thus there exists a function $g^\mu = g^\mu(x^\mu, q^i)$ such that the following condition holds

$$\frac{d\tilde{x}^\mu}{dx^\mu} \tilde{\mathcal{L}} = \mathcal{L} + \epsilon \partial_\mu g^\mu. \quad (21)$$

The derivative with respect to ϵ will give

$$\mathcal{L} \frac{\partial}{\partial \epsilon} \frac{d\tilde{x}^\mu}{dx^\mu} + \frac{d\tilde{x}^\mu}{dx^\mu} \frac{\partial \tilde{\mathcal{L}}}{\partial \epsilon} = \partial_\mu g^\mu, \quad (22)$$

and the transformations (17) allow us to calculate the various terms. That is,

$$\frac{d\tilde{x}^\mu}{dx^\mu} = \frac{\partial \tilde{x}^\mu}{\partial x^\mu} + \frac{\partial \tilde{x}^\mu}{\partial q^i} \partial_\mu q^i = 1 + \epsilon \frac{\partial \zeta^\mu}{\partial q^i} \partial_\mu q^i, \quad (23)$$

$$\frac{\partial}{\partial \epsilon} \frac{d\tilde{x}^\mu}{dx^\mu} = \frac{d}{dx^\mu} \left(\frac{\partial \tilde{x}^\mu}{\partial \epsilon} \right) = \partial_\mu \zeta^\mu, \quad (24)$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \epsilon} = \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{x}^\mu} \frac{\partial \tilde{x}^\mu}{\partial \epsilon} + \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{q}^i} \frac{\partial \tilde{q}^i}{\partial \epsilon} + \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_\mu \tilde{q}^i)} \frac{\partial (\partial_\mu \tilde{q}^i)}{\partial \epsilon}. \quad (25)$$

With the help of (19), we can replace (23), (24) and (25) into (22) and obtain

$$\left[\zeta^\mu \partial_\mu + \eta^i \frac{\partial}{\partial q^i} + (\partial_\mu \eta^i - \partial_\mu q^i \partial_\nu \zeta^\nu) \frac{\partial}{\partial (\partial_\mu q^i)} + \partial_\mu \zeta^\mu \right] \mathcal{L} = \partial_\mu g^\mu, \quad (26)$$

that is nothing else but (15). It is worth noticing that the associated Noether integral, which is the conserved quantity, is given by

$$j^\mu = - \frac{\partial \mathcal{L}}{\partial (\partial_\mu q^i)} \eta^i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu q^i)} \partial_\nu q^i \zeta^\nu - \mathcal{L} \zeta^\mu + g^\mu. \quad (27)$$

In particular, for spherical symmetry, where the metric only depends on r , Equations (15) and (14) acquire the form:

$$X^{[1]} = \xi(r, q^i) \partial_r + \eta^i(r, q^i) \frac{\partial}{\partial q^i} + [\partial_r \eta^i(r, q^i) - \partial_r q^i \partial_r \xi(r, q^i)] \frac{\partial}{\partial (\partial_r q^i)}, \quad (28)$$

$$X^{[1]} \mathcal{L} + \partial_r \xi(r, q^i) \mathcal{L} = \partial_r g(r, q^i). \quad (29)$$

With this considerations in mind, let us apply the Noether symmetry approach to the point-like Lagrangian (8).

4. Noether Symmetries in Gauss–Bonnet Gravity

The generator of the point transformations (17), in our case, is given by

$$X = \xi(r, \mathcal{G}, P, Q) \partial_r + \eta^{\mathcal{G}}(r, \mathcal{G}, P, Q) \partial_{\mathcal{G}} + \eta^P(r, \mathcal{G}, P, Q) \partial_P + \eta^Q(r, \mathcal{G}, P, Q) \partial_Q, \quad (30)$$

where ξ and η^i , with $i = \{\mathcal{G}, P, Q\}$, are the components of vector X . By applying the Noether theorem, Equation (15), we obtain a system of twelve equations which are not all independent. There are two non-trivial functions $f(\mathcal{G})$ determined by symmetries.

- Case 1: in $d\mathcal{G}eq3$ dimensions, we have $f(\mathcal{G}) = f_0 \mathcal{G}^n$ with $n \neq 1$. The Noether symmetry of this model is given by

$$X = c_1 r \partial_r - 4c_1 \mathcal{G} \partial_{\mathcal{G}} + (4n - d)c_1 P \partial_P \quad (31)$$

and $g = c_2$, with c_1 and c_2 being constants. The invariant quantity (27), related to the above symmetry (31), is

$$I = \frac{1}{Q^3} c_1 f_0 r^{d-4} \mathcal{G}^{n-2} \left[(1-n)r^4 \mathcal{G}^2 P Q^4 - 4n(n-1)(d-2)(d-1)r(Q^2-1)(rP' + (d-4n)P)\mathcal{G}' - \right. \\ \left. - (d-2)(d-1)n\mathcal{G}(Q^2-1)[16(n-1)rP' - (d-4)(d-3)P(Q^2-1)] \right] - c_2. \quad (32)$$

- Case 2: in $d = 4$ dimensions, there is also the possibility of having a linear model of the form $f(\mathcal{G}) = f_0 \mathcal{G}$. Its Noether symmetry reads

$$X = c_1 r \partial_r + c_2 \partial_P, \quad (33)$$

and $g = c_3 - \frac{8f_0 c_2 (3Q^2-1)}{Q^3}$, with c_1, c_2 and c_3 being constants. Respectively, the preserved quantity related to (33) is

$$I = -\frac{8f_0}{Q^3} \left(\frac{6r(Q^2-3)(c_1 r P' - c_2)Q'}{Q} + c_2 (3Q^2-5) \right) - c_3. \quad (34)$$

Spherically Symmetric Solutions

From the Noether theorem (15), we can build the following Lagrange system

$$\frac{dt}{\xi} = \frac{dq^i}{\eta^i} = \frac{d\dot{q}^i}{\eta^{[1],i}}. \quad (35)$$

From the above system applied in our case, i.e., for the symmetry (31), we get the zero and first order invariants that read respectively

$$W^{[0],\mathcal{G}}(r, \mathcal{G}) = \frac{dr}{c_1 r} - \frac{d\mathcal{G}}{-4c_1 \mathcal{G}} = \mathcal{G}r^4, \quad (36)$$

$$W^{[0],P}(r, P) = \frac{dr}{c_1 r} - \frac{dP}{(4n-d)c_1 P} = Pr^{d-4n}, \quad (37)$$

$$W^{[1],\mathcal{G}}(r, \mathcal{G}) = \frac{dr}{c_1 r} - \frac{d\mathcal{G}}{-4c_1 \mathcal{G}} - \frac{d\mathcal{G}'}{-5c_1 \mathcal{G}'} = \mathcal{G}'r^5, \quad (38)$$

$$W^{[1],P}(r, P) = \frac{dr}{c_1 r} - \frac{dP}{(4n-d)c_1 P} - \frac{dP'}{(4n-d-1)c_1 P'} = P'r^{1+d-4n}. \quad (39)$$

Using these, we can reduce the order of the equations of motion, from second to first and solve them.

The Lagrangian (8) for the case 1 of the previous section, i.e., $f = f_0 \mathcal{G}^n$, becomes

$$\begin{aligned} \mathcal{L} = \frac{1}{Q^4} f_0 r^{d-5} \mathcal{G}^{n-2} & \left\{ \mathcal{G} P Q \left[(d-4)(d-3)(d-2)(d-1)n(Q^4 - 2Q^2 + 1) - \mathcal{G}(n-1)Q^4 r^4 \right] + \right. \\ & \left. + 4(d-1)(d-2)n(Q^2 - 1)r \left[(d-3)\mathcal{G} P Q' + (n-1)r Q \mathcal{G}' P' \right] \right\}. \end{aligned} \quad (40)$$

and the associated Euler–Lagrange equations $\frac{\partial \mathcal{L}}{\partial q^i} = \frac{d}{dr} \frac{\partial \mathcal{L}}{\partial q'^i}$ are

$$\begin{aligned} \frac{1}{Q^4} f_0 r^{d-5} \mathcal{G}^n & \left\{ \frac{(d-2)(d-1)n(Q^2-1) \left[(d-3)\mathcal{G}^2((d-4)Q(Q^2-1) + 4rQ') - 4(n-1)(n-2)r^2 Q \mathcal{G}^2 \right]}{\mathcal{G}^3} - \right. \\ & \left. - (n-1)r^4 Q^5 + 4(d-2)(d-1)(n-1)nr \frac{\mathcal{G}' \left[(Q^2-3)rQ' - (d-3)Q(Q^2-1) \right] - Q(Q^2-1)r\mathcal{G}''}{\mathcal{G}^2} \right\} = 0, \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{1}{Q^4} f_0 r^{d-5} \mathcal{G}^{n-2} & \left\{ \mathcal{G}^2(n-1)PQ^4 r^4 + (d-2)(d-1)n \left[4(n-1)r\mathcal{G}'((d-3)P(Q^2-1) + (Q^2-3)rP') + \right. \right. \\ & \left. \left. + (d-3)\mathcal{G}(Q^2-1)(4rP' - (d-4)P(Q^2-1)) \right] \right\} = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{1}{Q^4} f_0(n-1)nr^{d-5} \mathcal{G}^{n-2} & \left\{ -\mathcal{G}PQ^5 r^4 + (d-2)(d-1) \left[(d-4)(d-3)PQ(Q^2-1)^2 - \right. \right. \\ & \left. \left. - 4r((Q^2-1)(QrP'' - (d-3)PQ') + P'((d-3)Q(Q^2-1) - (Q^2-3)rQ')) \right] \right\} = 0, \end{aligned} \quad (43)$$

for P , Q and $\mathcal{G}$ respectively. Solving Equation (43) for $\mathcal{G}(r)$ we find, as expected, that $\mathcal{G}(r) = \tilde{\mathcal{G}}$, given by (6). We simplify Equations (41) and (42) by setting $Q(r) = 1/P(r)$ and we end up with one equation of the form

$$\begin{aligned} (d-1)(d-2)\tilde{\mathcal{G}}^n & \left\{ (d-3)(P^2-1) \left[(d-4)(P^2-1) - 4(n-2)rPP' \right] - \right. \\ & \left. - 4(n-1)r^2 \left[(P^2-1)PP'' + (3P^2-1)P'^2 \right] \right\} = 0. \end{aligned} \quad (44)$$

Obviously, for $d = 1, 2$ Equation (44) is satisfied automatically. The rest of the equation accepts three solutions which read

$$P(r)^2 = 1 + e^{-2k_2} \sqrt{k_1 - 4r} r^{\frac{3-d}{2}} \text{ and } \mathcal{G}(r) = 0 \quad n \neq 1 \quad d\mathcal{G}e3, \quad (45)$$

$$P(r)^2 = P_0^2 \left(1 - \frac{k_3}{r^{\frac{d}{2}-2}} \right) \text{ and } \mathcal{G}(r) = 0 \quad n = 1 \quad d\mathcal{G}e4, \quad (46)$$

$$P(r)^2 = 1 \pm r^{2-\frac{d}{2}} \sqrt{\frac{4k_1 d}{120 \binom{d+1}{d-4}}} \pm r^2 \sqrt{\frac{\mathcal{G}_0(d-3)}{120 \binom{d+1}{d-4}}} \text{ and } \mathcal{G}(r) = \mathcal{G}_0, \quad f(\mathcal{G}) = f_0 \mathcal{G}^{\frac{d+1}{4}} \quad d\mathcal{G}e4, \quad (47)$$

with $k_1, k_2, k_3, P_0, \mathcal{G}_0$ constants. These are general black hole solutions for the theory (2) with $f(\mathcal{G}) = f_0 \mathcal{G}^n$; in particular, the first one is valid in arbitrary d dimension with $d\mathcal{G}e3$, while the other holds in more than four dimensions. Solution (47), which is the (A)dS equivalent of $f(\mathcal{G})$ gravity, holds for any $n = \frac{d+1}{4}$, in agreement with the trace Equation (4). For this reason, in four dimensions, it trivially provides a constant line element. In any case, the asymptotic flatness is always recovered in more than five dimensions; furthermore, solutions (46) and (45) admits as horizon $r_S \sim (GM)^{\frac{2}{d-4}}$.

Let us now see some more specific solutions of the system (41)–(43), analyzing the boundary cases $d = 3$ and $d = 4$. In $d = 3$ we have the following solution for any $P(r)$

$$Q(r) = \frac{1}{3} \left(A(r) - e^{q_0} P'(r) + \frac{e^{2q_0}}{A(r)} P'(r)^2 \right), \quad (48)$$

with

$$A(r) = \left(\frac{27e^{q_0}}{2} P'(r) - e^{3q_0} P'(r)^3 + \frac{3e^{q_0}}{2} P'(r) \sqrt{81 - 12e^{2q_0} P'(r)^2} \right)^{1/3}.$$

q_0 is an integration constant and the Gauss–Bonnet term vanishes in this case. As an example, by introducing the relation $Q(r) = P(r)^k$, we find that the field equations are satisfied by any $P(r)$ solving the equation

$$k(P^{2k} - 3)P'^2 - P(P^{2k} - 1)P'' = 0. \quad (49)$$

The limit $k = -1$ provides back solution (45); an interesting analytic solution of Equation (49) occurs for $k = -1/3$, where the components of the interval are:

$$P(r)^2 = -2c_1 \left[(r + c_2) \left(\frac{6r}{M(r)} + 1 \right) \right] + \frac{3}{8} \left[\frac{M(r)^2 + 9}{M(r)} + 3 \right]$$

with

$$M(r) = \sqrt[3]{128c_1^2 r^2 + 16(16c_1 c_2 - 9)c_1 r + 64\sqrt{c_1^3(c_2 + r)^3(4c_1 r + 4c_2 c_1 - 1) + 128c_2^2 c_1^2 - 144c_2 c_1 + 27}}. \quad (50)$$

Moreover, in $d = 4$ we only get the following solutions for constant $\mathcal{G}$,

$$P(r)^2 = -\frac{1}{2} \exp \left[\tanh^{-1} \left(\sqrt{\frac{\mathcal{G}_0}{30}} \frac{r^2}{2} \right) \right] \sqrt{4 - \frac{\mathcal{G}_0 r^4}{30}} \text{ and } Q(r)^{-2} = 1 + \frac{\sqrt{\mathcal{G}_0} r^2}{2\sqrt{30}} \text{ for } n = 5/4, \quad (51)$$

$$P(r)^2 = 1 = Q(r)^2, \text{ for } \mathcal{G}_0 = 0 \text{ and } \forall n. \quad (52)$$

If we Taylor expand $P(r)^2$ in the first solution (51), we find that $P(r)^2 = Q(r)^{-2}$, which is an AdS-like solution, where $k^2 \equiv \frac{\sqrt{\mathcal{G}_0}}{2\sqrt{30}}$ can be considered as the bulk cosmological constant. The second

one is Minkowski. In all of the above cases we set the integration constants so that we have the correct asymptotic behavior.

5. Conclusions

Higher dimensional theories have been extensively studied to provide solutions to address some shortcomings of GR. Gauss–Bonnet gravity is one of them [55]. In this paper, we studied a generalized Gauss–Bonnet gravity of the form of (2), in arbitrary $(d + 1)$ dimensions. Specifically, using the Noether symmetry approach, we found forms of the function f , which are invariant under point transformations, in a spherically symmetric background. As it turns out, the only possible form is a power-law $f(\mathcal{G}) = f_0 \mathcal{G}^n$. In this perspective, the standard action of GR is recovered for $f(\mathcal{G}) = f_0 \mathcal{G}^{1/2}$ as soon as the degrees of freedom related to R are dominant with respect to the others, like in the case of cosmology. Furthermore, we considered the above power-law model in arbitrary $(d + 1)$ dimensions, and found analytical static and spherically symmetric solutions of the form (5). The solutions we found are summarized in the following Table 1.

Table 1. Exact static and spherically symmetric solutions in $f(\mathcal{G})$ gravity, for $f(\mathcal{G}) = f_0 \mathcal{G}^n$ in arbitrary $d + 1$ dimensions.

$P(r)^2$	$Q(r)^2$	d	n
$1 + e^{-2c_2} \sqrt{c_1 - 4rr^{\frac{3}{2}-\frac{d}{2}}}$	$1/P(r)^2$	$d \geq 3$	$n > 0, \neq 1$
$P_0^2 \left(1 - \frac{k_3}{r^{\frac{d}{2}-2}}\right)$	$1/P(r)^2$	$d > 3$	$n = 1$
$1 \pm r^{2-\frac{d}{2}} \sqrt{\frac{4k_1 d}{120(d-4)}} \pm r^2 \sqrt{\frac{\mathcal{G}_0(d-3)}{120(d-4)}}$	$1/P(r)^2$	$d > 3$	$n = \frac{d+1}{4}$
$\forall P(r)$	$\frac{1}{3} \left(A(r) - e^{q_0} P'(r) + \frac{e^{2q_0}}{A(r)} P'(r)^2 \right)$	$d = 3$	$n > 0$
$-\frac{1}{2} \exp \left[\tanh^{-1} \left(\sqrt{\frac{\mathcal{G}_0}{30}} \frac{r^2}{2} \right) \right] \sqrt{4 - \frac{\mathcal{G}_0 r^4}{30}}$	$1 + \frac{\sqrt{\mathcal{G}_0} r^2}{2\sqrt{30}}$	$d = 4$	$n = 5/4$
1	1	$d = 4$	$\forall n$

In a future work, we will study the stability of the above solutions, as well as the possibility to find some compact object solutions for models with specified d and n .

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