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Oscillatory Behavior of Fourth-Order Differential Equations with Neutral Delay

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Abstract: In this paper, new sufficient conditions for oscillation of fourth-order neutral differential equations are established. One objective of our paper is to further improve and complement some well-known results which were published recently in the literature. Symmetry ideas are often invisible in these studies, but they help us decide the right way to study them, and to show us the correct direction for future developments. An example is given to illustrate the importance of our results.

Keywords: fourth-order differential equations; neutral delay; oscillation

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1. Introduction

Consider the fourth-order neutral differential equation of the form

$$\left(r(t)\left(N_u'''(t)\right)^{\alpha}\right)' + q(t)\,u^{\beta}(\sigma(t)) = 0, \tag{1}$$

where $t \geq t_0$ and $N_u(t) := u(t) + c(t)\,u(\tau(t))$. In this paper, we assume that

H1: α and β are quotients of odd positive integers and $\beta \geq \alpha$;
H2: $r \in C^1([t_0,\infty))$, $r(t) > 0$, $r'(t) \geq 0$ and $\int^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s = \infty$;
H3: $c, q \in C([t_0,\infty))$, $q(t) > 0$, $0 \leq c(t) < c_0 < \infty$ and $q(t)$ is not identically zero for large t;
H4: $\tau \in C^1([t_0,\infty))$, $\sigma \in C([t_0,\infty))$, $\tau'(t) > 0$, $\tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of (1), we mean a function $u \in C^3([t_y,\infty))$, $t_y \geq t_0$, which has the property $r(t)\left(N_u'''(t)\right)^{\alpha} \in C^1([t_y,\infty))$, and satisfies (1) on $[t_y,\infty)$. We consider only those solutions u of (1) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$, for all $T \geq t_y$. A solution u of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

The qualitative study of the neutral delay differential equations has, besides its theoretical interest, significant practical importance, see [1]. Lately, there has been a lot of research activities concerning the oscillation of differential equations with a different order, see [1–24].

Next, we quickly audit some significant oscillation criteria got for higher-order equations which can be viewed as an inspiration for this paper.

Theorem 1 (A. [23] (Theorem 2))**.** *Every solution u of*

$$N_u^{(n)}(t) + q(t)\,u(\sigma(t)) = 0 \tag{2}$$

is oscillatory, if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)\,2^{(n-1)(n-2)}}{\mathrm{e}} \tag{3}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)\,2^{(n-1)(n-2)},\ \ \sigma'(t)\geq 0,$$

where $Q(t) := \sigma^{n-1}(t)\,(1-c(\sigma(t)))\,q(t)$.

Theorem 2 (B. [24] (Corollary 1))**.** *If either*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}} \tag{4}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)!,\ \ \sigma(t)\geq 0$$

holds, then (2) is oscillatory.

Theorem 3 (C. [22] (Corollary 2.16))**.** *Equation (1) is oscillatory if*

$$\left(\sigma^{-1}(t)\right)'\geq\sigma_0>0,\ \tau'(t)\geq\tau_0>0,\ \tau^{-1}(\sigma(t))<t$$

and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\frac{\widehat{q}(s)}{r(s)}\left(s^{n-1}\right)^{\alpha}\mathrm{d}s > \left(\frac{1}{\sigma_0}+\frac{c_0^{\alpha}}{\sigma_0\tau_0}\right)\frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{5}$$

where $\widehat{q}(t) := \min\left\{q\left(\sigma^{-1}(t)\right), q\left(\sigma^{-1}(\tau(t))\right)\right\}$.

It's easy to see that results in [24] improved results of [23], where $(n-1)! < (n-1)\,2^{(n-1)(n-2)}$ for $n>3$. Using a different comparison approach Xing et al. [22], improved the results of [23,24].

In this paper, we obtain new oscillation criteria for fourth-order differential Equation (1) with neutral delay by using the Riccati transformations. Our results improve the results in [22–24]. An example is given to illustrate the importance of our results.

2. Main Results

Here, we consider the following notations:

$$c_1(t) = \frac{1}{c\left(\tau^{-1}(t)\right)}\left(1-\frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^3}{\left(\tau^{-1}(t)\right)^3 c\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right)$$

and

$$c_2(t) = \frac{1}{c\left(\tau^{-1}(t)\right)}\left(1-\frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{\left(\tau^{-1}(t)\right) c\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right).$$

All functional inequalities are assumed to hold eventually, that is, they are assumed to be satisfied for all t sufficiently large. We begin with the following auxiliary lemmas that can be found in [3,4,15], respectively.

Lemma 1. *Assume that* $u, v\geq 0$ *and* β *is a positive real number. Then*

$$(u+v)^{\beta}\leq 2^{\beta-1}\left(u^{\beta}+v^{\beta}\right),\ \textit{for}\ \beta\geq 1$$

and

$$(u+v)^{\beta} \leq u^{\beta}+v^{\beta}, \text{ for } \beta \leq 1.$$

Lemma 2. *If the function u satisfies $u^{(i)}(t)>0$, $i=0,1,\ldots,n$, and $u^{(n+1)}(t)<0$, then*

$$\frac{u(t)}{t^{N}/n!} \geq \frac{u'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 3. *Let $u \in C^{n}([t_0,\infty),(0,\infty))$. Assume that $u^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists a $t_1 \geq t_0$ such that $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} u(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_{\mu} \geq t_1$ such that*

$$u(t) \geq \frac{\mu}{(n-1)!} t^{n-1}\left|u^{(n-1)}(t)\right| \text{ for } t \geq t_{\mu}.$$

At studying the asymptotic properties of the positive solutions of (1), it is easy to verify—by [3] (Lemma 2.2.1)—that the function N_u has the following two possible cases:

Lemma 4. *Assume that u is an eventually positive solution of (1). Then, there exist two possible cases:*

$$\begin{array}{lll} (\mathbf{S}_1)\ N_u^{(\kappa)}(t) & > & 0 \text{ for } \kappa=0,1,2,3, \\ (\mathbf{S}_2)\ N_u^{(\kappa)}(t) & > & 0 \text{ for } \kappa=0,1,3, \text{ and } N_u''(u)<0, \end{array}$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Lemma 5. *If u is an eventually positive solution of (1) and*

$$\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{3} < \left(\tau^{-1}(t)\right)^{3} c\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right), \tag{6}$$

then

$$u(t) \geq \frac{1}{c(\tau^{-1}(t))}\left(N_u\left(\tau^{-1}(t)\right) - \frac{1}{c(\tau^{-1}(\tau^{-1}(t)))} N_u\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)\right). \tag{7}$$

Moreover,

$$\left(r(t)\left(N_u'''(t)\right)^{\alpha}\right)' \leq -q(t) c_1^{\beta}(\sigma(t)) N_u^{\beta}\left(\tau^{-1}(\sigma(t))\right), \text{ if } N_u \text{ satisfies } (\mathbf{S}_1) \tag{8}$$

and

$$N_u''(t) \leq -c_N^{\beta/\alpha} N_u^{\beta/\alpha}(t) \int_t^{\infty}\left(\frac{1}{r(\varrho)}\int_{\varrho}^{\infty} q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}\varrho, \text{ if } N_u \text{ satisfies } (\mathbf{S}_2). \tag{9}$$

Proof. Let u be an eventually positive solution of (1) on $[t_0,\infty)$. From the definition of $N_u(t)$, we see that

$$c(t)u(\tau(t)) = N_u(t) - u(t)$$

and so

$$c\left(\tau^{-1}(t)\right)u(t) = N_u\left(\tau^{-1}(t)\right) - u\left(\tau^{-1}(t)\right).$$

Repeating the same process, we obtain

$$u(t) = \frac{1}{c(\tau^{-1}(t))}\left(N_u\left(\tau^{-1}(t)\right) - \left(\frac{N_u(\tau^{-1}(\tau^{-1}(t)))}{c(\tau^{-1}(\tau^{-1}(t)))} - \frac{u(\tau^{-1}(\tau^{-1}(t)))}{c(\tau^{-1}(\tau^{-1}(t)))}\right)\right),$$

which yields

$$u(t) \geq \frac{N_u(\tau^{-1}(t))}{c(\tau^{-1}(t))} - \frac{1}{c(\tau^{-1}(t))} \frac{N_u(\tau^{-1}(\tau^{-1}(t)))}{c(\tau^{-1}(\tau^{-1}(t)))}.$$

Thus, (7) holds.

Next, it follows from Lemma 4 that there exist two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let $(\mathbf{S}_1)$ holds. Using Lemma 2, we get $N_u(t) \geq \frac{1}{3} t N_u'(t)$ and hence the function $t^{-3} N_u(t)$ is nonincreasing, which with the fact that $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$ gives

$$\left(\tau^{-1}(t)\right)^3 N_u\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^3 N_u\left(\tau^{-1}(t)\right). \tag{10}$$

From (7) and (10), we get that

$$\begin{aligned} u(t) &\geq \frac{N_u(\tau^{-1}(t))}{c(\tau^{-1}(t))} \left(1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^3}{(\tau^{-1}(t))^3 c(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\geq c_1(t) N_u\left(\tau^{-1}(t)\right). \end{aligned} \tag{11}$$

From (1) and (11), we obtain

$$\left(r(t)\left(N_u'''(t)\right)^\alpha\right)' + q(t) c_1^\beta(\sigma(t)) N_u^\beta\left(\tau^{-1}(\sigma(t))\right) \leq 0.$$

Thus, (8) holds.

In the case where $(\mathbf{S}_2)$ satisfies, by using Lemma 2, we find that

$$N_u(t) \geq t N_u'(t) \tag{12}$$

and hence $\left(t^{-1} N_u(t)\right)' \leq 0$. Therefore,

$$\tau^{-1}(t) N_u\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \tau^{-1}\left(\tau^{-1}(t)\right) N_u\left(\tau^{-1}(t)\right). \tag{13}$$

From (7) and (13), we have

$$\begin{aligned} u(t) &\geq \frac{1}{c(\tau^{-1}(t))} \left(1 - \frac{(\tau^{-1}(\tau^{-1}(t)))}{(\tau^{-1}(t)) c(\tau^{-1}(\tau^{-1}(t)))}\right) N_u\left(\tau^{-1}(t)\right) \\ &= c_2(t) N_u\left(\tau^{-1}(t)\right), \end{aligned}$$

which with (1) gives

$$\left(r(t)\left(N_u'''(t)\right)^\alpha\right)' \leq -q(t) c_2^\beta(\sigma(t)) N_u^\beta\left(\tau^{-1}(\sigma(t))\right).$$

Integrating this inequality from t to ϱ, we obtain

$$r(\varrho)\left(N_u'''(\varrho)\right)^\alpha - r(t)\left(N_u'''(t)\right)^\alpha \leq -\int_t^\varrho q(t) c_2^\beta(\sigma(t)) N_u^\beta\left(\tau^{-1}(\sigma(t))\right) \mathrm{d}s. \tag{14}$$

From (12), we get that

$$N_u\left(\tau^{-1}(\sigma(t))\right) \geq \frac{\tau^{-1}(\sigma(t))}{t} N_u(t). \tag{15}$$

Letting $\varrho \to \infty$ in (14) and using (15), we obtain

$$r(t)\left(N_u'''(t)\right)^\alpha \geq c_2^\beta(\sigma(t)) N_u^\beta(t) \int_t^\infty q(s) \left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^\beta \mathrm{d}s.$$

Integrating this inequality again from t to ∞, we get

$$N_u''(t) \leq -c_2^{\beta/\alpha} N_u^{\beta/\alpha}(t) \int_t^\infty \left(\frac{1}{r(\varrho)} \int_\varrho^\infty q(s) \left(\frac{\tau^{-1}(\sigma(s))}{s} \right)^\beta \mathrm{d}s \right)^{1/\alpha} \mathrm{d}\varrho,$$

for all $\mu_2 \in (0,1)$. This completes the proof. $\square$

Theorem 4. *Let $\sigma(t) \leq \tau(t)$ and (6) hold. If there exist positive functions $\theta, \rho \in C^1([t_0,\infty))$ such that*

$$\int_{t_0}^\infty \left(\Psi(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(\tau^{-1}(\sigma(t)))(\theta'(t))^{\alpha+1}}{\left(\mu_1 \theta(t) (\tau^{-1}(\sigma(t)))' (\sigma(t))' (\tau^{-1}(\sigma(t)))^2\right)^\alpha} \right) \mathrm{d}s = \infty \tag{16}$$

and

$$\int_{t_0}^\infty \left(\Phi(s) - \frac{(\rho'(s))^2}{4\rho(s)} \right) \mathrm{d}s = \infty, \tag{17}$$

for some $\mu_1 \in (0,1)$ and every $M_1, M_2 > 0$, where

$$\Psi(t) := M_1^{\beta-\alpha} \theta(t) q(t) c_1^\beta(\sigma(t))$$

and

$$\Phi(t) := c_2^{\beta/\alpha} \rho(t) M_2^{(\beta-\alpha)/\alpha} \int_t^\infty \left(\frac{1}{r(\varrho)} \int_\varrho^\infty q(s) \left(\frac{\tau^{-1}(\sigma(s))}{s} \right)^\beta \mathrm{d}s \right)^{1/\alpha} \mathrm{d}\varrho,$$

then (1) is oscillatory.

Proof. Let u be a non-oscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we can assume that u is eventually positive. It follows from Lemma 4 that there exist two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$. Let $(\mathbf{S}_1)$ holds. From Lemma 5, we arrive at (8). Next, we define a function ω by

$$\omega(t) := \theta(t) \frac{r(t)(N_u'''(t))^\alpha}{N_u^\alpha(\tau^{-1}(\sigma(t)))} > 0.$$

Differentiating and using (8), we obtain

$$\begin{aligned} \omega'(t) \leq\ & \frac{\theta'(t)}{\theta(t)} \omega(t) - \theta(t) q(t) c_1^\beta(\sigma(t)) N_u^{\beta-\alpha}\left(\tau^{-1}(\sigma(t))\right) \\ & -\alpha\theta(t) \frac{r(t)(N_u'''(t))^\alpha (\tau^{-1}(\sigma(t)))' (\sigma(t))' N_u'(\tau^{-1}(\sigma(t)))}{N_u^{\alpha+1}(\tau^{-1}(\sigma(t)))}. \end{aligned} \tag{18}$$

Recalling that $r(t)(N_u'''(t))^\alpha$ is decreasing, we get

$$r\left(\tau^{-1}(\sigma(t))\right)\left(N_u'''\left(\tau^{-1}(\sigma(t))\right)\right)^\alpha \geq r(t)(N_u'''(t))^\alpha.$$

This yields

$$\left(N_u'''\left(\tau^{-1}(\sigma(t))\right)\right)^\alpha \geq \frac{r(t)}{r(\tau^{-1}(\sigma(t)))} (N_u'''(t))^\alpha. \tag{19}$$

It follows from Lemma 3 that

$$N_u'\left(\tau^{-1}(\sigma(t))\right) \geq \frac{\mu_1}{2} \left(\tau^{-1}(\sigma(t))\right)^2 N_u'''\left(\tau^{-1}(\sigma(t))\right), \tag{20}$$

for all $\mu_1 \in (0,1)$ and every sufficiently large t. Thus, by (18)–(20), we get

$$\begin{aligned}\omega'(t) &\leq \frac{\theta'(t)}{\theta(t)}\omega(t) - \theta(t)q(t)c_N^{\beta}(\sigma(t))N_u^{\beta-\alpha}\left(\tau^{-1}(\sigma(t))\right)\\ &\quad -\alpha\theta(t)\frac{\mu_1}{2}\left(\frac{r(t)}{r(\tau^{-1}(\sigma(t)))}\right)^{1/\alpha}\frac{r(t)(N_u'''(t))^{\alpha+1}(\tau^{-1}(\sigma(t)))'(\sigma(t))'(\tau^{-1}(\sigma(t)))^2}{N_u^{\alpha+1}(\tau^{-1}(\sigma(t)))}.\end{aligned}$$

Hence,

$$\begin{aligned}\omega'(t) &\leq \frac{\theta'(t)}{\theta(t)}\omega(t) - \theta(t)q(t)c_N^{\beta}(\sigma(t))N_u^{\beta-\alpha}\left(\tau^{-1}(\sigma(t))\right)\\ &\quad -\alpha\frac{\mu_1}{2}\left(\frac{r(t)}{r(\tau^{-1}(\sigma(t)))}\right)^{1/\alpha}\frac{(\tau^{-1}(\sigma(t)))'(\sigma(t))'(\tau^{-1}(\sigma(t)))^2}{(r\theta)^{1/\alpha}(t)}\omega^{\frac{\alpha+1}{\alpha}}(t).\end{aligned}$$

Since $N_u'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that

$$N_u(t) > M, \tag{21}$$

for all $t \geq t_2$. Using the inequality

$$Uw - Vw^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}},\ V > 0,$$

with

$$U = \frac{\theta'(t)}{\theta(t)},\ V = \alpha\frac{\mu_1}{2}\left(\frac{r(t)}{r(\tau^{-1}(\sigma(t)))}\right)^{1/\alpha}\frac{(\tau^{-1}(\sigma(t)))'(\sigma(t))'(\tau^{-1}(\sigma(t)))^2}{(r\theta)^{1/\alpha}(t)}$$

and $w = \omega$, we get

$$\omega'(t) \leq -\Psi(t) + \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\tau^{-1}(\sigma(t)))(\theta'(t))^{\alpha+1}}{\left(\mu_1\theta(t)(\tau^{-1}(\sigma(t)))'(\sigma(t))'(\tau^{-1}(\sigma(t)))^2\right)^{\alpha}}.$$

This implies that

$$\int_{t_1}^{t}\left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\tau^{-1}(\sigma(t)))(\theta'(t))^{\alpha+1}}{\left(\mu_1\theta(t)(\tau^{-1}(\sigma(t)))'(\sigma(t))'(\tau^{-1}(\sigma(t)))^2\right)^{\alpha}}\right)\mathrm{d}s \leq \omega(t_1),$$

which contradicts (16).

On the other hand, let $(\mathbf{S}_2)$ holds. Using Lemma 5, we get that (9) holds. Now, we define

$$w(t) = \rho(t)\frac{N_u'(t)}{N_u(t)}.$$

Then $w(t) > 0$ for $t \geq t_1$. By differentiating w and using (9), we find

$$\begin{aligned}w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{N_u''(t)}{N_u(t)} - \rho(t)\left(\frac{N_u'(t)}{N_u(t)}\right)^2\\ &\leq \frac{\rho'(t)}{\rho(t)}w(t) - \frac{1}{\rho(t)}w^2(t)\\ &\quad -c_2^{\beta/\alpha}\rho(t)N_u^{\beta/\alpha-1}(t)\int_t^{\infty}\left(\frac{1}{r(\varrho)}\int_{\varrho}^{\infty}q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\varrho.\end{aligned}$$

Thus, we obtain

$$w'(t) \leq -\Phi(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{1}{\rho(t)} w^2(t)$$

and so

$$w'(t) \leq -\Phi(t) + \frac{(\rho'(t))^2}{4\rho(t)}.$$

Then, we get

$$\int_{t_1}^{t} \left(\Phi(s) - \frac{(\rho'(t))^2}{4\rho(t)} \right) \mathrm{d}s \leq w(t_1),$$

which contradicts (17). This completes the proof. □

Example 1. *Consider the equation*

$$(u(t) + c_0 u(\delta t))^{(4)} + \frac{q_0}{t^4} u(\lambda t) = 0, \tag{22}$$

where $t \geq 1$, $q_0 > 0$, $\delta \in \left(c_0^{-1/3}, 1\right)$ *and* $\lambda \in (0, \delta)$. *We note that* $r(t) = 1$, $c(t) = c_0$, $\tau(t) = \delta t$, $\sigma(t) = \lambda t$ *and* $q(t) = q_0/t^4$. *Thus, it's easy to see that (6) is satisfied. Moreover, we have*

$$c_1(t) = \frac{1}{c_0}\left(1 - \frac{1}{\delta^3 c_0}\right),\ c_2(t) = \frac{1}{c_0}\left(1 - \frac{1}{\delta c_0}\right),\ \Psi(t) = \frac{c_1 q_0}{t}$$

and

$$\Phi(t) = \frac{c_2 \lambda q_0}{6\delta t}.$$

Thus, (16) and (17) become

$$\int_{t_0}^{\infty} \left(\frac{c_1(t) q_0}{s} - \frac{9\delta^4}{2\lambda^4} \frac{1}{s} \right) \mathrm{d}s = \left(c_1(t) q_0 - \frac{9\delta^4}{2\lambda^4} \right)(+\infty)$$

and

$$\int_{t_0}^{\infty} \left(\Phi(s) - \frac{(\rho'(s))^2}{4\rho(s)} \right) \mathrm{d}s = \left(\frac{c_2 \lambda}{6\delta} q_0 - \frac{1}{4} \right)(+\infty),$$

respectively. Hence, from Theorem 4, we conclude that (22) is oscillatory if

$$q_0 \frac{1}{c_0}\left(1 - \frac{1}{\delta^3 c_0}\right) > \frac{9\delta^4}{2\lambda^4} \tag{23}$$

and

$$q_0 \frac{1}{c_0}\left(1 - \frac{1}{\delta c_0}\right) > \frac{3\delta}{2\lambda}. \tag{24}$$

In particular case that $c_0 = 16$, $\delta = 1/2$ *and* $\lambda = 1/3$, *Condition (23) yields* $q_0 > 41.14$. *Whereas, the criterion obtained from the results of [22] is* $q_0 > 4850.4$. *Hence, our results improve the results in [22].*

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