Article

On Spectral Properties of Doubly Stochastic Matrices

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Abstract: The relationship among eigenvalues, singular values, and quadratic forms associated with linear transforms of doubly stochastic matrices has remained an important topic since 1949. The main objective of this article is to present some useful theorems, concerning the spectral properties of doubly stochastic matrices. The computation of the bounds of structured singular values for a family of doubly stochastic matrices is presented by using low-rank ordinary differential equations-based techniques. The numerical computations illustrating the behavior of the method and the spectrum of doubly stochastic matrices is then numerically analyzed.

Keywords: doubly stochastic matrices; eigenvalues; singular values; sturctured singular values; ODEs

1. Introduction

A real stochastic matrix M is a matrix whose row sums or column sums are equal to 1. All the entries of a real stochastic matrix are non-negative. A real symmetric matrix with non-negative entries with row sums and column sums equal to 1 is called doubly stochastic matrix.

A list of n real numbers, i.e., $1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n$ is *s.d.s* realizable if there exists a symmetric doubly stochastic matrix M with its spectrum denoted by $\sigma(M)$. Doubly stochastic matrix describes the transitions corresponding to finite state symmetric Markov chains and this transition acts as a special class of this family. Doubly stochastic matrices are the convex hull for transition matrices with element set [1].

The inverse eigenvalue problems for non-negative doubly stochastic matrices have its origin in work of [2–5]. For more details on inverse eigenvalue problems, we refer the reader to [6–10] and references therein.

The non-negative matrix M has a real eigenvalue $\hat{\lambda}$ such that $\hat{\lambda} \geq |\hat{\lambda}_i|$ for all i. The eigenvalues $\hat{\lambda}_i$ for all i are the eigenvalues of M other than $\hat{\lambda}$. From the Perron–Frobenius theorem an eigenvector corresponding to $\hat{\lambda}$ is such that each of its entries are non-negative and sums to 1. For more details, we refer to [11–13].

[14] showed that the eigenvector has the form $x = \frac{1}{\sqrt{n}}(1,1,1,...,1)^t$ corresponding to eigenvalue $\hat{\lambda}$ for $x \in \mathbb{R}^n$, where $\mathbb{R}$ denotes the real line. The spectrum of a doubly stochastic matrix is bounded by 1, that is $\lambda_i \leq 1$ for all i. Three important eigenvalue problems for doubly stochastic matrices are considered in [14] whenever there is a possibility that eigenvalues can be placed in complex plane, denoted by $\mathbb{C}$. The first problem deals with necessary and sufficient conditions for $n-$tuples to be the spectrum of a given doubly stochastic matrix. The second problem is about the fact that which real numbers acts as the spectrum of the doubly stochastic matrix. The last problem deals with the study in which set of n real numbers act as the spectrum of the symmetric doubly stochastic inverse eigenvalue problems.

The matrix M over a field $\mathbb{F}$ such that each row sum and column sum is $\hat{\lambda}$, is called the generalization of doubly stochastic matrices by means of Lie theory and algebra $\Omega(n)$ of set of generalized doubly stochastic matrices studied in [15].

In the literature [6,7,10,14,15], much attention has been payed to study the eigenvalues or eigen-spectrum of doubly stochastic matrices. According to the best of our knowledge, however, the study of spectral properties, such as structured singular values for doubly stochastic matrices, is utterly missing from the literature. The current paper deals with the study of the spectral properties, such as singular values and structured singular values for a class of doubly stochastic matrices. The main contribution is to fill this gap which reflects the novelty of our results presented in paper.

The paper is arranged as follows: In Section 2 we provide definitions of symmetric stochastic matrices, singular values and structured singular values. In Section 3 we give a detailed explanation of the computation of singular values for symmetric doubly stochastic matrices. Section 4 of this article contains the geometrical interpretation of eigenvalues and singular values of doubly stochastic matrices. The computation of structured singular values for doubly stochastic matrices is addressed in Section 5. Whereas the numerical experimentation is discussed in Section 6. Section 7 summarizes the conclusions.

2. Preliminaries

Definition 1. *The $n-$dimensional matrix $M = (m_{ij})$ is said to be a doubly stochastic matrix if*

$$(i) \quad m_{ij} \geq 0, \quad \forall i, j = 1 : n$$

$$(ii) \quad \sum_{i=1}^{n} m_{ij} = 1, \ j = 1 : n \quad ; \quad \sum_{j=1}^{n} m_{ij} = 1, \ i = 1 : n.$$

Definition 2. *The $n-$dimensional matrix $M = (m_{ij})$ is said to be symmetric doubly stochastic matrix if its transpose is doubly stochastic matrix.*

Definition 3. *The singular values of a matrix M are the non-negative real numbers $\sigma_i = \sqrt{\lambda_i}$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, $\forall i = 1 : n$.*

Definition 4. *The set of block diagonal matrices is denoted by $\mathbb{B}$ and is defined as:*

$$\mathbb{B} := \{ diag(\delta_1 I_1, ..., \delta_S I_S; \Delta_1, ...\Delta_F) : \delta_i \in \mathbb{C}(\mathbb{R}), \ \Delta_j \in \mathbb{C}^{m_j, m_j}(\mathbb{R}^{m_j, m_j}) \ \forall i = 1 : S \ \& \ \forall j = 1 : F \}.$$

In the above definition S and F represent the number of repeated real or complex scalar blocks and the number of full real or complex blocks respectively.

Definition 5. *The structured singular value of $M \in \mathbb{C}^{n,n}$ with respect to set $\mathbb{B}$ is denoted by $\mu_{\mathbb{B}}(M)$ and is defined as:*

$$\mu_{\mathbb{B}}(M) = \begin{cases} 0, & if \ det(I - M\Delta) \neq 0, \forall \Delta \in \mathbb{B} \\ \frac{1}{min_{\Delta}\{\|\Delta\|_2 \ : \ det(I - M\Delta) = 0, \ \forall \ \Delta \in \mathbb{B}\}} & otherwise. \end{cases}$$

3. Computing Singular Values of Doubly Stochastic Matrices

Theorem 1. *Let D_1, D_2 be $n-$dimensional symmetric stochastic matrices with row and column sum equals to 1. Let $\{\sigma_i\}$ and $\{\hat{\sigma}_i\}$, $\forall i = 1 : n$ are singular values with $\{u_i\}$ and $\{v_i\}$, $\forall i = 1 : n$ as left and right singular vectors respectively with $\|u_i\|_2 = 1 = \|v_i\|_2$ for $\{\sigma_i\}$ and $\{\hat{u}_i\}$, $\{\hat{v}_i\}$ are the left and right singular vectors respectively with $\|\hat{u}_i\|_2 = 1 = \|\hat{v}_i\|_2$ for $\{\hat{\sigma}_i\}$. The leading singular vectors u_1 and $\hat{u}_2$ are orthogonal to $\{u_i\}$ and $\{v_i\}$ for all $i = 2 : n$, respectively. Let $e_n = \frac{1}{\sqrt{n}}(1, 1, 1, ..., 1)^t$ be the singular vector corresponding to σ_1 and $\hat{\sigma}_1$ then any vector $\vec{\eta} = \{\eta_1, \eta_2, \eta_3, ..., \eta_n\}$ which is orthogonal to e_n is not a singular vector to σ_1 and $\hat{\sigma}_1$.*

Proof. The result is proved by expanding and taking sum of all components of the singular value problem of the form:

$$D\vec{\eta} = \sigma\vec{\eta}. \tag{1}$$

In Equation (1), $\vec{\eta}$ is a singular vector corresponding to a singular value σ. The singular vector $\vec{\eta}$ is not a singular vector corresponding to singular values σ_1 and $\hat{\sigma}_1$.

We write

$$D\vec{\eta} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & \ddots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} d_{11}\eta_1 + d_{12}\eta_2 + \cdots + d_{1n}\eta_n \\ \cdots \\ \cdots \\ \cdots \\ d_{n1}\eta_1 + d_{n2}\eta_2 + \cdots + d_{nn}\eta_n \end{pmatrix}.$$

Taking sum S^{um} of all components of $D\vec{\eta}$, we have

$$S^{um}(D\vec{\eta}) = S^{um} \begin{pmatrix} d_{11}\eta_1 + d_{12}\eta_2 + \cdots + d_{1n}\eta_n \\ \cdots \\ \cdots \\ \cdots \\ d_{n1}\eta_1 + d_{n2}\eta_2 + \cdots + d_{nn}\eta_n \end{pmatrix} = \eta_1 S^{um}(d_i 1) + \ldots + \eta_n S^{um}(d_i n)$$

$$= \eta_1(\sigma_1) + \ldots + \eta_1(\sigma_1), \tag{2}$$

because D is a symmetric stochastic matrix with row and column sums equal to 1. Equation (2) implies that,

$$S^{um}(D\vec{\eta}) = \sigma_1(\eta_1 + \eta_2 + \ldots + \eta_n),$$

$$= \sigma_1 S^{um}(\vec{\eta}). \tag{3}$$

Now, by taking the sum S^{um} of righthand side of Equation (1) gives,

$$S^{um}(\sigma\vec{\eta}) = \sigma S^{um}(\vec{\eta}). \tag{4}$$

From Equation (3) and (4), we have $(\sigma - \sigma_1)S^{um}(\vec{\eta}) = 0$. In turn, this implies that $\sigma \neq \sigma_1$ and $S^{um}(\vec{\eta}) = 0$, which proves that any vector $\vec{\eta} = \{\eta_1, \eta_2, \eta_3, ..., \eta_n\}$ orthogonal to vector $e_n = \frac{1}{\sqrt{n}}(1, 1, 1, ..., 1)^t$ which is not a singular vector corresponding to singular values σ_1 and $\hat{\sigma}_1$. $\square$

Theorem 2. *Let D_1, D_2 be $n-$ dimensional symmetric stochastic matrices with row and column sum equals to 1. Let σ_i and $\hat{\sigma}_i$ be singular values corresponding to singular vectors as defined in Theorem 1. The matrix $\tilde{D}$ is the matrix with D_1 and D_2 along the main diagonal. The singular values of $\tilde{D}$ do not contain original singular values σ_1 and $\hat{\sigma}_1$ appearing along the main diagonal of a matrix $\tilde{D}_1$ with*

$$\tilde{D}_1 = \begin{pmatrix} \sigma_1 & 1 \\ 1 & \hat{\sigma}_1 \end{pmatrix}. \tag{5}$$

The off diagonal of $\tilde{D}_1$ contains the rank-1 matrices uv^t and vu^t consisting of leading left and right singular vectors corresponding to σ_1 and $\hat{\sigma}_1$, respectively.

Proof. We prove the result by computing the singular vectors of the singular value problem of the form:

$$\tilde{D}u_i = \sigma_i u_i, \quad \forall \ i = 2:n. \tag{6}$$

In Equation (6), the set of vectors $\{u_i\}$ for all $i = 2:n$. is the singular vectors corresponding to singular values $\{\sigma_i\} \forall \ i = 2:n$. The vector $(u_i, 0)^t$ for all $i = 2:n$. acts as a singular vector

corresponding to $\sigma_i \; \forall i = 2 : n$. The vector $(0, \hat{u}_i)^t$ acts as a singular vector corresponding to singular value $\hat{\sigma}_i \; \forall i = 2 : n$. From above discussion it is clear that the vectors of the form $(\alpha_i u_i, \beta_i \hat{u}_i)^t$ for all $i = 2 : n$. act as a singular vectors for matrix $\tilde{D}$ while the vectors $(\alpha_i, \beta_i)^t$ acts as singular vectors corresponding to singular values σ_1 and $\hat{\sigma}_i$ of $\tilde{D}_1$. $\square$

Theorem 3. *Let D_1, D_2 be two symmetric doubly stochastic matrices. Let σ_i and $\hat{\sigma}_i$ be leading singular values corresponding to singular vectors u_1 and $\hat{u}_1$, respectively.*
Let

$$\tilde{D} = \begin{pmatrix} D_1 + \rho I & 2\rho u_1 v_1^t \\ 2\rho v_1 u_1^t & D_2 + \rho I \end{pmatrix}. \tag{7}$$

with I is an identity matrix with the same dimension as of D_1 and D_2 and ρ is any constant. The singular values of $\tilde{D}$ does not contains the leading singular values of D_1 and D_2. The leading singular values are contained in $\tilde{D}_1$ with

$$\tilde{D}_1 = \begin{pmatrix} 3\rho + \sigma_1 & 0 \\ 0 & \rho - \hat{\sigma}_1 \end{pmatrix}. \tag{8}$$

Proof. We prove the result by computing singular values of singular value problem,

$$\tilde{D} \vec{u}_i = (\sigma_i + \rho) u_i, \quad \forall \; i = 2 : n. \tag{9}$$

The singular values of $\tilde{D}$ are $\alpha, \sigma_2 + \rho, \sigma_3 + \rho, \ldots, \sigma_n + \rho$, $\beta, \hat{\sigma}_2 + \rho, \hat{\sigma}_3 + \rho, \ldots, \hat{\sigma}_n + \rho$ where $\alpha, \beta \in \{3\rho + \sigma_1, \rho - \hat{\sigma}_1\}$ which are the singular values of

$$\begin{pmatrix} 3\rho + \sigma_1 & 0 \\ 0 & \rho - \hat{\sigma}_1 \end{pmatrix}.$$

Because $\{u_i\} \forall \; i = 2 : n$ represent the singular vectors corresponding to singular values $\{\sigma_i\} \forall i = 2 : n$. can be treated as $\begin{pmatrix} u_i \\ 0 \end{pmatrix} \forall i = 2 : m$. Similarly for $\{\hat{\sigma}_i\} \forall i = 2 : n$. the singular vector $\begin{pmatrix} 0 \\ \hat{u}_i \end{pmatrix}$ holds true. This show that the singular vector corresponding to $\tilde{D}$ can be expressed as $\begin{pmatrix} \alpha_i u_i \\ \beta_i \hat{u}_i \end{pmatrix}$ and finally the vectors $\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \forall i = 2 : n$. act as singular vectors to $3\rho + \sigma_1$ and $\rho - \hat{\sigma}$. $\square$

Theorem 4. *A two dimensional symmetric doubly stochastic matrix has singular values 1 and σ_1 if and only if $0 \leq \sigma_1 \leq 1$.*

Proof. To complete the prove its sufficient to see the fact that for $0 \leq \sigma_1 \leq 1$ the matrix

$$\begin{pmatrix} \frac{1+\sigma_1}{2} & \frac{1-\sigma_1}{2} \\ \frac{1-\sigma_1}{2} & \frac{1+\sigma_1}{2} \end{pmatrix},$$

is symmetric and doubly stochastic matrix. $\square$

Theorem 5. *A three dimensional symmetric doubly matrix has singular values $1, \sigma_1, \mu_1$ if and only if $0 \leq \sigma_1 \leq 1, 0 \leq \mu_1 \leq 1, \sigma_1 + 3\mu_1 + 2 \geq 0$ and $3\sigma_1 + \mu_1 + 2 \geq 0$.*

Proof. The proof is similar to the one in [15] and hence is omitted. $\square$

4. Geometrical Interpretation of Spectrum

In this section, we present the geometrical interpretation of the spectrum of symmetric doubly stochastic matrices. In particular, we discuss the geometry of the eigenspace and singular values and left, right singular vectors of such a class of matrices.

Example 1. *We take doubly stochastic matrices M_1 and M_2 taken from [16] as*

$$M_1 = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.5 & 0.3 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{pmatrix}.$$

The spectrum of M_1, M_2 is shown in Figure 1a,b of Example 1, respectively. The maximum eigenvalue for both M_1, M_2 is 1 and lies exactly on the spectral circle. The Figure 1c,d show the geometrical interpretation of the singular values and singular vectors obtained for M_1, M_2 of Example 1, respectively.
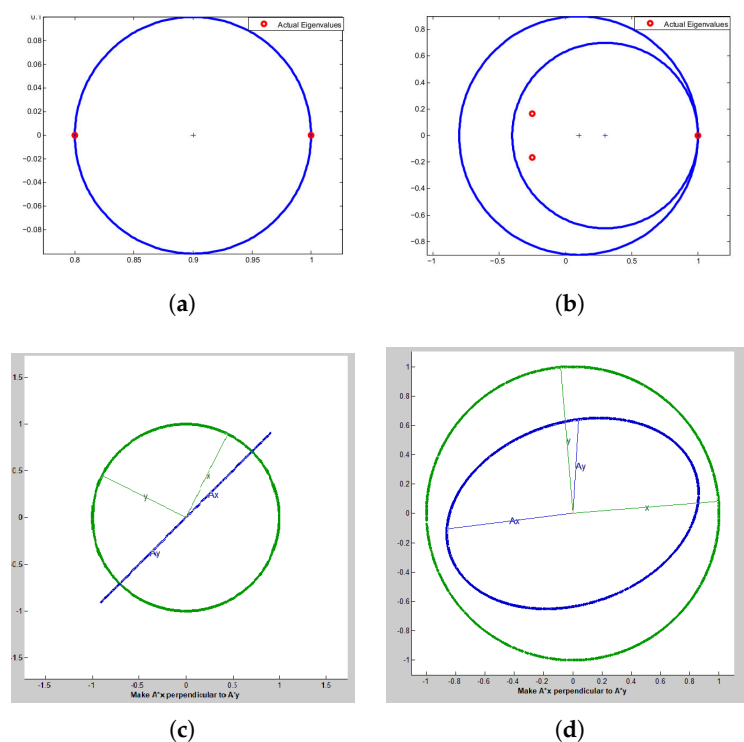

(a)

(b)

(c)

(d)

Figure 1. Geometrical interpretation of spectrum and singular values/vectors of Example 1. (**a**) and (**b**): spectrum of M_1, M_2 of Example 1, respectively; (**c**) and (**d**): geometrical interpretation of the singular values and singular vectors obtained for M_1, M_2 of Example 1, respectively.

Example 2. *We take five dimensional doubly stochastic matrices M_3, M_4 from [17] as*

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad M_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.25 & 0 & 0.75 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0.25 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The spectrum of M_3, M_3 is shown in Figure 2a,b of Example 2, respectively. The maximum eigenvalue for both M_3, M_3 is 1 and lies exactly on the spectral circle. Figure 2c,d show the

geometrical interpretation of the singular values and singular vectors obtained for M_3, M_4 of Example 2, respectively.
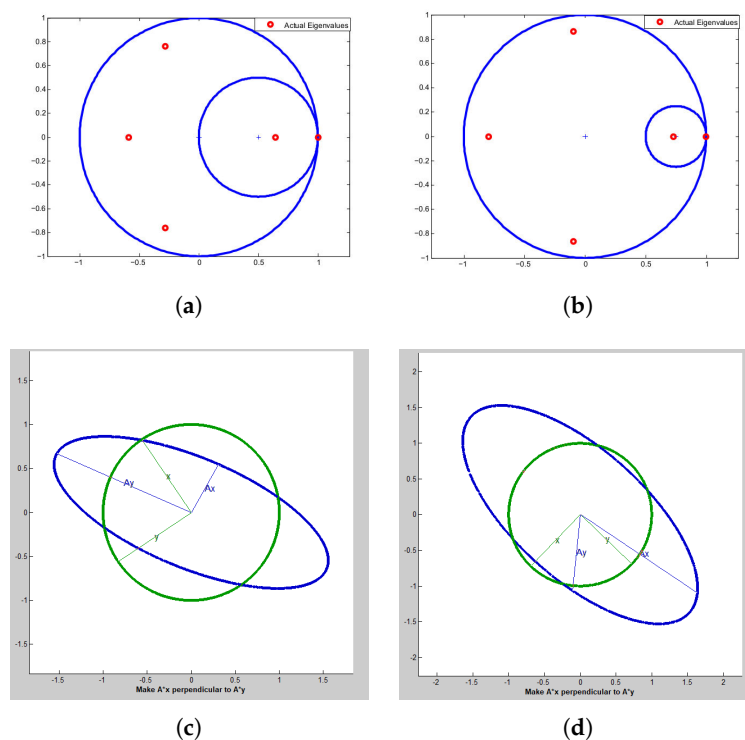

(a)

(b)

(c)

(d)

Figure 2. Geometrical interpretation of spectrum and singular values/vectors of Example 2. (**a**) and (**b**): spectrum of M_3, M_3 of Example 2, respectively; (**c**) and (**d**): geometrical interpretation of the singular values and singular vectors obtained for M_3, M_4 of Example 2, respectively

Example 3. *We take eight and nine dimensional doubly stochastic matrices from [18]. The largest eigenvalue corresponding to each matrix attains the maximum value 1. The spectrum is shown in Figure 3a,b of Example 3, respectively. The Figure 3c,d show the geometrical interpretation of the singular values and singular vectors of Example 3, respectively.*
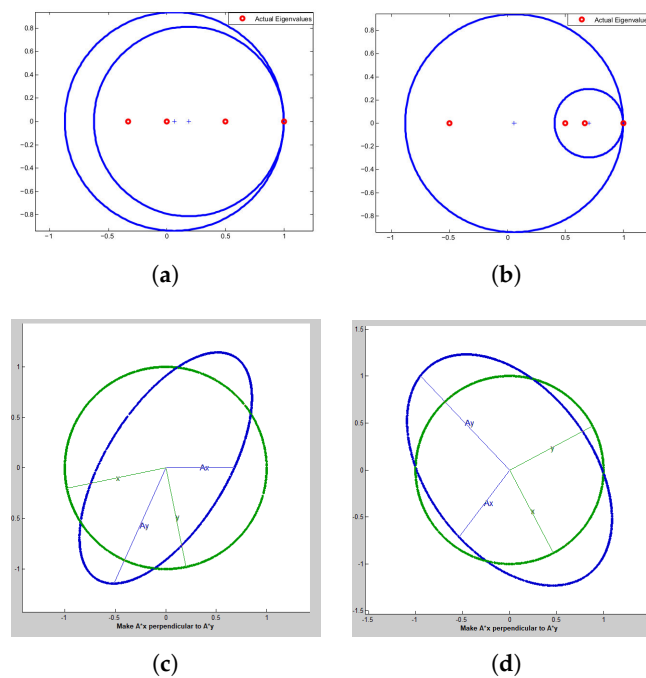
Figure 3. Geometrical interpretation of spectrum and singular values/vectors of Example 3. (**a**) and (**b**): spectrum of Example 3, respectively; (**c**) and (**d**): geometrical interpretation of the singular values and singular vectors of Example 3, respectively

5. Computing Structured Singular Values

In this section, our aim is to discuss the spectral properties of doubly stochastic matrices based on the computation of Structured Singular Values (SSV). For this purpose, we compute SSV for a class of matrices as considered in Section 4. SSV is the straight forward generalization of the singular values for the constant matrices. The computation of the exact value of SSV is NP-hard. For this reason, one needs to approximate its bounds, i.e., lower and upper bounds.

From an application point of view, the computation of lower bounds of SSV gives sufficient information about the instability of some feedback system while the upper bounds discuss the stability of feedback system under consideration. The computation of the bounds of SSV presented in this section is based on two powerful mathematical techniques: First technique is based on power method for approximating spectrum [19]. The upper bound of SSV is computed by means of the balanced/AMI technique [20] for computing the upper bound from [21]. The second technique [22] is based on the low rank ODEs-based techniques in order to approximate the lower bounds of SSV. This technique works on a two level algorithm, i.e., inner-outer algorithm. We give a brief description of inner-outer algorithms in the subsequent subsections.

5.1. Inner-Algorithm

The inner-algorithm is used to solve the minimization problem addressed in Equation (5). For this purpose, one needs to construct and then solve a gradient system of ordinary differential equations associated with the optimization problem. The construction of system of ODEs involves the approximation of the local extremizers of structured spectral values sets [22]. Following Theorem 6 helps us to approximate the local extremizer of structured spectral vale sets

Theorem 6. *[22]. For a perturbation $\Delta \in \mathbb{B}$ with the block diagonal structure*

$$\Delta = \{diag(\delta_1 I_1, \ldots \delta_{s'} I_{s'}, \delta_{s'+1} I_{s'+1}, \ldots \delta_S I_S; \Delta_1, \ldots, \Delta_F\},$$

with $||\Delta||_2 = 1$, *acts as a local extremizer of structured spectral value set. For a simple smallest eigenvalue* $\lambda = |\lambda|e^{\iota\theta}, \theta \in \mathbb{R}$ *of matrix valued function* $(I - \epsilon M\Delta)$ *with the right and left eigenvetors* x *and* y *scaled as* $S = e^{\iota\theta}y^*x$ *and let* $z = M^*y$. *The non-degeneracy conditions*

$$z_k^*x_k \neq 0, \forall = 1 : S'$$
$$Re(z_k^*x_k) \neq 0, \forall = 1 : S' + 1 : S$$
$$and \quad ||z_{s+h}||.||x_{s+h}|| \neq 0, \forall h = 1 : F,$$

hold. Then the magnitude of each complex scalar $\delta_i \forall \ i = 1 : s$ *appears to be exactly equal to* 1 *while each full block possesses a unit 2-norm.*

Proof. For proof we refer to [22].

The system of ordinary differential equations corresponding to a perturbation $\Delta \in \mathbb{B}$ is to approximate an extremizer of smallest eigenvalue in magnitude, i.e., $\lambda = |\lambda|e^{\iota\theta}$ which is obtained as,

$$\dot{\delta}_i = \nu_i(x_i^*z_i - Re(x_i^*z_i\bar{\delta}_i)\delta_i); \quad i = 1 : s'$$
$$\dot{\delta}_l = sign(Re(z_l^*x_l)\Psi_{(-1,1)}(\delta_l); \quad l = s' + 1 : s$$
$$\dot{\Delta}_j = \nu_j(z_{s+j}x_{s+j}^* - Re\langle\Delta_j; z_{s+j}x_{s+j}^*\rangle); \quad j = 1 : F,$$

where $\delta_i \in \mathbb{C}, \forall \ i = 1 : s'$, $\delta_l \in \mathbb{R}$ for $l = s' + 1$ and $\Psi_{(-1,1)}$ is a characteristic function. For further discussion on the construction of system of ordinary differential equations in above equations, we refer to [22]. □

5.2. Outer-Algorithm

The main aim of the outer-algorithm is to vary a small positive parameter $\epsilon > 0$ known as the perturbation level. To vary perturbation level a fast Newton's iteration was used in [22]. The quantity $1/\epsilon$ provides the approximation of lower bound to SSV.

The fast Newton's iteration to solve a problem is

$$|\lambda(\epsilon)| = 1, \qquad \epsilon > 0. \tag{10}$$

To solve Equation (10), we compute the derivative

$$\frac{d}{d\epsilon}(|\lambda(\epsilon)|).$$

To compute $\frac{d}{d\epsilon}(|\lambda(\epsilon)|)$, one need following Theorem 7

Theorem 7. *[22]. Let* $\Delta \in \mathbb{B}$ *be the matrix valued function and let* x *and* y *as a function of* $\epsilon > 0$ *are right and left eigenvectors of the perturbed matrix* $(\epsilon M\Delta)$. *Consider the scaling of vectors* x *and* y *accordingly to Theorem 6. Let* $z = M^*y$ *and consider that the non-degenracy conditions as discussed in Theorem 6 holds true, then it yields that*

$$\frac{d}{d\epsilon}(|\lambda(\epsilon)|) = \frac{1}{|y(\epsilon^*)x(\epsilon)|}\left(\sum_{i=1}^{s}|z_i(\epsilon)^*x_i(\epsilon)| + \sum_{j=1}^{F}||z_{s+j}(\epsilon)||.||y_{s+j}(\epsilon)||\right) > 0. \tag{11}$$

For the proof of the above statement, we refer to [22].

6. Numerical Experimentation

The aim of this section is to present numerical eperimentations for lower bounds of SSV for a class of doubly stochastic matrices obtained in Section 4. The numerical experimentations show that the obtained lower bounds with the help of algorithm [22] are either tighter or equal to one approximated with MATLAB function **mussv**.

Example 4. *We consider a three dimensional real doubly stochastic valued matrix M_2 taken from [16]*

$$M_2 = \begin{bmatrix} 0.1000 & 0.2000 & 0.7000 \\ 0.5000 & 0.3000 & 0.2000 \\ 0.4000 & 0.5000 & 0.1000 \end{bmatrix}.$$

　　We take block uncertainties $\mathbb{B} = diag\{\delta_1 I_1, \delta_2 I_1, \delta_3 I_1 : \delta_1, \delta_3 \in \mathbb{C}, \delta_2 \in \mathbb{R}\}$. The admissible perturbation E is approximated as

$$E = \begin{bmatrix} 1.0000 + 0.0000i & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 + 0.0000i \end{bmatrix},$$

*with $\|E\|_2 = 1$. The lower bound of SSV by using algorithm [22] is obtained as 1 which is equal to the lower bound approximated by **mussv** function.*

　　*Moreover, by using **mussv** function, the admissible perturbation ∇ is obtained as*

$$\nabla = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix},$$

*such that $\|\nabla\|_2 = 1$. The **mussv** function approximates the same lower and upper bounds of SSV, i.e., 1.*

Example 5. *We consider a five dimensional real doubly stochastic valued matrix M_3 taken from [17]*

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0.5000 & 0 & 0.5000 \\ 0 & 0.5000 & 0.5000 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 & 0.5000 \\ 1.0000 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

　　We take block uncertainties $\mathbb{B} = diag\{\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1, \delta_5 I_1 : \delta_1, \delta_3, \delta_5 \in \mathbb{C}, \delta_2, \delta_4 \in \mathbb{R}\}$. The admissible perturbation E is approximated as

$$E = \begin{bmatrix} 1.0000 + 0.0000i & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 + 0.0000i & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 + 0.0000i \end{bmatrix},$$

*with $\|E\|_2 = 1$. The lower bound of SSV by using algorithm [22] is obtained as 1 which is same as the upper bound approximated by **mussv** function.*

　　*Moreover, by using **mussv** function, the admissible perturbation ∇ is obtained as*

$$\nabla = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1.2361 & 0 & 0 & 0 \\ 0 & 0 & 1.2361 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

*such that $\|\nabla\|_2 = 1.2361$ The **mussv** function approximates the lower bounds as 0.8090 and upper bound 1 for SSV.*

Example 6. *We consider a five dimensional real doubly stochastic valued matrix M_4 taken from [17]*

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0.2500 & 0 & 0.7500 \\ 0 & 0.2500 & 0.7500 & 0 & 0 \\ 0 & 0.7500 & 0 & 0 & 0.2500 \\ 1.0000 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We take block uncertainties $\mathbb{B} = diag\{\Delta_1, \delta_1 I_1, \delta_2 I_1, \delta_3 I_1 : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1, \delta_3 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$. The admissible perturbation E is approximated as

$$E = \begin{bmatrix} 0.5000 & 0.5000 & 0 & 0 & 0 \\ 0.5000 & 0.5000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 + 0.0000i & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix},$$

with $\|E\|_2 = 1$. The lower bound of SSV by using algorithm [22] is obtained as 1.

*Moreover, by using **mussv** function, the admissible perturbation ∇ is obtained as*

$$\nabla = \begin{bmatrix} 0.6631 & 0.6028 & 0 & 0 & 0 \\ 0.6028 & 0.5480 & 0 & 0 & 0 \\ 0 & 0 & 1.2111 & 0 & 0 \\ 0 & 0 & 0 & 1.2111 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

*such that $\|\nabla\|_2 = 1.2111$ The **mussv** function approximates the lower bounds as 0.8257 and upper bound 1 for SSV.*

7. Conclusions

In this article, we presented some useful theorems concerning the spectral properties such as singular values and structured singular values for a class of doubly stochastic matrices. We used low-rank ordinary differential equations-based techniques and MATLAB function mussv to approximate bounds of structured singular values corresponding to doubly stochastic matrices. The numerical experimentations show the behavior of singular values and structured singular values which agree with the fact that the largest value of each singular value and structured singular value for doubly stochastic matrix is bounded above by 1. The obtained results for singular values and structured singular values agree with the results obtained for eigenvalues of doubly stochastic matrices, that is:

- The doubly stochastic matrix has an eigenvalue 1.
- The absolute value of any eigenvalue corresponding to a doubly stochastic matrix is less than or equal to 1. The results achieved in this study for structured singular values of doubly stochastic matrices could lead the way to discuss the stability and instability analysis of:
- Stochastic optimal control systems.
- Linear feedback systems in control.

Author Contributions: M.-U.R. and A.H. both contributed in conceptualization, methodology, writing manuscript and validation of results, while J.H.B. and J.A. contributed in writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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