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On a Generalization of a Lucas' Result and an Application to the 4-Pascal's Triangle †

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Abstract: The Pascal's triangle is generalized to "the k-Pascal's triangle" with any integer $k \geq 2$. Let p be any prime number. In this article, we prove that for any positive integers n and e, the n-th row in the p^e-Pascal's triangle consists of integers which are congruent to 1 modulo p if and only if n is of the form $\dfrac{p^{em}-1}{p^e-1}$ with some integer $m \geq 1$. This is a generalization of a Lucas' result asserting that the n-th row in the (2-)Pascal's triangle consists of odd integers if and only if n is a Mersenne number. As an application, we then see that there exists no row in the 4-Pascal's triangle consisting of integers which are congruent to 1 modulo 4 except the first row. In this application, we use the congruence $(x+1)^{p^e} \equiv (x^p+1)^{p^{e-1}} (\bmod\ p^e)$ of binomial expansions which we could prove for any prime number p and any positive integer e. We think that this article is fit for the Special Issue "Number Theory and Symmetry," since we prove a symmetric property on the 4-Pascal's triangle by means of a number-theoretical property of binomial expansions.

Keywords: the p^e-Pascal's triangle; Lucas' result on the Pascal's triangle; congruences of binomial expansions

MSC: 11A99.

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1. Introduction

As it is known, Pascal's triangle is constructed in the following way: Write the first row "1 1". Then each member of each subsequent row is given by taking the sum of the just above two members, regarding any blank as 0.

Example 1. *Here is the Pascal's triangle from the first row to the 7-th row:*

$$
\begin{array}{c}
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

Remark 1. *For any integers $n \geq 1$ and $r \geq 0$, we put*

$$ {}_nC_r := \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r\cdots 1}, $$

where we put $0! = 1$. Then it is well-known that the n-th row in the Pascal's triangle is equal to the sequence

$$ {}_nC_0, {}_nC_1, \ldots, {}_nC_{n-1}, {}_nC_n $$

consisting of $n+1$ terms.

In ([1], Section 1.4), the construction above is generalized as follows:

Definition 1. *Let $k \geq 2$ be any integer. The k-Pascal's triangle is constructed in the following way: Write the first row "$\overbrace{1\,1\,\cdots\,1}^{k}$". Then each member of each subsequent row is given by taking the sum of the just above k members regarding the blank as 0.*

Example 2. *In the case where $k = 4$, the 4-Pascal's triangle from the first row to the 5-th row is the following:*

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & 1 & 1 & 1 & 1 & & & & & & & \\
 & & & & 1 & 2 & 3 & 4 & & 3 & 2 & 1 & & & & & \\
 & & 1 & 3 & 6 & 10 & 12 & & 12 & 10 & 6 & 3 & 1 & & & & \\
 & 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 & & & \\
1 & 5 & 15 & 35 & 65 & 101 & 135 & 155 & 155 & 135 & 101 & 65 & 35 & 15 & 5 & 1 &
\end{array}
$$

Remark 2. (1) *In ([1], Section 1.4), for any integers $k \geq 2$ and $n \geq 1$, it is mentioned that the n-th row in the k-Pascal's triangle consists of $n(k-1)+1$ integers*

$$ {}_nC_0^{(k)}, {}_nC_1^{(k)}, \ldots, {}_nC_{n(k-1)-1}^{(k)}, {}_nC_{n(k-1)}^{(k)} $$

satisfying the equation

$$
\begin{aligned}
&(x^{k-1} + x^{k-2} + \cdots + x + 1)^n \\
&= {}_nC_0^{(k)} x^{n(k-1)} + {}_nC_1^{(k)} x^{n(k-1)-1} + \cdots + {}_nC_{n(k-1)-1}^{(k)} x + {}_nC_{n(k-1)}^{(k)}
\end{aligned}
$$

of polynomials with indeterminate x and integral coefficients. A detailed proof of this fact is described in ([2], Lemma 1.1).

(2) *In ([1], Section 9.10), the following formula for ${}_nC_i^{(k)}$ is described:*

$$ {}_nC_i^{(k)} = \sum_{j=0}^{\left[\frac{i}{k}\right]} (-1)^j \, {}_{n+i-jk-1}C_{n-1} \cdot {}_nC_j, $$

where $\left[\dfrac{i}{k}\right]$ is the greatest integer that is less than or equal to $\dfrac{i}{k}$.

In Example 1, we can see that the n-th row consists of odd integers when n is equal to the Mersenne number 1, 3 or 7. Actually, Lucas showed the following

Theorem 1 ([3], Exemple I in Section 228)**.** *Let $n \geq 1$ be any integer. Then ${}_nC_r$ is odd for any $0 \leq r \leq n$ if and only if n is a Mersenne number, i.e., n is of the form $2^m - 1$ with some integer $m \geq 1$.*

In Section 2 in this article, we generalize the Lucas' result above as the following

Theorem 2. *Let p be any prime number and e any positive integer. For any integer $n \geq 1$, the n-th row in the p^e-Pascal's triangle consists of integers which are congruent to 1 modulo p if and only if n is of the form $\dfrac{p^{em}-1}{p^e-1}$ with some integer $m \geq 1$.*

Remark 3. (1) *Theorem 2 is a generalization of ([2], Theorem 0.2) which is in the case where $e = 1$.*

(2) *We can see that Example 2 gives a partial example of Theorem 2 in the case where $p = 2$, $e = 2$ and $m = 1, 2$.*

As an application of Theorem 2, we can prove that ([2], Conjecture 0.3) holds for $k = 4$, i.e., there exists no row in the 4-Pascal's triangle consisting of integers which are congruent to 1 modulo 4 except the first row as follows:

By Theorem 2, in the case where $k = 4$, we see that for any integer $n \geq 1$, the n-th row in the 4-Pascal's triangle consists of odd integers if and only if n is of the form $\dfrac{4^m-1}{3}$ with some integer $m \geq 1$.

Moreover, we can see an essential property of the $\dfrac{4^m-1}{3}$-th row in the 4-Pascal's triangle for any integer $m \geq 2$ as in the following theorem proved in Section 3.2:

Theorem 3. *For any integer $m \geq 2$, the $\dfrac{4^m-1}{3}$-th row in the 4-Pascal's triangle is congruent to the sequence*

$$\overbrace{1133\cdots1133}^{2^{2m-3}}\overbrace{3311\cdots3311}^{2^{2m-3}}$$

modulo 4, which consists of the repeated 1133's and 3311's whose numbers are the same 2^{2m-3}.

Therefore we can obtain the following

Corollary 1. *([2], Conjecture 0.3) holds for $k = 4$, i.e., there exists no row in the 4-Pascal's triangle consisting of integers which are congruent to 1 modulo 4 except the first row.*

Remark 4. (1) *By Example 2, in the case where $m = 2$, we can see that the 5-th row in the 4-Pascal's triangle is congruent to the sequence*

$$1\,1\,3\,3\,1\,1\,3\,3\,3\,3\,1\,1\,3\,3\,1\,1$$

modulo 4, which matches the assertion of Theorem 3.

(2) *It seems that one could obtain the forms of the sequenece to which the $\left(\dfrac{4^m-1}{3} \pm \ell\right)$-th row in the 4-Pascal's triangle is congruent modulo 4 for some positive integers ℓ by means of Theorem 3. We would like to do these calculations in the future.*

In the proof of Theorem 3 in Section 3.2, we shall use the following lemma proved in Section 3.1:

Lemma 1. *For any prime number p and any positive integer e, we have the following coefficient-wise congruence*

$$(x+1)^{p^e} \equiv (x^p+1)^{p^{e-1}} \pmod{p^e}$$

of binomial expansions with indetermiate x.

2. A Proof of Theorem 2

Although Theorem 2 can be proved by the same argument as the proof of ([2], Theorem 0.2), we shall describe its detailed proof here to make this article self-contained.

Let n and e be any positive integers and p be any prime number.

Firstly, we assume that n is of the form $n = \dfrac{p^{em}-1}{p^e-1}$ with some integer $m \geq 1$. In the algebra $\mathbb{F}_p[x]$ of polynomials of one varible x with coefficients in the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements, we see that for any positive integer ℓ,

$$\begin{aligned}(x-1)^{p^\ell-1} = \frac{(x-1)^{p^\ell}}{x-1} &= \frac{x^{p^\ell}-1}{x-1} \\ &= x^{p^\ell-1} + x^{p^\ell-2} + \cdots + x + 1.\end{aligned}$$

Therefore we see that

$$\begin{aligned}(x^{p^e-1} + x^{p^e-2} + \cdots + x + 1)^n &= (x^{p^e-1} + x^{p^e-2} + \cdots + x + 1)^{\frac{p^{em}-1}{p^e-1}} \\ &= \left((x-1)^{p^e-1}\right)^{\frac{p^{em}-1}{p^e-1}} \\ &= (x-1)^{p^{em}-1} \\ &= x^{p^{em}-1} + x^{p^{em}-2} + \cdots + x + 1 \\ &= x^{n(p^e-1)} + x^{n(p^e-1)-1} + \cdots + x + 1\end{aligned}$$

in $\mathbb{F}_p[x]$. By Remark 2 (1), this implies that the n-th row in the p^e-Pascal's triangle consists of integers which are congruent to 1 modulo p as desired.

Conversely, we now assume that n is of the form

$$n = 1 + p^e + \cdots + p^{e(m-1)} + k$$

with some integers $m \geq 1$ and $1 \leq k \leq p^{em} - 1$. Moreover, we assume that we have

$$(x^{p^e-1} + x^{p^e-2} + \cdots + x + 1)^n = x^{n(p^e-1)} + x^{n(p^e-1)-1} + \cdots + x + 1$$

in $\mathbb{F}_p[x]$ to obtain some contradiction. Since the left hand side is equal to $(x-1)^{n(p^e-1)}$ and the right hand side is equal to $\dfrac{x^{n(p^e-1)+1}-1}{x-1}$, we then have the equality

$$(x-1)^{n(p^e-1)+1} = x^{n(p^e-1)+1} - 1$$

in $\mathbb{F}_p[x]$. Since $n = \dfrac{p^{em}-1}{p^e-1} + k$, this implies that

$$(x-1)^{p^{em}+k(p^e-1)} = x^{p^{em}+k(p^e-1)} - 1.$$

Let $v_p(a)$ be the p-adic valuation of any non-zero integer a, i.e., $p^{v_p(a)} \mid a$ and $p^{v_p(a)+1} \nmid a$. Since $1 \leq k \leq p^{em} - 1$, we see that $v_p(k) < em$ and then

$$v_p(p^{em} + k(p^e-1)) = v_p(k).$$

Therefore we can put

$$p^{em} + k(p^e-1) = p^{v_p(k)} t$$

with some positive integer t which is prime to p. Then we have

$$(x-1)^{p^{v_p(k)}t} = x^{p^{v_p(k)}t} - 1 = (x^t - 1)^{p^{v_p(k)}}$$

which implies that

$$(x-1)^{p^{v_p(k)}(t-1)} = (x^{t-1} + x^{t-2} + \cdots + x + 1)^{p^{v_p(k)}},$$

since $\mathbb{F}_p[x]$ is an integral domain. Since $p^{v_p(k)} < p^{em}$, we see that $t \geq 2$. Therefore substituting $x = 1$ leads a contradiction $t = 0$ in $\mathbb{F}_p$ as desired, and Theorem 2 is proved.

3. An Application to the 4-Pascal's Triangle

By Theorem 2, in the case where $p = 2$ and $e = 2$, we see that for any integer $n \geq 1$, the n-th row in the 4-Pascal triangle consists of odd integers if and only if n is of the form $\dfrac{4^m - 1}{3}$ with some integer $m \geq 1$.

In this section, we shall prove Theorem 3 asserting that for any integer $m \geq 2$, the $\dfrac{4^m - 1}{3}$-th row in the 4-Pascal's triangle is congruent to the sequence

$$\overbrace{1133 \cdots 1133}^{2^{2m-3}} \overbrace{3311 \cdots 3311}^{2^{2m-3}}$$

modulo 4. Here we should note that 2^{2m-3} is the number of 1133's and 3311's, respectively.

Then Theorems 2 and 3 imply that ([2], Conjecture 0.3) holds in the case where $k = 4$, i.e., there exists no row in the 4-Pascal's triangle consisting of integers which are congruent to 1 modulo 4 except the first row as we have seen in Corollary 1.

3.1. On a Congruence of Binomial Expansions

Before proving Theorem 3, we shall prove Lemma 1 on a congruence of binomial expansions in this subsection.

Let p be any prime number and e any positive integer. In order to prove the congruence

$$(x+1)^{p^e} \equiv (x^p + 1)^{p^{e-1}} \pmod{p^e}$$

of binomial expansions with indeterminate x, it suffices to see the following two congruences hold:

(1) For any integer $1 \leq \ell \leq p^e - 1$ which is prime to p,

$$_{p^e}C_\ell \equiv 0 \pmod{p^e}.$$

(2) In the case where $e \geq 2$, for any integers $0 \leq f \leq e-2$ and i such that $1 \leq ip^f \leq p^{e-1} - 1$ and $(i, p) = 1$,

$$_{p^e}C_{ip^{f+1}} \equiv {}_{p^{e-1}}C_{ip^f} \pmod{p^e}.$$

Firstly, we shall prove the part (1). In the case where $\ell = 1$, we see that

$$_{p^e}C_1 = p^e \equiv 0 \pmod{p^e}.$$

Moreover, in the case where $2 \leq \ell \leq p^e - 1$, we see that

$$_{p^e}C_\ell = \frac{p^e}{\ell} \prod_{j=1}^{\ell-1} \frac{p^e - j}{j}.$$

Since $v_p(p^e - j) = v_p(j)$ for any $1 \le j \le \ell - 1 < p^e$ and ℓ is prime to p, we then see that

$$\begin{aligned} v({}_{p^e}C_\ell) &= e - v_p(\ell) + \sum_{j=1}^{\ell-1} v_p\left(\frac{p^e - j}{j}\right) \\ &= e + \sum_{j=1}^{\ell-1} (v_p(p^e - j) - v_p(j)) \\ &= e. \end{aligned}$$

Therefore ${}_{p^e}C_\ell \equiv 0 \pmod{p^e}$, and part (1) is proved.

Secondly, we shall prove part (2). We see that

$$\begin{aligned} &{}_{p^e}C_{ip^{f+1}} - {}_{p^{e-1}}C_{ip^f} \\ &= \frac{p^e}{ip^{f+1}} \cdot \frac{\prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1} + (p^e - ip^{f+1}))\right)}{\prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1})\right)} \cdot {}_{p^{e-1}-1}C_{ip^f-1} \\ &\quad - \frac{p^{e-1}}{ip^f} \cdot {}_{p^{e-1}-1}C_{ip^f-1} \\ &= \frac{p^{e-f-1}}{i} \cdot {}_{p^{e-1}-1}C_{ip^f-1} \left(\frac{\prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1} + (p^e - ip^{f+1}))\right)}{\prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1})\right)} - 1 \right) \end{aligned}$$

and that

$$\begin{aligned} &\prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1} + (p^e - ip^{f+1}))\right) \\ &\equiv \prod_{j=0}^{i-1}\left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} (k + jp^{f+1})\right) \\ &\equiv \left(\prod_{1\le k\le p^{f+1}-1,\ (k,p)=1} k\right)^i \pmod{p^{f+1}}. \end{aligned}$$

Since $(i, p) = 1$, we then see that ${}_{p^e}C_{ip^{f+1}} - {}_{p^{e-1}}C_{ip^f}$ is divisible by $p^{e-f-1} \cdot p^{f+1} = p^e$ as desired.

3.2. A Proof of Theorem 3

Now we shall prove Theorem 3 by means of Lemma 1 with $p = 2$ and $e = 2$, i.e., the congruence of binomial expansions

$$(x+1)^4 \equiv (x^2+1)^2 \pmod 4. \cdots (*)$$

By Remark 2 (1), proving Theorem 3 is equivalent to proving that for any integer $m \geq 2$, the coefficient-wise congruence

$$\begin{aligned}
&(x^3+x^2+x+1)^{\frac{4^m-1}{3}} \\
&\equiv x^{4^m-1}+x^{4^m-2}-x^{4^m-3}-x^{4^m-4}+\cdots+x^{\frac{4^m}{2}+3}+x^{\frac{4^m}{2}+2}-x^{\frac{4^m}{2}+1}-x^{\frac{4^m}{2}} \\
&-x^{\frac{4^m}{2}-1}-x^{\frac{4^m}{2}-2}+x^{\frac{4^m}{2}-3}+x^{\frac{4^m}{2}-4}-\cdots-x^3-x^2+x+1 \pmod 4 \quad \cdots \ (**)
\end{aligned}$$

holds with indeterminate x by the induction on m.

Before doing this, we see the following

Lemma 2. *The polynomial in the right hand side of the congruence relation* $(**)$ *can be decomposed as*

$$(x+1)(x^2-1)(x^4+1)\cdots(x^{2^{2m-2}}+1)(x^{2^{2m-1}}-1).$$

Proof. By a direct calculation, we can see that there exists some positive integer ℓ such that the polynomial in the right hand side of the congruence relation $(**)$ can be decomposed as

$$\begin{aligned}
&(x+1)(x^{4^m-2}-x^{4^m-4}+x^{4^m-6}-x^{4^m-8}+\cdots+x^{\frac{4^m}{2}+6}-x^{\frac{4^m}{2}+4}+x^{\frac{4^m}{2}+2}-x^{\frac{4^m}{2}} \\
&\qquad -x^{\frac{4^m}{2}-2}+x^{\frac{4^m}{2}-4}-x^{\frac{4^m}{2}-6}+x^{\frac{4^m}{2}-8}-\cdots-x^6+x^4-x^2+1) \\
&= (x+1)(x^2-1)(x^{4^m-4}+x^{4^m-8}+\cdots+x^{\frac{4^m}{2}+4}+x^{\frac{4^m}{2}} \\
&\qquad\qquad -x^{\frac{4^m}{2}-4}-x^{\frac{4^m}{2}-8}-\cdots-x^4-1) \\
&= \cdots \\
&= (x+1)(x^2-1)(x^4+1)\cdots(x^{2^\ell}+1)(x^{3\cdot 2^{\ell+1}}+x^{2\cdot 2^{\ell+1}}-x^{2^{\ell+1}}-1) \\
&= (x+1)(x^2-1)(x^4+1)\cdots(x^{2^\ell}+1)(x^{2^{\ell+1}}+1)(x^{2^{\ell+2}}-1).
\end{aligned}$$

Since the degree of the polynomial in the right hand side of the congruence relation $(**)$ is equal to 4^m-1, we then see that

$$\begin{aligned}
4^m-1 &= 1+2+2^2+\cdots+2^\ell+2^{\ell+1}+2^{\ell+2} \\
&= 2^{\ell+3}-1,
\end{aligned}$$

which implies that $\ell = 2m-3$ as desired. □

Let us start to prove Theorem 3 by the induction on $m \geq 2$. Firstly, in the case where $m=2$, since

$$(x^2+1)^4 \equiv (x^4+1)^2 \pmod 4$$

and

$$\begin{aligned}
(x+1)^4 &\equiv (x^2+1)^2 \equiv x^4+2x^2+1 \equiv x^4-2x^2+1 \\
&\equiv (x^2-1)^2 \pmod 4
\end{aligned}$$

by the congruence relation $(*)$, we see that

$$\begin{aligned}
(x^3+x^2+x+1)^5 &\equiv (x+1)(x^2+1)(x+1)^4(x^2+1)^4 \\
&\equiv (x+1)(x^2+1)(x^2-1)^2(x^4+1)^2 \\
&\equiv (x+1)(x^2-1)(x^4+1)(x^8-1) \pmod 4.
\end{aligned}$$

Therefore the congruence relation $(**)$ holds for $m = 2$ by Lemma 2.

Secondly, we assume that the congruence relation $(**)$ holds for some $m \geq 2$. By the congruence relation $(*)$, we see that

$$\begin{aligned}(x+1)^{4^m} &\equiv (x+1)^{2^{2m}} \equiv ((x+1)^4)^{2^{2m-2}} \\ &\equiv (x^2+1)^{2^{2m-1}} \equiv ((x^2+1)^4)^{2^{2m-3}} \\ &\equiv (x^{2^2}+1)^{2^{2m-2}} \\ &\equiv \cdots \\ &\equiv (x^{2^{2m-1}}+1)^2 \\ &\equiv (x^{\frac{4^m}{2}}+1)^2 \pmod 4.\end{aligned}$$

By Lemma 2, we then see that

$$\begin{aligned}&(x^3+x^2+x+1)^{\frac{4^{m+1}-1}{3}} \\ \equiv\ &(x^3+x^2+x+1)^{\frac{4^m-1}{3}+4^m} \\ \equiv\ &(x+1)(x^2-1)(x^4+1)\cdots(x^{2^{2m-2}}+1)(x^{2^{2m-1}}-1)(x^2+1)^{4^m}(x+1)^{4^m} \\ \equiv\ &(x+1)(x^2-1)(x^4+1)\cdots(x^{2^{2m-2}}+1)(x^{\frac{4^m}{2}}-1)(x^{4^m}+1)^2(x^{\frac{4^m}{2}}+1)^2 \\ \equiv\ &(x+1)(x^2-1)(x^4+1)\cdots(x^{2^{2m-2}}+1)(x^{2^{2m-1}}+1)(x^{2^{2m}}+1)(x^{2\cdot 4^m}-1) \\ \equiv\ &(x+1)(x^2-1)(x^4+1)\cdots(x^{2^{2m}}+1)(x^{2^{2m+1}}-1) \pmod 4,\end{aligned}$$

i.e., the congruence relation $(**)$ also holds for $m+1$ as desired. This proves Theorem 3.

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References

1. Matsuda, O.; Tsuyama Math Club in National Institute of Technology, Tsuyama College. 11 *kara Hajimaru sūgaku—k-Pasukaru Sankakkei, k-Fibonatchi Sūretsu, chōōgonsū*; [Mathematics that begins with 11—*k*-Pascal's Triangles, *k*-Fibonacci Sequences and the Super Golden Numbers]; Tokyo Tosho Co. Ltd.: Tokyo, Japan, 2008. (In Japanese)
2. Yamagami, A.; Harada, H. On a generalization of a Lucas' result on the Pascal triangle and Mersenne numbers. *JP J. Algebra Number Theory Appl.* **2019**, *42* 159–169. [CrossRef]
3. Lucas, É. *Théorie des Nombres*; Gauthier-Villars et Fils; Libraires, du Bureau des Longitudes, de l'École Polytechnique, Quai des Grands-Augustins: Paris, France, 1891; Volume 55.