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Mean Value of the General Dedekind Sums over Interval $[1, \frac{q}{p})$

Lei Liu and Zhefeng Xu *

School of Mathematics, Northwest University Xi'an, Xi'an 710069, China; liulei@stumail.nwu.edu.cn

* Correspondence: zfxu@nwu.edu.cn

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Abstract: Let $q > 2$ be a prime, p be a given prime with $p < q$. The main purpose of this paper is using transforms, the hybrid mean value of Dirichlet L-functions with character sums and the related properties of character sums to study the mean value of the general Dedekind sums over interval $[1, \frac{q}{p})$, and give some interesting asymptotic formulae.

Keywords: general Dedekind sum; character sums; L-functions; mean value

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1. Introduction

For a positive integer k and an arbitrary integer h with $(h, k) = 1$, the classical Dedekind sum $S(h, k)$ is defined by

$$S(h,k) = \sum_{a=1}^{k} \left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ not be an integer;} \\ 0, & \text{if } x \text{ be an integer.} \end{cases}$$

It plays a significant role in the transformation theory of the Dedekind η function. In [1–5], many researchers have investigated the various properties of $S(h, k)$. Perhaps the most well-known property of the Dedekind sums is the reciprocity formula

$$S(h,k) + S(k,h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4}.$$

Conrey, J.B. et al. [2] studied the $2m$-th power mean of $S(h, k)$, and proved the following important asymptotic formula

$$\sum_{h=1}^{k}{}' S^{2m}(h,k) = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{9/5} + k^{2m-1+1/(m+1)}\right)\log^3 k\right),$$

where $\sum_h'$ denotes summation over all h such that $(h, k) = 1$ and $f_m(k)$ is defined by the Dirichlet series

$$\sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s).$$

For $m \geq 2$, Jia, C. [3] reduced the error terms to $O(k^{2m-1})$. While for $m = 1$, Zhang, W. [5] showed

$$\sum_{h=1}^{k}{}' S^2(h,k) = \frac{5}{144}k\phi(k)\prod_{p^\alpha \| k}\left(1 + \frac{1}{p} + \frac{1}{p^2}\right)^{-1}\left[\left(1 + \frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}\right] + O\left(k\exp\left(\frac{4\log k}{\log\log k}\right)\right).$$

Zhang, W. and Yi, Y. [6] studied the first mean value of $S(h,k)$, and obtained an asymptotic formula

$$\sum_{n\le N}{}' S(n,k)=\frac{1}{12}\phi(k)\left(\log N+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)+O\left(\frac{k2^{\omega(k)}}{N}+Nk^{\epsilon}\right)$$

for positive integer k and $1<N\le\frac{1}{2}k$.

Zhang, W. [7] defined the general Dedekind sum $S(h,n,k)$ as follows:

$$S(h,n,k)=\sum_{a=1}^{k}\overline{B_n}\left(\frac{a}{k}\right)\overline{B_n}\left(\frac{ah}{k}\right),$$

where

$$\overline{B_n}(x)=\begin{cases}B_n(x-[x]), & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

called the n-th periodic Bernoulli polynomials defined on $0<x\le 1$, and $B_n(x)$ is the n-th Bernoulli polynomials. Clearly, $S(h,1,k)=S(h,k)$ is the classical Dedekind sum.

Recently, Kim et al. [8–10] studied the poly-Dedekind sums given by

$$S_n^{(m)}(h,k)=\sum_{a=1}^{k}\left(\frac{a}{k}\right)\overline{B}_n^{(m)}\left(\frac{ah}{k}\right),$$

where $\overline{B}_n^{(m)}(x)=B_n^{(m)}(x-[x])$ are the type 2 poly-Bernoulli functions of index m, and obtained some interesting identities. Obviously, $S_1^{(1)}=S(h,k)$ is the classical Dedekind sum.

Let $q>2$ be a prime, p be a given prime with $p<q$. Using the similar method of Shparlinski, I.E. [11] and combining with the mean value of L-functions and estimate of character sums, the authors and Wang, N. [12] studied the mean value distribution of the general Dedekind sums over short interval, that is

$$\sum_{a\le N}\sum_{b\le N}a^l b^k S(a\bar{b},n,q),$$

here n and N be two positive integers with $q^{\epsilon}\le N\le q^{1-\epsilon}$, l,k be two non-negative integers and $\bar{b}$ denote the multiplicative inverse of b modulo q. However, in the final remarks, Shparlinski, I.E. [11] pointed out that "the author sees no reason why an appropriate asymptotic formula cannot hold for even larger values of N, up to $q/2$". In this paper we can take N to q/p, then through transform, mean value of Dirichlet L-functions and the properties of character sums to study the mean value of the general Dedekind sums over interval $[1,\frac{q}{p})$, and obtain some sharper asymptotic formulae for it.

Now we give the main conclusion.

Theorem 1. *Let $q>2$ be a prime, p be a given prime with $p<q$, n be a positive integer. Then we have*

(i) when n be an even number,

$$\sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}}S(a\bar{b},n,q)=\frac{(n!)^2q^2}{2^{2n-2}\pi^{2n}}\left(\frac{1}{2\pi^2}C_{p,n}-\frac{\zeta^2(n)}{p^2}\right)+O(q^{1+\epsilon}),$$

here $C_{p,n}=\sum_{u=1}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}$, $\gamma_p(u,n)=\sum_{d_1d_2=u}\sin\frac{2\pi d_1}{p}\cdot d_2^{1-n}$.

(ii) when n be an odd number,

$$\sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}}S(a\bar{b},n,q)=\frac{(n!)^2q^2}{2^{2n-2}\pi^{2n+2}}T_{p,n}+O(q^{1+\epsilon}),$$

here

$$\begin{aligned} T_{p,n} = \quad & \frac{1}{2}\sum_{u=1}^{\infty}\frac{\nu_p^2(u,n)}{u^2} + \frac{1}{2}\left(1+\frac{1}{p^2}\right)\sum_{u=1}^{\infty}\frac{\nu^2(u,n)}{u^2} \\ + \quad & \frac{1}{p^2}\sum_{u=1}^{\infty}\frac{\nu(u,n)\nu_p(pu,n)}{u^2} - \frac{1}{p^2}\sum_{u=1}^{\infty}\frac{\nu(u,n)\nu(pu,n)}{u^2} \\ - \quad & \sum_{u=1}^{\infty}\frac{\nu(u,n)\nu_p(u,n)}{u^2}, \end{aligned}$$

$\nu_p(u,n) = \sum_{d_1 d_2 = u} \cos\frac{2\pi d_1}{p} \cdot d_2^{1-n}$, $\nu(u,n) = \sum_{d|u} d^{1-n}$.

It is clear that $C_{p,n}$ and $T_{p,n}$ are constants depending on p and n. From our theorem we may immediately deduce the following corollaries:

Corollary 1. *Let $q > 2$ be a prime, we have*

$$\sum_{a<\frac{q}{2}}\sum_{b<\frac{q}{2}} S(a\bar{b},2,q) = -\frac{q^2}{144} + O(q^{1+\epsilon}),$$

$$\sum_{a<\frac{q}{2}}\sum_{b<\frac{q}{2}} S_2^{(1)}(a\bar{b},q) = \frac{\pi}{144i}q^2 + O(q^{1+\epsilon}).$$

Corollary 2. *Let $q > 2$ be a prime, we have*

$$\sum_{a<\frac{q}{2}}\sum_{b<\frac{q}{2}} S(a\bar{b},4,q) = -\frac{q^2}{3600} + O(q^{1+\epsilon}),$$

$$\sum_{a<\frac{q}{2}}\sum_{b<\frac{q}{2}} S_4^{(1)}(a\bar{b},q) = \frac{\pi^3 i}{10800}q^2 + O(q^{1+\epsilon}).$$

For the general index m, the method of our article does not obtain the expected result. It would be an interesting question to continue to study the mean value of $S_n^{(m)}(h,k)$.

2. Some Lemmas

To prove the theorem, We need the following lemmas.

Lemma 1. *Let k and r be integers with $k \geq 2$ and $(r,k) = 1$, χ be a Dirichlet character modulo k. Then we have*

$$\sum_{\chi \bmod k}{}^{*} \chi(r) = \sum_{d|(k,r-1)} \mu\left(\frac{k}{d}\right)\phi(d)$$

where $\sum^*_{\chi \bmod k}$ denotes the summation over all primitive characters modulo k.

Proof. See Lemma 4 of reference [13]. □

Lemma 2. *Let $k \geq 3$ and h be two integers with $(h,k) = 1$, n be positive integer. Then we have*

$$S(h,n,k) = \frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(h)\left|L(n,\chi)\right|^2$$

for odd n and

$$S(h,n,k) = \frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}} \sum_{d|k} \frac{d^{2n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=1}} \chi(h) \left| L(n,\chi) \right|^2 - \frac{(n!)^2}{4^{n-1}\pi^{2n}} \zeta^2(n)$$

for even n.

Proof. See Theorem of reference [7]. □

Lemma 3. *Let χ be a primitive Dirichlet character modulo k. Then for any real number $\lambda \in [0,1)$ with $\lambda \neq \frac{r}{k}$, we have*

$$\sum_{a=1}^{[\lambda k]} \chi(a) = \begin{cases} \frac{\tau(\chi)}{\pi} \sum\limits_{n=1}^{\infty} \frac{\bar{\chi}(n)\sin(2\pi n \lambda)}{n}, & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi)}{\pi i} \sum\limits_{n=1}^{\infty} \frac{\bar{\chi}(n)(1-\cos(2\pi n \lambda))}{n}, & \text{if } \chi(-1) = -1. \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x, $\tau(\chi) = \sum_{a=1}^{k} \chi(a) e(\frac{a}{k})$ is the Gauss sum and $e(y) = e^{2\pi i y}$.

Proof. See Section 3.1 of [14]. □

Lemma 4. *Let $k > 3$ be an integer and p be a prime with $p \nmid k$ and $p < k$. Then we have the identity*

(i) for even primitive character $\chi \bmod k$,

$$\sum_{a<\frac{k}{p}} \chi(a) = \frac{\tau(\chi)}{\pi i \phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) L(1, \xi\bar{\chi}).$$

(ii) for odd primitive character $\chi \bmod k$,

$$\sum_{a<\frac{k}{p}} \chi(a) = \frac{\tau(\chi)}{\pi i} \left(1 - \frac{\bar{\chi}(p)}{p}\right) L(1,\bar{\chi}) + \frac{i\tau(\chi)}{\pi\phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1, \xi\bar{\chi}).$$

Proof. From Lemma 3, (i) when $\chi(-1) = 1$, we can write

$$\sum_{a<\frac{k}{p}} \chi(a) = \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\sin(2\pi n/p)}{n} = \frac{\tau(\chi)}{\pi} \sum_{h=1}^{p-1} \sin\frac{2\pi h}{p} \sum_{\substack{n=1 \\ n \equiv h (\bmod p)}}^{\infty} \frac{\bar{\chi}(n)}{n}.$$

Now

$$\sum_{\substack{n=1 \\ n \equiv h (\bmod p)}}^{\infty} \frac{\bar{\chi}(n)}{n} = \frac{1}{\phi(p)} \sum_{\xi \bmod p} \bar{\xi}(h) \sum_{n=1}^{\infty} \frac{\xi\bar{\chi}(n)}{n} = \frac{1}{\phi(p)} \sum_{\xi \bmod p} \bar{\xi}(h) L(1, \xi\bar{\chi}).$$

Furthermore,

$$\begin{aligned} \sum_{h=1}^{p-1} \bar{\xi}(h) \sin\frac{2\pi h}{p} &= \frac{1}{2i} \sum_{h=1}^{p-1} \bar{\xi}(h) \left(e\left(\frac{h}{p}\right) - e\left(\frac{-h}{p}\right) \right) \\ &= \frac{1}{2i} (1 - \xi(-1)) \tau(\bar{\xi}), \end{aligned}$$

where $e(x) = e^{2\pi i x}$.

Thus, we obtain the identity for $\chi(-1) = 1$,

$$\sum_{a<\frac{k}{p}} \chi(a) = \frac{\tau(\chi)}{\pi i \phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) L(1, \xi\bar{\chi}).$$

(ii) when $\chi(-1) = -1$, we can write

$$\begin{aligned}
\sum_{a<\frac{k}{p}} \chi(a) &= \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{1-\cos(2\pi n/p)}{n} \\
&= \frac{\tau(\chi)}{\pi i} \left(L(1,\bar{\chi}) - \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\cos(2\pi n/p)}{n} \right) \\
&= \frac{\tau(\chi)}{\pi i} L(1,\bar{\chi}) + \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{i\cos(2\pi n/p)}{n}.
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{n=1}^{\infty} \bar{\chi}(n) \frac{i\cos(2\pi n/p)}{n} \\
= \; & i \sum_{h=1}^{p-1} \cos \frac{2\pi h}{p} \sum_{\substack{n=1 \\ n\equiv h(\bmod p)}}^{\infty} \frac{\bar{\chi}(n)}{n} \\
= \; & i \sum_{h=1}^{p-1} \cos \frac{2\pi h}{p} \cdot \frac{1}{\phi(p)} \sum_{\xi \bmod p} \bar{\xi}(h) L(1,\xi\bar{\chi}) + \frac{i\bar{\chi}(p)}{p} L(1,\bar{\chi}) \\
= \; & \frac{i}{\phi(p)} \sum_{\xi \bmod p} \left(\sum_{h=1}^{p-1} \bar{\xi}(h) \cos \frac{2\pi h}{p} \right) L(1,\xi\bar{\chi}) + \frac{i\bar{\chi}(p)}{p} L(1,\bar{\chi}),
\end{aligned}$$

noting that

$$e\left(\frac{h}{p}\right) + e\left(\frac{-h}{p}\right) = 2\cos\frac{2\pi h}{p},$$

we have

$$\sum_{h=1}^{p-1} \bar{\xi}(h) \cos \frac{2\pi h}{p} = \frac{1}{2} \sum_{h=1}^{p-1} \bar{\xi}(h) \left[e\left(\frac{h}{p}\right) + e\left(\frac{-h}{p}\right) \right] = \frac{1+\bar{\xi}(-1)}{2} \tau(\bar{\xi}).$$

So, we can get

$$\sum_{n=1}^{\infty} \bar{\chi}(n) \frac{i\cos(2\pi n/p)}{n} = \frac{i}{\phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) + \frac{i\bar{\chi}(p)}{p} L(1,\bar{\chi}).$$

Thus, we obtain the identity for $\chi(-1) = -1$,

$$\sum_{a<\frac{k}{p}} \chi(a) = \frac{\tau(\chi)}{\pi i} \left(1 - \frac{\bar{\chi}(p)}{p}\right) L(1,\bar{\chi}) + \frac{i\tau(\chi)}{\pi\phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}).$$

This proves Lemma 4. □

Lemma 5. *Let $q > 2$ be a prime, p be a given prime with $p < q$, n be a positive integer. Then we have*

$$\sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(a\bar{b}) \, |L(n,\chi)|^2 = \frac{q^2}{2\pi^2} C_{p,n} + O(q^{1+\epsilon}),$$

here $C_{p,n} = \sum_{u=1}^{\infty} \frac{\gamma_p^2(u,n)}{u^2}$, $\gamma_p(u,n) = \sum_{d_1 d_2 = u} \sin \frac{2\pi d_1}{p} \cdot d_2^{1-n}$. *and*

$$\sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a\bar{b}) \, |L(n,\chi)|^2 = \frac{q^2}{\pi^2} T_{p,n} + O(q^{1+\epsilon}),$$

here

$$\begin{aligned} T_{p,n} \quad = \quad & \frac{1}{2} \sum_{u=1}^{\infty} \frac{\nu_p^2(u,n)}{u^2} + \frac{1}{2}\left(1 + \frac{1}{p^2}\right) \sum_{u=1}^{\infty} \frac{\nu^2(u,n)}{u^2} \\ & + \frac{1}{p^2} \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu_p(pu,n)}{u^2} - \frac{1}{p^2} \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu(pu,n)}{u^2} \\ & - \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu_p(u,n)}{u^2}, \end{aligned}$$

$\nu_p(u,n) = \sum_{d_1 d_2 = u} \cos \frac{2\pi d_1}{p} \cdot d_2^{1-n}$, $\nu(u,n) = \sum_{d|u} d^{1-n}$.

Proof. From Lemma 4, for $\chi(-1) = 1$, we have

$$\begin{aligned} & \sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(a\bar{b}) \, |L(n,\chi)|^2 \\ = \quad & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(n,\chi)|^2 \sum_{a<\frac{q}{p}} \chi(a) \sum_{b<\frac{q}{p}} \chi(\bar{b}) \\ = \quad & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(n,\chi)|^2 \frac{\tau(\chi)}{\pi i \phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) \frac{\tau(\bar{\chi})}{\pi i \phi(p)} \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) L(1,\bar{\lambda}\chi) \\ = \quad & \frac{-q}{\pi^2 \phi^2(p)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(n,\chi)|^2 \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) L(1,\bar{\lambda}\chi) \\ = \quad & \frac{-q}{\pi^2 \phi^2(p)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) \\ & \times \left(\sum_{u=1}^{\infty} \frac{\bar{\chi}(u) \sum\limits_{d_1 d_2 = u} \xi(d_1) d_2^{1-n}}{u} \right) \left(\sum_{v=1}^{\infty} \frac{\chi(v) \sum\limits_{d_1' d_2' = v} \bar{\lambda}(d_1') {d_2'}^{1-n}}{v} \right) \\ := \quad & \frac{-q}{\pi^2 \phi^2(p)} M. \end{aligned}$$

For convenience, we put

$$A(\bar{\chi},\xi,y)=\sum_{N<u\le y}\bar{\chi}(u)\sum_{d_1d_2=u}\xi(d_1)d_2^{1-n},$$

where N is a parameter with $q\le N<q^5$. Then from Abel's identity we have

$$\begin{aligned}
M &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{u=1}^{\infty}\frac{\bar{\chi}(u)\sum\limits_{d_1d_2=u}\xi(d_1)d_2^{1-n}}{u}\right)\left(\sum_{v=1}^{\infty}\frac{\chi(v)\sum\limits_{d_1'd_2'=v}\bar{\lambda}(d_1'){d_2'}^{1-n}}{v}\right)\\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{u\le N}\frac{\bar{\chi}(u)\sum\limits_{d_1d_2=u}\xi(d_1)d_2^{1-n}}{u}+\int_N^{\infty}\frac{A(\bar{\chi},\xi,y)}{y^2}dy\right)\\
&\quad\times\left(\sum_{v\le N}\frac{\chi(v)\sum\limits_{d_1'd_2'=v}\bar{\lambda}(d_1'){d_2'}^{1-n}}{v}+\int_N^{\infty}\frac{A(\chi,\bar{\lambda},y)}{y^2}dy\right)\\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{u\le N}\frac{\bar{\chi}(u)\sum\limits_{d_1d_2=u}\xi(d_1)d_2^{1-n}}{u}\right)\left(\sum_{v\le N}\frac{\chi(v)\sum\limits_{d_1'd_2'=v}\bar{\lambda}(d_1'){d_2'}^{1-n}}{v}\right)\\
&\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{u\le N}\frac{\bar{\chi}(u)\sum\limits_{d_1d_2=u}\xi(d_1)d_2^{1-n}}{u}\right)\left(\int_N^{\infty}\frac{A(\chi,\bar{\lambda},y)}{y^2}dy\right)\\
&\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{v\le N}\frac{\chi(v)\sum\limits_{d_1'd_2'=v}\bar{\lambda}(d_1'){d_2'}^{1-n}}{v}\right)\left(\int_N^{\infty}\frac{A(\bar{\chi},\xi,y)}{y^2}dy\right)\\
&\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\left(\int_N^{\infty}\frac{A(\bar{\chi},\xi,y)}{y^2}dy\right)\left(\int_N^{\infty}\frac{A(\chi,\bar{\lambda},y)}{y^2}dy\right)\\
&:= M_1+M_2+M_3+M_4, \qquad (1)
\end{aligned}$$

we shall calculate each term in the above expression.

(i) From Lemma 1 we have

$$\begin{aligned}
\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(\bar{u}v) &= \frac{1}{2} \sum_{\chi \bmod q} (1+\chi(-1))\chi(\bar{u}v) \\
&= \frac{1}{2} \sum_{\chi \bmod q} \chi(\bar{u}v) + \frac{1}{2} \sum_{\chi \bmod q} \chi(-\bar{u}v) \\
&= \frac{1}{2} \sum_{d|(q,\bar{u}v-1)} \mu\left(\frac{q}{d}\right) \phi(d) + \frac{1}{2} \sum_{d|(q,\bar{u}v+1)} \mu\left(\frac{q}{d}\right) \phi(d).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi})\xi(d_1) &= \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \sum_{a=1}^{p} \bar{\xi}(a) e^{2\pi i \frac{a}{p}} \xi(d_1) \\
&= \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \sum_{a=1}^{p} \bar{\xi}(a) e^{2\pi i \frac{ad_1}{p}} \\
&= \frac{\phi(p)}{2} e^{2\pi i \frac{d_1}{p}} - \frac{\phi(p)}{2} e^{2\pi i \frac{(p-1)d_1}{p}} \\
&= i\phi(p) \sin \frac{2\pi d_1}{p}.
\end{aligned}$$

Similarly, we can also get

$$\sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda)\bar{\lambda}(d_1') = i\phi(p) \sin \frac{2\pi d_1'}{p}.$$

So, we have

$$\begin{aligned}
& \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) \sum_{d_1 d_2 = u} \xi(d_1) d_2^{1-n} \sum_{d_1' d_2' = v} \bar{\lambda}(d_1') {d_2'}^{1-n} \\
= \; & -\phi^2(p) \sum_{d_1 d_2 = u} \sin \frac{2\pi d_1}{p} \cdot d_2^{1-n} \sum_{d_1' d_2' = v} \sin \frac{2\pi d_1'}{p} \cdot {d_2'}^{1-n} \\
= \; & -\phi^2(p) \gamma_p(u,n) \gamma_p(v,n).
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
M_1 &= \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) \\
&\quad \times \left(\sum_{u \le N} \frac{\bar{\chi}(u) \sum_{d_1 d_2 = u} \xi(d_1) d_2^{1-n}}{u} \right) \left(\sum_{v \le N} \frac{\chi(v) \sum_{d_1' d_2' = v} \bar{\lambda}(d_1') {d_2'}^{1-n}}{v} \right) \\
&= \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda)
\end{aligned}$$

$$
\begin{aligned}
&\times \sideset{}{'}\sum_{1\le u\le N}\sideset{}{'}\sum_{1\le v\le N}\frac{\sum\limits_{d_1d_2=u}\xi(d_1)d_2^{1-n}\sum\limits_{d_1'd_2'=v}\bar{\lambda}(d_1')d_2'^{\,1-n}}{uv}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(\bar{u}v)\\
=\;& -\frac{1}{2}\sideset{}{'}\sum_{1\le u\le N}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\sum_{d|(q,\bar{u}v-1)}\mu\left(\frac{p}{d}\right)\phi(d)\\
& -\frac{1}{2}\sideset{}{'}\sum_{1\le u\le N}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\sum_{d|(q,\bar{u}v+1)}\mu\left(\frac{q}{d}\right)\phi(d)\\
=\;& -\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
& -\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u\equiv -v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv},
\end{aligned}
\tag{2}
$$

where $\sideset{}{'}\sum\limits_{1\le n\le N}$ denotes the summation over n from 1 to N such that $(n,q)=1$.

For calculation convenience, we divide the sum over u or v into four cases: (i) $d\le u,v\le N$; (ii) $d\le u\le N$ and $1\le v\le d-1$; (iii) $1\le u\le d-1$ and $d\le v\le N$; iv) $1\le u,v\le d-1$. So we have

$$
\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{d\le u\le N\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{d\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
\ll\;& \sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{N}{d}}\sideset{}{'}\sum_{\substack{l_1=1\\ l_1\equiv l_2(\bmod d)}}^{d-1}\sideset{}{'}\sum_{l_2=1}^{d-1}\frac{\phi^2(p)\gamma_p(r_1d+l_1,n)\gamma_p(r_2d+l_2,n)}{(r_1d+l_1)(r_2d+l_2)}\\
\ll\;& \phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{N}{d}}\sideset{}{'}\sum_{l_1=1}^{d-1}\frac{\sum\limits_{d_2|(r_1d+l_1)}d_2^{1-n}\sum\limits_{d_2'|(r_2d+l_1)}d_2'^{\,1-n}}{(r_1d+l_1)(r_2d+l_1)}\\
\ll\;& \phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{N}{d}}\sideset{}{'}\sum_{l_1=1}^{d-1}\frac{\tau_2(r_1d+l_1)\tau_2(r_2d+l_1)}{(r_1d+l_1)(r_2d+l_1)}\\
\ll\;& \phi^2(p)\sum_{d|q}\frac{\phi(d)}{d}\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le r_2\le \frac{N}{d}}\frac{[(r_1d+1)(r_2d+1)]^{\epsilon}}{r_1r_2}\\
\ll\;& \phi^2(p)q^{\epsilon}.
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{d\le u\le N\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le d-1}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
\ll\;& \sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sideset{}{'}\sum_{\substack{1\le v\le d-1\\ v\equiv l_1(\bmod d)}}\sideset{}{'}\sum_{l_1=1}^{d-1}\frac{\phi^2(p)\gamma_p(r_1d+l_1,n)\gamma_p(v,n)}{(r_1d+l_1)v}\\
\ll\;& \phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sideset{}{'}\sum_{1\le v\le d-1}\frac{\tau_2(r_1d+v)\tau_2(v)}{(r_1d+v)v}\\
\ll\;& \phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le r_1\le \frac{N}{d}}\sum_{1\le v\le d-1}(r_1vd)^{\epsilon-1}
\end{aligned}
$$

$$\ll \quad \phi^2(p)q^{\epsilon}. \tag{4}$$

and

$$\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le d-1\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{d\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
\ll\quad & \phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le u\le d-1}\sum_{1\le r_2\le\frac{N}{d}}(ur_2d)^{\epsilon-1}\\
\ll\quad & \phi^2(p)q^{\epsilon},
\end{aligned}\tag{5}$$

where we have used the estimate $\tau_2(n)\ll n^{\epsilon}$.

For the case $1\le u,v\le d-1$, the solution of the congruence $u\equiv v(\bmod d)$ is $u=v$. Hence,

$$\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le d-1\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le d-1}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
=\quad & \phi^2(p)\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{1\le u\le d-1}\frac{\gamma_p^2(u,n)}{u^2}\\
=\quad & \phi^2(p)\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{u=1\\ (u,q)=1}}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}+O(\phi^2(p)q^{\epsilon})\\
=\quad & (q-2)\phi^2(p)\sum_{\substack{u=1\\ (u,q)=1}}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}+O(\phi^2(p)q^{\epsilon})\\
=\quad & (q-2)\phi^2(p)\left(\sum_{u=1}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}-\sum_{\substack{u=1\\ q|u}}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}\right)+O(\phi^2(p)q^{\epsilon})\\
=\quad & (q-2)\phi^2(p)\sum_{u=1}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}+O(\phi^2(p)q^{\epsilon}).
\end{aligned}\tag{6}$$

Then from (3)–(6), we have

$$\begin{aligned}
&-\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u\equiv v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
=\quad & -\frac{\phi^2(p)(q-2)}{2}\sum_{u=1}^{\infty}\frac{\gamma_p^2(u,n)}{u^2}+O(\phi^2(p)q^{\epsilon})\\
=\quad & -\frac{\phi^2(p)(q-2)}{2}C_{p,n}+O(\phi^2(p)q^{\epsilon}).
\end{aligned}\tag{7}$$

Similarly, we can also get the estimate

$$\begin{aligned}
&-\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u\equiv -v(\bmod d)}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
=\quad & -\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u+v=d}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le u\le N\\ u+v=ld,l\ge 2}}\sideset{}{'}\sum_{1\le v\le N}\frac{\phi^2(p)\gamma_p(u,n)\gamma_p(v,n)}{uv}\\
\ll\ &\phi^2(p)\sum_{d|q}\phi(d)\sum_{1\le u\le d-1}\frac{\gamma_p(u,n)\gamma_p(d-u,n)}{u(d-u)}\\
&+\phi^2(p)\sum_{d|q}\phi(d)\sideset{}{'}\sum_{1\le u\le N}\sum_{l=\left[\frac{u}{d}\right]+2}^{\left[\frac{N+u}{d}\right]}\frac{\gamma_p(u,n)\gamma_p(ld-u,n)}{ldu-u^2}\\
\ll\ &\phi^2(p)\sum_{d|q}\frac{\phi(d)}{d}\sum_{1\le u\le d-1}\frac{u^\epsilon(d-u)^\epsilon}{u}\\
&+\phi^2(p)\sum_{d|q}\frac{\phi(d)}{d}\sideset{}{'}\sum_{1\le u\le N}\sum_{l=\left[\frac{u}{d}\right]+2}^{\left[\frac{N+u}{d}\right]}\frac{u^\epsilon(ld-u)^\epsilon}{lu-\frac{u^2}{d}}\\
\ll\ &\phi^2(p)\left(q^\epsilon+\sum_{d|q}\frac{\phi(d)}{d}\sum_{u=1}^{N}\sum_{l=1}^{N}\frac{u^\epsilon l^\epsilon}{ul}\right)\\
\ll\ &\phi^2(p)q^\epsilon. \qquad (8)
\end{aligned}
$$

Then combining (2), (7) and (8), we have

$$M_1=-\tfrac{\phi^2(p)(q-2)}{2}C_{p,n}+O(\phi^2(p)q^\epsilon). \qquad (9)$$

(ii) From Lemma 4 of [15], we have the estimate

$$\sum_{\chi\ne\chi_0}|A(y,\chi)|^2\ll y^{1+\epsilon}q^2,$$

where χ_0 denotes the principal character modulo q, $A(y,\chi)=\sum_{N<n\le y}\chi(n)\tau_2(n)$. Then from the Cauchy inequality we can easily get

$$\sum_{\chi(-1)=1}|A(y,\chi)|\ll\sum_{\chi\ne\chi_0}|A(y,\chi)|\ll y^{\frac{1}{2}+\epsilon}q^{\frac{3}{2}}.$$

Using this estimate we have

$$
\begin{aligned}
M_2 &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{\substack{\xi \bmod p\\ \xi(-1)=-1}}\tau(\bar{\xi})\sum_{\substack{\lambda \bmod p\\ \lambda(-1)=-1}}\tau(\lambda)\\
&\quad\times\left(\sum_{u\le N}\frac{\bar{\chi}(u)\sum_{d_1d_2=u}\xi(d_1)d_2^{1-n}}{u}\right)\left(\int_N^\infty\frac{A(\chi,\bar{\lambda},y)}{y^2}dy\right)\\
&= -\phi^2(p)\sum_{1\le u\le N}\frac{\bar{\chi}(u)\gamma_p(u,n)}{u}\int_N^\infty\frac{1}{y^2}\left(\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\sum_{N<v\le y}\chi(v)\gamma_k(v,n)\right)dy\\
&\ll \phi^2(p)\sum_{1\le u\le N}\frac{\bar{\chi}(u)d(u)}{u}\int_N^\infty\frac{1}{y^2}\left(\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}A(y,\chi)\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&\ll \phi^2(p) \sum_{1\le u\le N} u^{\epsilon-1} \int_N^\infty \frac{1}{y^2} \left(\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |A(y,\chi)| \right) dy \\
&\ll \phi^2(p) N^\epsilon \int_N^\infty \frac{q^{\frac{3}{2}} y^{\frac{1}{2}+\epsilon_1}}{y^2} dy \ll \phi^2(p) \frac{q^{\frac{3}{2}}}{N^{\frac{1}{2}-\epsilon}}. \qquad (10)
\end{aligned}
$$

(iii) Similar to (ii), we can also get

$$
M_3 \ll \phi^2(p) \frac{q^{\frac{3}{2}}}{N^{\frac{1}{2}-\epsilon}}. \qquad (11)
$$

(iv) Using the same discussion in (ii), and making use of the absolute convergent properties of the integral, we can calculate

$$
\begin{aligned}
M_4 &= \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \sum_{\substack{\xi \bmod p \\ \xi(-1)=-1}} \tau(\bar{\xi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=-1}} \tau(\lambda) \left(\int_N^\infty \frac{A(\bar{\chi},\xi,y)}{y^2} dy \right) \left(\int_N^\infty \frac{A(\chi,\bar{\lambda},y)}{y^2} dy \right) \\
&= \phi^2(p) \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \left(\int_N^\infty \frac{1}{y^2} \sum_{N<u\le y} \bar{\chi}(u)\gamma_p(u,n) dy \right) \left(\int_N^\infty \frac{1}{y^2} \sum_{N<v\le y} \chi(v)\gamma_p(v,n) dy \right) \\
&\ll \phi^2(p) \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \left(\int_N^\infty \frac{A(y,\bar{\chi})}{y^2} dy \right) \left(\int_N^\infty \frac{A(y,\chi)}{y^2} dy \right) \\
&\le \phi^2(p) \int_N^\infty \int_N^\infty \frac{1}{y^2 z^2} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |A(y,\bar{\chi})| |A(y,\chi)| dy dz \\
&\ll \phi^2(p) \int_N^\infty \frac{1}{y^2} \int_N^\infty \frac{1}{z^2} \left(\sum_{\chi\ne\chi_0} |A(y,\bar{\chi})|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi\ne\chi_0} |A(y,\chi)|^2 \right)^{\frac{1}{2}} dy dz \\
&\ll \phi^2(p) \left(\int_N^\infty \frac{1}{y^2} \left(\sum_{\chi\ne\chi_0} |A(y,\chi)|^2 \right)^{\frac{1}{2}} dy \right)^2 \\
&\ll \phi^2(p) \left(\int_N^\infty \frac{q}{y^{\frac{3}{2}-\epsilon}} dy \right)^2 \ll \phi^2(p) \frac{q^2}{N^{1-\epsilon}}. \qquad (12)
\end{aligned}
$$

Now taking $N = q^4$ and $\epsilon < \frac{1}{2}$, combining (1) and (9)–(12), we have

$$
M = -\frac{\phi^2(p)(q-2)}{2} C_{p,n} + O(\phi^2(p) q^\epsilon).
$$

Thus we obtain the asymptotic formula for $\chi(-1) = 1$,

$$
\sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(a\bar{b}) |L(n,\chi)|^2 = \frac{-q}{\pi^2\phi^2(p)} M = \frac{q^2}{2\pi^2} C_{p,n} + O(q^{1+\epsilon}).
$$

For $\chi(-1) = -1$, from Lemma 4, we have

$$
\begin{aligned}
&\sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a\bar{b}) \, |L(n,\chi)|^2 \\
= &\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(n,\chi)|^2 \left[\frac{\tau(\chi)}{\pi i} \left(1 - \frac{\bar{\chi}(p)}{p}\right) L(1,\bar{\chi}) + \frac{i\tau(\chi)}{\pi\phi(p)} \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) \right] \\
&\times \left[\frac{\tau(\bar{\chi})}{\pi i} \left(1 - \frac{\chi(p)}{p}\right) L(1,\chi) + \frac{i\tau(\bar{\chi})}{\pi\phi(p)} \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=1}} \tau(\lambda) L(1,\bar{\lambda}\chi) \right] \\
= &\frac{q}{\pi^2} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left(1 - \frac{\bar{\chi}(p)}{p}\right) \left(1 - \frac{\chi(p)}{p}\right) |L(n,\chi)|^2 \, |L(1,\chi)|^2 \\
&+ \frac{-q}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left(1 - \frac{\bar{\chi}(p)}{p}\right) |L(n,\chi)|^2 \, L(1,\bar{\chi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=1}} \tau(\lambda) L(1,\bar{\lambda}\chi) \\
&+ \frac{-q}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left(1 - \frac{\chi(p)}{p}\right) |L(n,\chi)|^2 \, L(1,\chi) \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) \\
&+ \frac{q}{\pi^2\phi^2(p)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(n,\chi)|^2 \sum_{\substack{\xi \bmod p \\ \xi(-1)=1}} \tau(\bar{\xi}) L(1,\xi\bar{\chi}) \sum_{\substack{\lambda \bmod p \\ \lambda(-1)=1}} \tau(\lambda) L(1,\bar{\lambda}\chi) \\
:= &A + B + C + D.
\end{aligned}
$$

Using the same method as proving $\chi(-1) = 1$, we can easily get

$$
A = \frac{q^2}{2\pi^2} \left(1 + \frac{1}{p^2}\right) \sum_{u=1}^{\infty} \frac{\nu^2(u,n)}{u^2} - \frac{q^2}{\pi^2 p^2} \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu(pu,n)}{u^2} + O(q^{1+\epsilon}),
$$

$$
B = C = \frac{q^2}{2\pi^2 p^2} \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu_p(pu,n)}{u^2} - \frac{p^2}{2\pi^2} \sum_{u=1}^{\infty} \frac{\nu(u,n)\nu_p(u,n)}{u^2} + O(q^{1+\epsilon}),
$$

$$
D = \frac{q^2}{2\pi^2} \sum_{u=1}^{\infty} \frac{\nu_p^2(u,n)}{u^2} + O(q^{1+\epsilon}).
$$

Thus we obtain the asymptotic formula for $\chi(-1) = -1$,

$$
\sum_{a<\frac{q}{p}} \sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a\bar{b}) \, |L(n,\chi)|^2 = \frac{q^2}{\pi^2} T_{p,n} + O(q^{1+\epsilon}).
$$

This proves Lemma 5. □

3. Proof of Theorem and Corollaries

In this section we will accomplish the proof of the theorem and corollaries. From Lemmas 2 and 5, we have:

(i) when n be an even number,

$$\begin{aligned}
&\sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}} S(a\bar{b},n,q) \\
= &\frac{(n!)^2 q}{4^{n-1}\pi^{2n}\phi(q)} \sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(a\bar{b})\,|L(n,\chi)|^2 + \frac{(n!)^2\zeta^2(n)}{4^{n-1}\pi^{2n}}\left(\frac{1}{q^{2n-1}}-1\right)\sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}} 1 \\
= &\frac{(n!)^2 q^2}{2^{2n-2}\pi^{2n}}\left(\frac{1}{2\pi^2}C_{p,n} - \frac{\zeta^2(n)}{p^2}\right) + O(q^{1+\epsilon}).
\end{aligned}$$

(ii) when n be an odd number,

$$\begin{aligned}
\sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}} S(a\bar{b},n,q) &= \frac{(n!)^2 q}{4^{n-1}\pi^{2n}\phi(q)} \sum_{a<\frac{q}{p}}\sum_{b<\frac{q}{p}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a\bar{b})\,|L(n,\chi)|^2 \\
&= \frac{(n!)^2 q}{4^{n-1}\pi^{2n}\phi(q)}\left(\frac{q^2}{\pi^2}T_{p,n} + O(q^{1+\epsilon})\right) \\
&= \frac{(n!)^2 q^2}{2^{2n-2}\pi^{2n+2}}T_{p,n} + O(q^{1+\epsilon}).
\end{aligned}$$

This completes the proof of the Theorem.

Taking $p = 2$ and $n = 2$ or 4 in the Theorem, we get $C_{2,2} = C_{2,4} = 0$. When n be an even number, from reference [7], we can easily calculate

$$S_n^{(1)}(h,k) = \frac{(2\pi)^{n-1}}{i^{n+1}\cdot n!}S(h,n,k).$$

Noting that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we easily get Corollarys 1 and 2.

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